PSEUDOLINDELÒF SPACES AND HEWITT REALCOMPACTIFICATION OF PRODUCTS

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ABSTRACT. The concept of pseudoLindelöf spaces is introduced. It is shown that the followings are equivalent:

(a) for any two disjoint zero-sets in $X$, at least one of them is Lindelöf,
(b) $|vX - X| \leq 1$, and
(c) for any space $T$ with $X \subseteq T$, there is an embedding $f : vX \rightarrow vT$ such that $f(x) = x$ for all $x \in X$ and that if $X \times Y$ is a $z$-embedded pseudoLindelöf subspace of $vX \times vY$, then $v(X \times Y) = vX \times vY$.

1. Introduction

For any Tychonoff space $X$, $\beta X$ denotes the Stone-Čech comactification of $X$ and $vX$ denotes the Hewitt realcompactification of $X$. Glicksberg [5] showed that for any infinite spaces $X$ and $Y$, $\beta X \times \beta Y = \beta(X \times Y)$ if and only if $X \times Y$ is pseudocompact. An important open question in the theory of Hewitt realcompactifications of Tychonoff spaces concerns when the equality $vX \times vY = v(X \times Y)$ is valid (cf. [6]). Comfort [3] showed that if $X \times Y$ is $C^*$-embedded in $vX \times vY$, then $vX \times vY = v(X \times Y)$ and that if card$(X)$ or card$(Y)$ is non-measurable and $X \times Y$ is $C^*$-embedded in $X \times \beta Y$, then $vX \times vY = v(X \times Y)$. McArthur [7] has shown that $X \times Y$ is $C^*$-embedded in $X \times \beta Y$ if and only if the projection $\pi_X : X \times Y \rightarrow X$ is $z$-closed.

In this paper, we introduce the concept of pseudoLindelöf spaces and show that for a pseudoLindelöf space $X$, the followings are equivalent (cf. Theorem 2.6):

(a) $|vX - X| \leq 1$.
(b) For any two disjoint zero-sets in $X$, at least one of them is Lindelöf.
(c) For any space \( T \) with \( X \subseteq T \), \( vX \subseteq vT \).

Moreover, we will show that if \( X \times Y \) is a \( z \)-embedded pseudoLindelöf subspace of \( vX \times vY \), then \( v(X \times Y) = vX \times vY \) and that if \( X \times Y \) is a pseudoLindelöf space such that \( \text{card} (X) \) or \( \text{card} (Y) \) is non-measurable and \( X \) is a \( P \)-space, then \( v(X \times Y) = vX \times vY \) if and only if the projection \( \pi_X : X \times Y \to X \) is \( z \)-closed. For the terminology, we refer to Gillman-Jerison [4] and Porter-Woods [8].

2. PseudoLindelöf spaces

All topological spaces discussed in this paper are assumed to be Tychonoff spaces. For a space \( X \), \( C(X) \) denotes the ring of all continuous real-valued functions on \( X \) and \( C^*(X) \) denotes the subring of bounded functions. A subspace \( S \) of a space \( X \) is said to be \( C \)-embedded in \( X \) if every function in \( C(S) \) extends to a function in \( C(X) \). \( C^* \)-embedding is defined analogously. For a space \( X \), \( \beta X \) denotes the Stone-Čech compactification of \( X \), which is characterized as a compact space in which \( X \) is densely \( C^* \)-embedded and \( vX \) denotes the Hewitt realcompactification of \( X \), which is characterized as a realcompact space in which \( X \) is densely \( C \)-embedded. Both of the spaces \( \beta X \) and \( vX \) are unique up to a homeomorphism which extends the identity on \( X \).

Definition 2.1. A space \( X \) is called pseudoLindelöf if \( vX \) is Lindelöf.

Every Lindelöf space is pseudoLindelöf. A separable space \( X \) is pseudoLindelöf if and only if every base is complete (cf. [2]). If \( X \) is a pseudocompact space, then \( vX = \beta X \) and hence \( X \) is a pseudoLindelöf space. PseudoLindelöf spaces are not productive and a \( C \)-embedded subspace of a pseudoLindelöf space is again pseudoLindelöf.

Example 2.2. Let \( w_1 \) be the first uncountable ordinal and \( D(w_1) \) the discrete space of cardinality \( w_1 \). Let \( S = D(w_1) \cup \{p\} \), topologized as follows. Each point of \( D(w_1) \) is isolated, and a subset \( G \) of \( S \) that contains \( p \) is open in \( S \) if and only if \( |S - G| \leq \aleph_0 \). Then \( S \) is a zero-dimensional Hausdorff space and hence Tychonoff. Let \( N^* = N \cup \{w\} \) denote the one-point compactification of \( N \) and \( X = S \times N^* - \{(p, w)\} \). Then \( X \) is called Dieudonné plank and \( vX = S \times N^* \) (cf. [8]). Since \( S \) is Lindelöf, \( X \) is pseudoLindelöf. But \( X \) is neither Lindelöf nor pseudocompact.
For a space $X$ and $f \in C(X)$, $f^{-1}(0)$, denoted by $Z(f)$, is called a zero-set in $X$ and $X - f^{-1}(0)$ is called a cozero-set in $X$. It is well-known that for any $f \in C(X)$, $\text{cl}_{vX}(Z(f)) = Z(f^v)$, where $f^v$ is the extension of $f$ to $vX$ (cf. [4]).

**Lemma 2.3.** Let $X$ be a pseudolindelöf space and $A$ a zero-set in $X$. Then $A$ is closed in $vX$ if and only if $A$ is Lindelöf.

**Proof.** Suppose that $A$ is Lindelöf. Let $p \in vX - A$. If $p \in X$, then $p \notin \text{cl}_{vX}(A)$. Suppose that $p \notin X$. For any $a \in A$, there is a cozero-set neighborhood $C_a$ of $a$ in $vX$ such that $p \notin C_a$. Since $A$ is Lindelöf, there is a countable subfamily $\mathcal{U}$ of $\{C_a : a \in A\}$ with $A \subseteq \bigcup \mathcal{U}$. Let $C = \bigcup \mathcal{U}$ and $Z = vX - C$. Then $p \in Z$, $Z$ is a zero-set in $vX$ and $A \cap Z = \emptyset$.

Since $X$ is $C^*$-embedded in $vX$, $\text{cl}_{vX}(A) \cap \text{cl}_{vX}(Z \cap X) = \emptyset$ and since $\text{cl}_{vX}(Z \cap X) = Z$, $p \notin \text{cl}_{vX}(A)$ and hence $A = \text{cl}_{vX}(A)$. The converse is trivial. \( \square \)

For a space $X$, $Z(X)$ denotes the set of zero-sets in $X$. A non-empty subfamily $\mathcal{F}$ of $Z(X)$ is called a z-filter on $X$ if

(i) $\emptyset \notin \mathcal{F}$,

(ii) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, and

(iii) if $Z \in \mathcal{F}$ and $Z \subseteq A \in Z(X)$, then $A \in \mathcal{F}$.

A maximal z-filter on $X$ is called a z-ultrafilter on $X$ and a z-ultrafilter on $X$ is called real if it has the countable intersection property.

**Definition 2.4.** Let $X$ be a dense subspace of a space $T$, $\mathcal{F}$ a z-filter on $X$ and $p \in T$. Then $\mathcal{F}$ converges to the limit $p$ if every neighborhoods of $p$ in $T$ contains a member of $\mathcal{F}$.

**Lemma 2.5** [4]. Let $X$ be a dense subspace of $T$. The following are equivalent:

(a) $X$ is $C$-embedded in $T$.

(b) Every point of $T$ is the limit of a unique real z-ultrafilter on $X$.

(c) $vX = vT$, that is, there is a homeomorphism $h : vX \rightarrow vT$ such that $h(x) = x$ for all $x \in X$.

For any space $X$ and $\mathcal{F} \subseteq 2^X$, $\bigcap \text{cl}_X(\mathcal{F})$ denotes the set $\bigcap \{ \text{cl}_X(F) : F \in \mathcal{F} \}$.

**Theorem 2.6.** Let $X$ be a pseudolindelöf space. Then the following are equivalent:

(a) For any two disjoint zero-sets in $X$, at least one of them is Lindelöf.
(b) \(|vX - X| \leq 1\).

(c) For any space \(T\) with \(X \subseteq T\), there is an embedding \(f : vX \to vT\) such that \(f(x) = x\) for all \(x \in X\).

Proof. (a) \(\Rightarrow\) (b) Suppose that \(2 \leq |vX - X|\). Pick \(p, q \in vX - X\) with \(p \neq q\). By Lemma 2.5, there are \(z\)-ultrafilters \(\mathcal{A}^p\) and \(\mathcal{A}^q\) on \(X\) such that \(p\), (resp.) is a limit of \(\mathcal{A}^p\) (\(\mathcal{A}^q\), resp.) and since \(p \neq q\), \(\mathcal{A}^p \neq \mathcal{A}^q\) and hence there are disjoint zero-sets \(A, B\) in \(X\) such that \(A \in \mathcal{A}^p\) and \(B \in \mathcal{A}^q\). Note that \(p \in \text{cl}_{vX}(A)\) and \(q \in \text{cl}_{vX}(B)\). We may assume that \(A\) is Lindelöf. By Lemma 2.3, \(A\) is closed in \(vX\) and hence \(p \notin A = \text{cl}_{vX}(A)\). This is a contradiction.

(b) \(\Rightarrow\) (a) Suppose that \(vX - X = \{p\}\). Take any disjoint zero-sets \(A\) and \(B\) in \(X\). Then \(\text{cl}_{vX}(A) \cap \text{cl}_{vX}(B) = \emptyset\) and hence \(p \notin \text{cl}_{vX}(A)\) or \(p \notin \text{cl}_{vX}(B)\). So \(\text{cl}_{vX}(A) = A\) or \(\text{cl}_{vX}(B) = B\). Hence \(A\) is Lindelöf or \(B\) is Lindelöf.

(b) \(\Rightarrow\) (c) Suppose that \(vX - X = \{p\}\). Take any space \(T\) with \(X \subseteq T\). Then there is a continuous map \(f : vX \to vT\) such that \(f(x) = x\) for all \(x \in X\) (cf. [4]). Let \(q = f(p)\) and \(Y = X \cup \{q\}\). Then \(X\) is a dense subspace of \(Y\). Let \(g\) be the corestriction of \(f\) to \(Y\), then \(g : vX \to Y\) is one-to-one, onto, and continuous.

We will show that \(g\) is a homeomorphism. Since \(vX\) is Lindelöf, \(Y\) is Lindelöf and so \(Y\) is a realcompactification of \(X\). Since \(X\) is \(C\)-embedded in \(vX\), there is a unique real \(z\)-ultrafilter \(\mathcal{A}^p\) on \(X\) such that \(p\) is a limit point of \(\mathcal{A}^p\). Take any neighborhood \(V\) of \(p\) in \(Y\). Then \(g^{-1}(V)\) is a neighborhood of \(p\) in \(vX\). Since \(p\) is a limit point of \(\mathcal{A}^p\), there is \(A \in \mathcal{A}^p\) with \(A \subseteq g^{-1}(V)\) and so \(g(A) = A \subseteq V\). Hence \(q\) is a limit point of \(\mathcal{A}^p\).

Suppose that \(\mathcal{F}\) is a real \(z\)-ultrafilter on \(X\) such that \(q\) is a limit point of \(\mathcal{F}\). If \(\cap \mathcal{F} \neq \emptyset\), then \(\cap \mathcal{F} = \{x\}\) for some \(x \in X\). Since \(x \neq q\), there are disjoint zero-set neighborhoods \(C\) of \(x\) and \(D\) of \(q\) in \(Y\). Since \(C \cap X \in \mathcal{F}\) and \((C \cap X) \cap (D \cap X) = \emptyset\), \(q\) is not a limit point of \(\mathcal{F}\). Hence \(\cap \mathcal{F} = \emptyset\). Since \(\mathcal{F}\) is real, \(\text{cl}_{vX}(\mathcal{F}) = \{\text{cl}_{vX}(F) : F \in \mathcal{F}\}\) is a \(z\)-filter on \(vX\) with the countable intersection property and since \(vX\) is Lindelöf, \(\cap \text{cl}_{vX}(\mathcal{F}) \neq \emptyset\). Hence \(\cap \text{cl}_{vX}(\mathcal{F}) = \{p\}\). Let \(F \in \mathcal{F}\) and suppose that \(F \notin \mathcal{A}^p\). Then there is \(B \in \mathcal{A}^p\) with \(F \cap B = \emptyset\) and so \(\text{cl}_{vX}(F) \cap \text{cl}_{vX}(B) = \emptyset\). Since \(p \in \text{cl}_{vX}(B)\), this is a contradiction. Hence \(\mathcal{F} = \mathcal{A}^p\). Thus every point of \(Y\) is the limit of a unique real \(z\)-ultrafilter on \(X\). By Lemma 2.5, \(X\) is \(C\)-embedded in \(Y\) and therefore, \(g\) is a homeomorphism.

(c) \(\Rightarrow\) (b) Suppose that there are \(p, q \in vX - X\) with \(p \neq q\). Let \(Y = X \cup \{p, q\}\) and \(R = \{(x, x) : x \in Y\} \cup \{(p, q), (q, p)\}\). Then \(R\) is an equivalence relation on \(Y\).
Let $K$ be the quotient space $Y/R$ and $\pi : Y \to K$ the quotient map. Clearly, $K$ is a Tychonoff space and $X$ is a dense subspace of $K$. By the assumption, there is an embedding $f : \nu X \to \nu K$ such that $f(x) = x$ for all $x \in X$. Since $X$ is dense in $Y$ and $(j \circ \pi)|_X = f|_X$, $j \circ \pi = f|_Y$, where $j : K \to \nu K$ is the dense embedding. Since $f$ is one-to-one and $p \neq q$, $f(p) \neq f(q)$ but $j(\pi(p)) = \pi(p) = [p] = [q] = \pi(q) = j(\pi(q))$. This is a contradiction. $\square$

A subspace $Y$ of a space $X$ is $z$-embedded in $X$ if for any zero-set $A$ in $Y$, there is a zero-set $Z$ in $X$ with $A = Z \cap Y$. It is known that a space $X$ is $z$-embedded in each of its compactifications if and only if for any two disjoint zero-sets in $X$, one of them is Lindelöf (cf. [1]). Using this, we have the following:

**Corollary 2.7.** Let $X$ be a pseudoLindelöf space. Then $|\nu X - X| \leq 1$ if and only if $X$ is $z$-embedded in each of its compactifications.

Recall that a space $X$ is called quasi-$F$ if every dense cozero-set in $X$ is $C^*$-embedded in $X$, equivalently, every dense $z$-embedded subspace of $X$ is $C^*$-embedded in $X$.

**Corollary 2.8.** Let $X$ be a pseudoLindelöf space. If $|\nu X - X| \leq 1$, then $\beta X$ is the unique compactification of $X$ which is quasi-$F$.

### 3. Hewitt realcompactification of a product space.

The equality $\nu(X \times Y) = \nu X \times \nu Y$ is to be interpreted to mean that $X \times Y$ is $C$-embedded in $\nu X \times \nu Y$.

**Lemma 3.1** [3]. Let $X$ and $Y$ be spaces. Then $\nu(X \times Y) = \nu X \times \nu Y$ if and only if $X \times Y$ is $C^*$-embedded in $\nu X \times \nu Y$.

**Theorem 3.2.** Let $X$ and $Y$ be spaces such that $X \times Y$ is a pseudoLindelöf spaces. Then $X \times Y$ is $z$-embedded in $\nu X \times \nu Y$ if and only if $\nu(X \times Y) = \nu X \times \nu Y$.

**Proof.** Suppose that $X \times Y$ is $z$-embedded in $\nu X \times \nu Y$. Since $\nu X \times \nu Y$ is a realcompact space, there is a continuous map $f : \nu(X \times Y) \to \nu X \times \nu Y$ such that $f((x, y)) = (x, y)$ for all $(x, y) \in X \times Y$. Take any $(p, q) \in (\nu X \times \nu Y) - (X \times Y)$. Then there is a $z$-ultrafilter $\mathcal{A}^p$ on $X$ ($\mathcal{A}^q$ on $Y$, resp.) such that $p$ ($q$, resp.) is the
limit of $\mathcal{A}^p$ ($\mathcal{A}^q$, resp.) and hence
\[
\{(p, q)\} = \left(\bigcap \text{cl}_u X (\mathcal{A}^p)\right) \times \left(\bigcap \text{cl}_u Y (\mathcal{A}^q)\right).
\]

Let $\mathcal{F}$ be the $z$-filter on $X \times Y$ generated by $\{A \times B : A \in \mathcal{A}^p, B \in \mathcal{A}^q\}$. Then $\mathcal{F}$ has the countable intersection property and $\bigcap \mathcal{F} = \emptyset$. Since $v(X \times Y)$ is Lindelöf, $\bigcap \text{cl}_v (X \times Y) (\mathcal{F}) \neq \emptyset$. Pick $x \in \bigcap \text{cl}_v (X \times Y) (\mathcal{F})$. Then for any $A \in \mathcal{A}^p$ and $B \in \mathcal{A}^q$,
\[
f(x) \in f (\text{cl}_v (X \times Y) (A \times B)) \subseteq \text{cl}_v (X \times Y) (f (A \times B)) = \text{cl}_v (X \times Y) (A \times B) = \text{cl}_v X (A) \times \text{cl}_v Y (B).
\]

Hence $f(x) \in (\bigcap \text{cl}_v X (\mathcal{A}^p)) \times (\bigcap \text{cl}_v Y (\mathcal{A}^q))$. So $f(x) = (p, q)$. Thus $f$ is onto.

Take any two zero-sets $A$ and $B$ in $X \times Y$ with $A \cap B = \emptyset$. Then there are zero-sets $C$ and $D$ in $vX \times vY$ with $A = C \cap (X \times Y)$ and $B = D \cap (X \times Y)$. Since $f^{-1} (C \cap D) \cap (X \times Y) = \emptyset$ and $f^{-1} (C \cap D)$ is a zero-set in $v(X \times Y)$, $f^{-1} (C \cap D) = \emptyset$ and since $f$ is onto, $C \cap D = \emptyset$. So $\text{cl}_{vX \times vY} (A) \cap \text{cl}_{vX \times vY} (B) = \emptyset$. By Urysohn's extension theorem, $X \times Y$ is $C^{*}$-embedded in $vX \times vY$. By Lemma 3.1, $v(X \times Y) = vX \times vY$. The converse is trivial. □

**Definition 3.3.** Let $X$ and $Y$ be spaces. Then $f : X \rightarrow Y$ is called $z$-closed if for any zero-set $Z$ in $X$, $f(Z)$ is closed in $Y$.

Recall that a space $X$ is called a $P$-space if every $G_\delta$-set in $X$ is open in $X$.

**Remark 3.4.** (1) If the projection $\pi_X : X \times Y \rightarrow X$ is $z$-closed, then $X$ is a $P$-space or $Y$ is a pseudocompact space (cf. [8]).

(2) The projection $\pi_X : X \times Y \rightarrow X$ is $z$-closed if and only if $X \times Y$ is $C^{*}$-embedded in $X \times \beta Y$ (cf. [3]).

(3) If $\text{card}(X)$ or $\text{card}(Y)$ is non-measurable and $X \times Y$ is $C^{*}$-embedded in $X \times \beta Y$, then $v(X \times Y) = vX \times vY$ (cf. [3]).

**Theorem 3.5.** Let $X$ be a $P$-space and $X \times Y$ a pseudoLindelöf space. If $v(X \times Y) = vX \times vY$, then the projection $\pi_X : X \times Y \rightarrow X$ is $z$-closed.

**Proof.** Take any zero-set $A$ in $X \times Y$ and $x \notin \pi_X (A)$. Then $\{x\} \times Y \cap A = \emptyset$. We will show that $\{x\} \times Y$ is $C$-embedded in $X \times Y$. Take any continuous map $f : \{x\} \times Y \rightarrow R$. Note that the map $h : Y \rightarrow \{x\} \times Y$, defined by $h(y) = (x, y)$, is a homeomorphism. Let $k = f \circ h$ and define a map $0 : X \rightarrow R$ by $0(x) = 0$
for all $x \in X$. Then the map $l : X \times Y \to R$, defined by $l((z, y)) = \varnothing(z) + k(y)$, is continuous and $l|_{\{x\} \times Y} = f$. Hence $\{x\} \times Y$ is $C$-embedded in $X \times Y$. Thus $\{x\} \times Y$ and $A$ are completely separated in $X \times Y$ (cf. [4]).

Since $\nu(X \times Y) = \nu X \times \nu Y$, $(\{x\} \times \nu Y) \cap \text{cl}_{\nu X \times \nu Y}(A) = \emptyset$. For any $y \in \nu Y$, there are open neighborhoods $C_y$ of $x$ in $X$ and $D_y$ of $y$ in $Y$ such that $(C_y \times D_y) \cap A = \emptyset$. Since $\nu Y$ is Lindelöf, there is a sequence $(y_n)$ in $\nu Y$ with $\{x\} \times \nu Y \subseteq \bigcup\{C_{y_n} \times D_{y_n} : n \in N\}$. Let $Z = \bigcap\{C_{y_n} : n \in N\}$. Since $X$ is a $P$-space, $Z$ is open in $X$ and $\{x\} \times \nu Y \subseteq Z \times \bigcup\{D_{y_n} : n \in N\}$. Moreover, $(Z \times \nu Y) \cap A = \emptyset$. Thus $Z \cap \pi(A) = \emptyset$ and so $x \notin \text{cl}_X(\pi(A))$. Therefore $\pi_X(A)$ is closed in $X$. □

**Corollary 3.6.** Suppose that $X \times Y$ is a pseudolindelöf space such that $\text{card}(X)$ or $\text{card}(Y)$ is non-measurable and $X$ is a $P$-space. Then $\pi_X$ is $z$-closed if and only if $\nu(X \times Y) = \nu X \times \nu Y$.

**References**


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