ON THE $w$-DERIVED SET

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ABSTRACT. We introduce the notion of the $w$-derived set and $w$-dense, and investigate some of their properties.

1. Introduction

We define the notion of the $w$-derived set which is more general than that of the derived set, and examine the relation between the derived set and the $w$-derived set. And we investigate some properties of the $w$-derived set.

Also, we introduce the notion of $w$-dense, and study its property.

2. $w$-derived set

We denote by $A'$ and $\text{cl} A$ the derived set and the closure of the set $A$, respectively.

Definition 2.1. Let $X$ be a space. For a subset $A$ of $X$, the $w$-derived set $A'_w$ of $A$ is defined by

$$A'_w = \{ x \in X | (A - \{ x \}) \cap \text{cl}U \neq \emptyset \text{ for all neighborhoods } U \text{ of } x \}.$$

It is obvious that $A' \subset A'_w$, but as the following example illustrates, there exists a subset $A$ of a space $X$ such that $A' \neq A'_w$.

Example 2.2. Consider the topology $\tau = \{ \emptyset, \{ 1 \}, X \}$ on $X = \{ 0, 1 \}$. Let $A = \{ 0 \}$. Then $A' = \emptyset$ and $A'_w = \{ 1 \}$. Therefore, $A' \neq A'_w$.
Theorem 2.3. Let $X$ be a $T_1$-space and let $V$ be an open subset of $X$. Then $V' = V'_w$.

Proof. Let $x \in V'_w$. Then for any neighborhood $U$ of $x$, $(V - \{x\}) \cap \text{cl}U \neq \emptyset$. Take $y \in (V - \{x\}) \cap \text{cl}U$. Since $y \in \text{cl}U$ and $(V - \{x\})$ is a neighborhood of $y$, $(V - \{x\}) \cap U \neq \emptyset$. Thus $x \in V'$, so $V'_w \subseteq V'$. Since $V' \subseteq V'_w$, $V' = V'_w$. \[\square\]

Theorem 2.4. Let $A$ and $B$ be subsets of a space $X$. Then the followings hold.

1. If $A \subseteq B$, then $A'_w \subseteq B'_w$.
2. $(A \cup B)'_w = A'_w \cup B'_w$.

Proof. (1) Let $x \in A'_w$ and let $V$ be any neighborhood of $x$. Then $(A - \{x\}) \cap \text{cl}V \neq \emptyset$. Since $(A - \{x\}) \cap \text{cl}V \subseteq (B - \{x\}) \cap \text{cl}V$, $(B - \{x\}) \cap \text{cl}V \neq \emptyset$. Thus $x \in B'_w$, so $A'_w \subseteq B'_w$.

(2) Suppose $x \notin A'_w \cup B'_w$. Then $x \notin A'_w$ and $x \notin B'_w$. Therefore there exist neighborhoods $U$ and $V$ of $x$ such that $(A - \{x\}) \cap \text{cl}U = \emptyset$ and $(B - \{x\}) \cap \text{cl}V = \emptyset$. Now $U \cap V$ is a neighborhood of $x$ and $(A \cup B - \{x\}) \cap \text{cl}(U \cap V) = \emptyset$. Therefore $x \notin (A \cup B)'_w$, so $(A \cup B)'_w \subseteq A'_w \cup B'_w$. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (1) $A'_w \subseteq (A \cup B)'_w$ and $B'_w \subseteq (A \cup B)'_w$. Therefore $A'_w \cup B'_w \subseteq (A \cup B)'_w$. Hence $(A \cup B)'_w = A'_w \cup B'_w$. \[\square\]

Definition 2.5. Let $X$ be a space and let $A$ be a subset of $X$. The $w$-closure \(\text{cl}_w(A)\) of $A$ is defined by

\[\text{cl}_w(A) = \{x \in X| A \cap \text{cl}U \neq \emptyset \text{ for all neighborhoods } U \text{ of } x\}.\]

It is clear that $A \subseteq \text{cl}A \subseteq \text{cl}_w(A)$.

Theorem 2.6 [2]. For any open subset $U$ of $X$, $\text{cl}U = \text{cl}_w(U)$.

Theorem 2.7. For subsets $A$ and $B$ of a space $X$, the followings hold.

1. $\text{cl}_w \emptyset = \emptyset$.
2. $A \subseteq \text{cl}_w(A)$.
3. $\text{cl}_w(A) \subseteq \text{cl}_w(B)$ whenever $A \subseteq B$.
4. $\text{cl}_w(A \cup B) = \text{cl}_w(A) \cup \text{cl}_w(B)$.
5. $\text{cl}_w(A \cap B) \subseteq \text{cl}_w(A) \cap \text{cl}_w(B)$.

Proof. (1) Since $\emptyset$ is an open set and $\text{cl} \emptyset = \emptyset$, by Theorem 2.6, $\text{cl} \emptyset = \text{cl}_w \emptyset$. Therefore $\text{cl}_w \emptyset = \emptyset$. 

(2) Since $A \subseteq A \cup B$, by (3), $A \subseteq \text{cl}_w(A \cup B)$.

(3) If $A \subseteq B$, then $\text{cl}_w(A) \subseteq \text{cl}_w(B)$.

(4) $\text{cl}_w(A \cup B) = \text{cl}_w(A) \cup \text{cl}_w(B)$.

(5) $\text{cl}_w(A \cap B) \subseteq \text{cl}_w(A) \cap \text{cl}_w(B)$.
(2) Since $A \subseteq \text{cl}_w(A) \subseteq \text{cl}_u(A)$, $A \subseteq \text{cl}_w(A)$.

(3) Let $x \in \text{cl}_w(A)$ and let $U$ be any neighborhood of $x$. Then $A \cap \text{cl}_U \neq \emptyset$. Since $A \subseteq B$, $B \cap \text{cl}_U \neq \emptyset$. Therefore $x \in \text{cl}_w(B)$, so $\text{cl}_w(A) \subseteq \text{cl}_w(B)$.

(4) By (3), $\text{cl}_w(A) \subseteq \text{cl}_w(A \cup B)$ and $\text{cl}_w(B) \subseteq \text{cl}_w(A \cup B)$. Hence $\text{cl}_w(A) \cup \text{cl}_w(B) \subseteq \text{cl}_w(A \cup B)$. Now suppose $x \notin \text{cl}_w(A) \cup \text{cl}_w(B)$. Then $x \notin \text{cl}_w(A)$ and $x \notin \text{cl}_w(B)$. Therefore there are neighborhoods $U$ and $V$ of $x$ such that $A \cap \text{cl}_U = \emptyset$ and $B \cap \text{cl}_V = \emptyset$. Now $U \cap V$ is a neighborhood of $x$ and $(A \cup B) \cap \text{cl}(U \cap V) = \emptyset$. Thus $x \notin \text{cl}_w(A \cup B)$, so $\text{cl}_w(A \cup B) \subseteq \text{cl}_w(A) \cup \text{cl}_w(B)$. Hence $\text{cl}_w(A \cup B) = \text{cl}_w(A) \cup \text{cl}_w(B)$.

(5) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (3) $\text{cl}_w(A \cap B) \subseteq \text{cl}_w(A)$ and $\text{cl}_w(A \cap B) \subseteq \text{cl}_w(B)$. Therefore $\text{cl}_w(A \cap B) \subseteq \text{cl}_w(A) \cap \text{cl}_w(B)$. □

In the following example, we show that there exist subsets $A$ and $B$ of a space $X$ such that $\text{cl}_w(A \cap B) \neq \text{cl}_w(A) \cap \text{cl}_w(B)$.

**Example 2.8.** Let $\tau$ be a topology $\{\emptyset, \{1\}, X\}$ on $X = \{0, 1\}$, and let $A = \{0\}$ and $B = \{1\}$. Then $A \cap B = \emptyset$, so $\text{cl}_w(A \cap B) = \emptyset$. However, $\text{cl}_w(A) = \text{cl}_w(B) = X$, so $\text{cl}_w(A) \cap \text{cl}_w(B) = X$. Therefore $\text{cl}_w(A \cap B) \neq \text{cl}_w(A) \cap \text{cl}_w(B)$.

The following result is a consequence of Theorem 2.7.

**Corollary 2.9.** For subsets $A$ and $B$ of a space $X$,

$$X - \text{cl}_w(A \cup B) = (X - \text{cl}_w(A)) \cap (X - \text{cl}_w(B)).$$

**Theorem 2.10.** Let $A$ be a subset of a space $X$. Then $\text{cl}_w(A) = A \cup A'_w$.

*Proof.* Let $x \in \text{cl}_w(A)$ and let $U$ be any neighborhood of $x$. Then $A \cap \text{cl}_U \neq \emptyset$. If $x \notin A$, then $(A - \{x\}) \cap \text{cl}_U \neq \emptyset$ and hence $x \in A'_w$. If $x \in A$, we are through. Thus $\text{cl}_w(A) \subseteq A \cup A'_w$. On the other hand, since $A \subseteq \text{cl}_w(A)$ and $A'_w \subseteq \text{cl}_w(A)$, we obtain $A \cup A'_w \subseteq \text{cl}_w(A)$. □

**Theorem 2.11.** A space $X$ is regular if and only if for each subset $A$ of $X$, $A' = A'_w$.

*Proof.* For each subset $A$ of $X$, suppose $A' = A'_w$. Let $x \in X$ and let $U$ be a neighborhood of $x$. Since $x \notin X - U$ and $X - U = \text{cl}(X - U) = (X - U) \cup (X - U)' = (X - U) \cup (X - U)'_w$, we obtain $X - U = \text{cl}_w(X - U)$ from Theorem 2.10, so $x \notin \text{cl}_w(X - U)$. Therefore there is a neighborhood $V$ of $x$ such that $(X - U) \cap \text{cl}V = \emptyset$, so $\text{cl}V \subseteq X - (X - U) = U$. Hence $X$ is regular.
Conversely, suppose \( X \) is regular. Let \( x \in A'_w \). Then for any neighborhood \( U \) of \( x \), there exists a neighborhood \( V \) of \( x \) such that \( \text{cl} V \subset U \). Since \((A - \{x\}) \cap \text{cl} V \neq \emptyset\), \((A - \{x\}) \cap U \neq \emptyset\). Thus \( x \in A' \), so \( A'_w \subset A' \). Since \( A' \subset A'_w \), \( A' = A'_w \). □

**Definition 2.12.** A subset \( A \) of a space \( X \) is \( w \)-dense in \( X \) provided \( \text{cl}_w(A) = X \).

If a subset \( A \) of a space \( X \) is dense in \( X \), then \( A \) is \( w \)-dense in \( X \). However, in the Example 2.8, since \( \text{cl} A = \{0\} \) and \( \text{cl}_w(A) = X \), the converse does not hold generally, but in every regular space it holds.

**Theorem 2.13** [2]. A space \( X \) is regular if and only if for any subset \( A \) of \( X \), we have \( \text{cl} A = \text{cl}_w(A) \).

As a consequence of Theorem 2.13, we obtain the following corollary.

**Corollary 2.14.** Let \( A \) be a subset of a regular space \( X \). Then a set \( A \) is \( w \)-dense in \( X \) if and only if \( A \) is dense in \( X \).

**REFERENCES**


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