論 文 14-1-26

수치해석(미분구적법 DQM)을 이용한 곡선보의 진동분석

Vibration Analysis of Curved Beams Using Differential Quadrature

강 기 준*

Ki-Jun Kang

(1998년 7월 22일 접수, 1999년 1월 15일 채택)

ABSTRACT

The differential quadrature method (DQM) is applied to computation of eigenvalues of the equations of motion governing the free in-plane and out-of-plane vibrations for circular curved beams. Fundamental frequencies are calculated for the members with various end conditions and opening angles. The results are compared with existing exact solutions and numerical solutions by other methods (Rayleigh-Ritz, Galerkin or FEM) for cases in which they are available. The differential quadrature method gives good accuracy even when only a limited number of grid points is used.

국 문 요 약

곡선보(curved beam)의 평면내(in-plane)와 평면외(out-of-plane)의 자유진동을 해석하는데 differential quadrature method (DQM)를 이용하여 다양한 경계조건(boundary conditions)과 굽힘각 (opening angles)에 따른 진동수(frequencies)를 계산하였다. DQM의 결과는 엄밀해(exact solution) 또는 다른 수치해석(Rayleigh-Ritz, Galerkin 또는 FEM) 결과와 비교하였으며, DQM은 적은 요소(grid points)를 사용하여 정확한 해석결과를 보여주었다.

1. Introduction

The increasing use of curved beams in highway bridges, ships, and aircraft has resulted in considerable effort being directed toward developing na accurate method for anal-

* 호서대학교 기계공학부

yzing the dynamic behavior of such structures. Accurate knowledge of the vibration response of curved beams in of great importance in many engineering applications such as the design of machines and structures. Owing to their importance, the dynamic behavior of curved beams has been the subject of a large number of investigations. Solutions of the relevant differential equations have traditionally been obtained by the standard finite difference method or finite element method. These techniques require a great deal of computation time as the number of discrete nodes becomes relatively large under conditions of complex geometry and loading. In many cases, the moderately accurate solution which can be calculated rapidly is desired at a few points in physical domain. However, in order to get results with even only limited accuracy at or near a point of interest for a reasonably complicated problem, solutions ofter have dependence of the accuracy and stability of the mentioned methods on the nature and refinement of the discretization of the domain.

The early investigators into the in-plane vibration of rings were Hoppe[1] and Love[2]. Love[2] improved on Hoppe's theory by allowing for stretching of the ring. Lamb[3] investigated the statics of incomplete ring with various boundary conditions and the dynamics of an incomplete free-free ring of small curvature. Den Hartog[4] used the Rayleigh-Ritz method for finding the lowest natural frequency of circular arcs with simply supported or clamped ends and his work was extended by Volterra and Morell[5] for the vibrations of arches having center lines in the form of cycloids, catenaries or parabolas. Archer[6] carried out for a mathematical study of the in-plane inextensional vibrations of an incomplete circular ring of small cross section with the basic equations of motion as given in Love[2] and gave a prescribed time-dependent displacement at the other end for the case of clamped ends. Nelson[7] applied the Rayleigh-Ritz method in conjunction with Lagrangian multipliers to the case of a circular ring segment having simply supported ends. Auciello and De Rosa[8] reviewed the free vibrations of circular arches and briefly illustrated a number of other approaches. Out-of-plane vibrations of complete and incomplete rings have been the subject of interest for several research workers. Ojalvo[9] obtained the equations governing three-dimensional linear motions of elastic rings and results for generalized loadings and viscous damping making use of usual classical beam-theory assumptions for the clamped ends. Rodgers and Warner[10] calculated the frequencies of curved elastic rods with simply supported ends.

A rather efficient alternate procedure for the solution of partial differential equations is the method of differential quadrature which was introduced by Bellman and Casti[11]. This simple direct technique can be applied to a large number of cases to circumvent the difficulties of programming complex algorithms for the computer, as well as excessive use of storage. This method is used in the present work to analyze the free in-plane inextensional vibrations and the out-of-plane twist-bending vibrations of curved beams with various boundary conditions and opening angles. The lowest frequencies are calculated for the member. The curved beams considered are of uniform cross section and mass per unit of length and are either clamped or simply supported at both ends. Numerical results are compared with existing exact solutions and numerical solutions by the Rayleigh-Ritz, Galerkin or the finite element methods.

2. Governing Equations

The uniform curved beam considered is shown in Fig. 1. A point on the centroidal axis is defined by the angle θ, measured from the left support. The tangential and radial displacements of the arch axis are v and w, respectively. u is the displacement at right angles to the plane of the arch, a is the radius of the centroidal axis, and β is the angular rotation of a cross section of the principal axes about the tangential axis. These displacements are considered to be positive in the directions indicated.

2.1 In-plane inextensional vibrations of thin curved beams

A mathematical study of the in-plane inextensional vibrations of a curved beam of small cross section is carried out starting with the basic equations of motion as given by Love[2]. Following Love, the analysis is simplified by restricting attention to problems where there is no extension of the center line. This condition requires that w and v be related by

$$w = -\frac{\partial v}{\partial \theta} \quad \cdots\cdots (1)$$

If rotatory inertia and shear deformation are neglected, the differential equation governing the free flexural vibration of this curved beam, in terms of the displacement v, can be written as (Love[2])

$$\frac{EI}{a^4}\left(\frac{\partial^6 v}{\partial \theta^6} + 2\frac{\partial^4 v}{\partial \theta^4} + \frac{\partial^2 v}{\partial \theta^2}\right) = m\frac{\partial^2}{\partial t^2}\left(v - \frac{\partial^2 v}{\partial \theta^2}\right) \quad \cdots\cdots (2)$$

or

$$\frac{v^{vi}}{\theta_0^6} + 2\frac{v^{iv}}{\theta_0^4} + \frac{v''}{\theta_0^2} = \frac{ma^4\omega^2}{EI}\left(\frac{v''}{\theta_0^2} - v\right) \quad \cdots\cdots (3)$$

in which each prime denotes one differentiation with respect to the dimensionless distance coordinate, X, defined as

$$X = \frac{\theta}{\theta_0} \quad \cdots\cdots (4)$$

Here, m is the mass per unit length, v is the displacement in the direction of increasing θ, θ_0 is the opening angle for the curved beam, ω is the circular frequency of vibration of the system, E is the Young's modulus of elasticity for the material of the arch, and I is the area moment of inertia of the cross section.

If the curved beam is clamped at $\theta = 0$ and $\theta = \theta_0$, then the boundary conditions take the form

$$v = 0 \quad \cdots\cdots (5)$$

$$w = -\frac{\partial v}{\partial \theta} = 0 \quad \cdots\cdots (6)$$

$$\frac{\partial^2 v}{\partial \theta^2} + v = 0 \quad \cdots\cdots (7)$$

at $\theta = 0$ and $\theta = \theta_0$, or

$$v(0) = v'(0) = v''(0) = v(\theta_0) = v'(\theta_0) = v''(\theta_0) = 0 \quad \cdots\cdots (8)$$

If the curved beam is simply supported, then the boundary conditions can be expressed in the following form

$$v = 0 \quad \cdots\cdots (9)$$

$$w = -\frac{\partial v}{\partial \theta} = 0 \quad \cdots\cdots (10)$$

$$\frac{\partial^2 w}{\partial \theta^2} + w = 0 \quad \cdots\cdots (11)$$

or

$$v(0) = v'(0) = v'''(0) = v(\theta_0) = v'(\theta_0) = v'''(\theta_0) = 0 \quad \cdots\cdots (12)$$

2.2 Coupled twist-bending vibrations of thin curved beams

The differential equation can be written as (Ojalvo[9])

$$\frac{\partial^4 u}{\partial \theta^4} - a\frac{\partial^2 \beta}{\partial \theta^2} - \kappa\left(\frac{\partial^2 u}{\partial \theta^2} + a\frac{\partial^2 \beta}{\partial \theta^2}\right) = \frac{ma^4}{EI}\frac{\partial^2 u}{\partial t^2} \quad \cdots\cdots (13)$$

$$\frac{\partial^2 u}{\partial \theta^2} - a\beta + k\left(a\frac{\partial^2 \beta}{\partial \theta^2} + \frac{\partial^2 u}{\partial \theta^2}\right) = 0 \quad \cdots (14)$$

or

$$\frac{\beta^{iv}}{\theta_0^4} + 2\frac{\beta''}{\theta_0^2} + \beta = -\chi \frac{1+k}{a} u \cdots\cdots\cdots (15)$$

$$\frac{u''}{\theta_0^2} = \frac{a}{1+k}\left(\beta - k\frac{\beta''}{\theta_0^2}\right) \cdots\cdots\cdots\cdots\cdots (16)$$

where k is the stiffness parameter GJ/EI, G is the shear modulus, J is the torsion constant of the cross section, χ is a dimensionless parameter, related to the circular frequency of vibration of the system, ω, and the moment-displacement relation can be expressed as

$$\chi = \omega^2 \frac{ma^4}{GJ}, \quad M = \frac{EI}{a^2}(a\beta - u'') \cdots\cdots (17)$$

The boundary conditions for simply supported and clamped ends are, respectively,

$$\beta = u = u'' = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (18)$$

$$\beta = u = u' = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (19)$$

3. Differential Quadrature Method

The Differential Quadrature Method was introduced by Bellman and Casti[11]. By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals in their introductory paper, they proposed the differential quadrature method as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. It was applied for the first time to static analysis of structural components by Jang et al.[12]. The versatility of the DQM to engineering analysis in general and to structural analysis in particular is becoming increasingly evident by the related publications of recent years. Kukreti et al.[13] calculated the fundamental frequencies of tapered plates, and Farsa et al.[14] applied the method to analysis and detailed parametric evaluation of the fundamental frequencies of general anisotropic and laminated plates. In another development, the quadrature method was introduced in lubrication mechanics by Malik and Bert[15]. Han and Kang[16] applied the method to the buckling analysis of circular curved beams. From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$L\{f(x)\}_i = \sum_{j=1}^{N} W_{ij} f(x_j)$$

$$\text{for } i, j = 1, 2, \cdots, N \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (20)$$

where L denotes a differential operator, χ_j are the discrete points considered in the domain, $f(x_j)$ are the function values at these points, W_{ij} are the weighting coefficients attached to these function values, and N denotes the number of discrete points in the domain. This equation, thus, can be expressed as the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

The general form of the function f(x) is taken as

$$f_k(x) = x^{k-1} \quad \text{for } k = 1, 2, 3, \cdots, N \quad \cdots (21)$$

If the differential operator L represents an n^{th} derivative, then

$$\sum_{j=1}^{N} W_{ij} x_j^{k-1} = (k-1)(k-2)\cdots(k-n) x_i^{k-n-1}$$

$$\text{for } i, k = 1, 2, \cdots, N \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (22)$$

This expression represents N sets of N linear algebraic equations, giving a unique solution for the weighting coefficients, W_{ij}, since the coefficient matrix is a Vandermonde matrix which always has an inverse, as described by Hamming[17].

4. Application

The method of differential quadrature is applied to the determination of the in-plane and out-of-plane vibrations of curved beams. The differential quadrature approximations of the governing equations and boundary conditions are shown.

4.1 In-plane inextensional vibrations of thin curved beams

Applying the differential quadrature method to equation(3) gives

$$\frac{1}{\theta_0^6}\sum_{j=1}^{N} F_{ij} v_j + \frac{2}{\theta_0^4}\sum_{j=1}^{N} D_{ij} v_j + \frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij} v_j$$

$$= \frac{ma^4\omega^2}{EI}\left(\frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij} v_j - v_i\right) \cdots\cdots\cdots (23)$$

where B_{ij}, D_{ij} and F_{ij} are the weighting coefficients for the second-, fourth-, and sixth-order derivatives, respectively, along the dimensionless axis.

The boundary conditions for clamped ends, given by equation(8), can be expressed in differential quadrature form as follows:

$$v_1 = 0 \quad \text{at} \quad X=0 \cdots\cdots\cdots\cdots (24)$$

$$v_N = 0 \quad \text{at} \quad X=1 \cdots\cdots\cdots\cdots\cdots (25)$$

$$\sum_{j=1}^{N} A_{2j} v_j = 0 \quad \text{at} \quad X=0+\delta \cdots\cdots\cdots (26)$$

$$\sum_{j=1}^{N} A_{(N-1)j} v_j = 0 \quad \text{at} \quad X=1-\delta \cdots\cdots\cdots (27)$$

$$\sum_{j=1}^{N} B_{3j} v_j = 0 \quad \text{at} \quad X=0+2\delta \cdots\cdots\cdots (28)$$

$$\sum_{j=1}^{N} B_{(N-2)j} v_j = 0 \quad \text{at} \quad X=1-2\delta \cdots\cdots\cdots (29)$$

Similarly, the boundary conditions for simply supported ends given by equation(12), can be expressed in differential quadrature form as follows:

$$v_1 = 0 \quad \text{at} \quad X=0 \cdots\cdots\cdots\cdots\cdots (30)$$

$$v_N = 0 \quad \text{at} \quad X=1 \cdots\cdots\cdots\cdots\cdots (31)$$

$$\sum_{j=1}^{N} A_{2j} v_j = 0 \quad \text{at} \quad X=0+\delta \cdots\cdots\cdots (32)$$

$$\sum_{j=1}^{N} A_{(N-1)j} v_j = 0 \quad \text{at} \quad X=1-\delta \cdots\cdots\cdots (33)$$

$$\sum_{j=1}^{N} C_{3j} v_j = 0 \quad \text{at} \quad X=0+2\delta \cdots\cdots\cdots (34)$$

$$\sum_{j=1}^{N} C_{(N-2)j} v_j = 0 \quad \text{at} \quad X=1-2\delta \cdots\cdots\cdots (35)$$

where A_{ij} and C_{ij} are the weighting coefficients for the first- and third-order derivatives. Here, δ denotes a very small distance measured along the dimensionless axis from the boundary ends. This set of equations together with the appropriate boundary conditions can be solved for the in-plane inextensional free vibrations.

4.2 Coupled twist-bending vibrations of thin curved beams

Applying the differential quadrature method to equations(15) and (16) gives

$$\frac{1}{\theta_0^4}\sum_{j=1}^{N} D_{ij}\beta_j + \frac{2}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\beta_j + \beta_i$$

$$= -x\frac{1+k}{a}u_i \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (36)$$

$$\frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij} u_j$$

$$= \frac{a}{1+k}\left(\beta_i - k\frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\beta_j\right) \cdots (37)$$

The boundary conditions for simply supported ends, given by equation(18), can be expressed in differential quadrature form as follows:

$$\beta_1 = 0 \ \text{and} \ u_1 = 0 \quad \text{at} \quad X=0 \cdots\cdots\cdots\cdots (38)$$

$$\beta_N = 0 \ \text{and} \ u_N = 0 \quad \text{at} \quad X=1 \cdots\cdots\cdots\cdots (39)$$

$$\sum_{j=1}^{N} B_{2j} u_j = 0 \quad \text{at} \quad X=0+\delta \cdots\cdots (40)$$

$$\sum_{j=1}^{N} B_{(N-1)j} u_j = 0 \quad \text{at} \quad X=1-\delta \cdots\cdots (41)$$

Similarly, the boundary conditions for clamped ends, given by equation(19), can be expressed in differential quadrature form as follows:

$$\beta_1 = 0 \ \text{and} \ u_1 = 0 \quad \text{at} \quad X=0 \cdots\cdots\cdots (42)$$

$\beta_N=0$ and $u_N=0$ at $X=1$ .... (43)

$$\sum_{j=1}^{N} A_{2j} u_j = 0 \quad \text{at} \quad X=0+\delta \quad \cdots (44)$$

$$\sum_{j=1}^{N} A_{(N-1)j} u_j = 0 \quad \text{at} \quad X=1-\delta \quad \cdots (45)$$

5. Numerical results and comparisons

Based on the above derivations, the fundamental natural frequencies of the in-plane and out-of-plane vibrations are calculated by the differential quadrature method and are presented together with existing exact and numerical solutions by other methods. All results are computed with thirteen discrete points along the dimensionless axis since the optimal value for N is found to be thirteen discrete points.

5.1 In-plane inextensional vibrations of thin curved beams

The fundamental frequency parameter of this curved beam is calculated by differential quadrature and is presented together with results from other methods: exact solutions by Archer[6], the Lagrangian multiplier technique by Nelson[7], Galerkin, Rayleigh-Ritz, or finite element methods. The results are summarized in Tables 3, 5, and 6 using $\delta=1\times10^{-5}$.

Tables 1 and 2 present the results of convergence studies relative to the number of grid points N and the δ parameter, respectively. Table 1 shows that the accuracy of the numerical solution increases with increasing N and passes through a maximum. Then, numerical instabilities arise if N becomes too large. The optimal value for N is found to be 11 to 13. Table 2 shows the sensitivity of the numerical solution to the choice of δ. The optimal value for δ is found to be 1×10^{-5} to 1×10^{-6}, which is obtained from trial-and-error calculations. The solution accuracy decreases due to numerical instabilities if δ becomes too small.

Table 1 Fundamental frequency parameters, $\Omega=\omega(ma^4/EI)^{1/2}$, for in-plane vibration of thin curved beams with clamped ends including a range of grid point, $\theta_0=180°$

Archer[6] (Exact)	Number of grid points			
$\Omega=\omega(ma^4/EI)^{1/2}$	7	9	11	13
4.3841	5.0586	4.1740	4.3975	4.3844

Table 2 Fundamental frequency parameters, $\Omega=\omega(ma^4/EI)^{1/2}$, for in-plane vibration of thin curved beams with clamped ends including a range of δ, $\theta_0=180°$

Archer[6] (Exact)	δ				
$\Omega=\omega(ma^4/EI)^{1/2}$	1×10^{-2}	1×10^{-3}	1×10^{-4}	1×10^{-5}	1×10^{-6}
4.3841	4.8845	4.4301	4.3885	4.3844	4.3840

Table 3 Fundamental frequency parameters, $\Omega=\omega(ma^4/EI)^{1/2}$, for in-plane vibration of thin curved beams with clamped ends

θ_0, degrees	$\Omega=\omega(ma^4/EI)^{1/2}$				
	Archer[6] (Exact)	Galerkin	Rayleigh-Ritz	SAP IV finite element	DQM
30		228.18	222.36	222.36	222.358
60		55.221			53.737
90		23.295			22.624
120		12.225			11.847
150		7.194			6.958
180	4.384	4.539			4.384
270	1.395				1.395
324	0.789				0.789
360	0.566				0.566

Auciello and De Rosa[8] determined the natural frequencies of the arches using the SAP IV or SAP 90 finite element method (FEM). Tables 3 and 5 show that the numerical results by the DQM are in excellent agreement with those by the SAP IV FEM.

However, the SAP IV FEM was quite expensive because 60 finite elements were employed, as described by Auciello and De Rosa[8]. Table 4 presents the first four nondimensional free vibration frequencies by DQM without comparison, since no data are available.

Table 4 First four frequency parameters, $\Omega=\omega(ma^4/EI)^{1/2}$, for in-plane vibration of thin curved beams with simply supported ends, $\theta_0=30^\circ$

$\Omega=\omega(ma^4/EI)^{1/2}$	Rayleigh-Ritz	DQM
Ω_1	141.53	141.536
Ω_2		306.634
Ω_3		589.323
Ω_4		804.053

Table 5 Fundamental frequency parameters, $\Omega=\omega(ma^4/EI)^{1/2}$, for in-plane vibration of thin curved beams with simply supported ends

θ_0, degrees	$\Omega=\omega(ma^4/EI)^{1/2}$				
	Nelson[7]	Galerkin	Rayleigh-Ritz	SAP IV finite element	DQM
30		141.53	141.53	141.53	141.536
60	33.636	33.727			33.627
90	13.764	13.765			13.764
120		6.928			6.927
150		3.860			3.859
180	2.267	2.268			2.267
240	0.818				0.818

Table 6 Fundamental frequency parameters, $\Omega=\omega(ma^4/EI)^{1/2}$, for in-plane vibration of thin curved beams with clamped-simply supported ends

θ_0, degrees	$\Omega=\omega(ma^4/EI)^{1/2}$	
	SAP90, finite element	DQM
30		178.945
60	42.942	2.942
90		17.871
120	9.216	9.210
150		5.299
180	3.258	3.254

From Tables 3, 5, and 6, the natural frequencies of the member with clamped ends are much higher than those of the member with simply supported ends and those of the member with simply supported clamped ends, and the frequencies can be increased by decreasing the opening angle.

5.2 Coupled twist-bending vibrations of thin curved beams

The values of χ corresponding to the natural frequencies of out-of-plane vibration are calculated by the differential quadrature method and presented together with results obtained by other methods: the exact solutions by Ojalvo[9] and Rodgers and Warner[10]. Tables 7 and 8 show the fundamental natural frequency parameters with variations in stiffness parameter k for clamped ends and simply supported ends.

Table 7 Fundamental frequency parameters, $\chi=\omega^2 ma^4/GJ$, for out-of-plane vibration of thin curved beams with clamped ends

θ_0	k=GJ/EI	$\chi=\omega^2 ma^4/GJ$	
		Ojalvo[9]	DQM
180°	0.005	47.60	47.60
	0.2	13.36	13.36
	0.5	6.334	6.334
	1.0	3.375	3.375
	1.625	2.134	2.131
270°	0.005	3.304	3.305
	0.2	1.646	1.646
	0.5	0.955	0.955
	1.0	0.578	0.578
	1.625	0.394	0.394
360°	0.005	0.454	0.454
	0.2	0.335	0.335
	0.5	0.253	0.253
	1.0	0.192	0.192
	1.625	0.153	0.153

Table 8 Fundamental frequency parameters, $\chi = \omega^2 ma^4/GJ$, for out-of-plane vibration of thin curved beams with simply supported ends; $\theta_0 = 90^\circ$

θ_0	k=GJ/EI	$\chi = \omega^2 ma^4/GJ$	
		Rodgers and Warner[10]	DQM
90°	0.005	35.29	35.29
	0.2	20.0	20.0
	0.5	12.0	12.0
	1.0	7.20	7.20
	1.625	4.80	4.80

6. Conclusions

The differential quadrature method was used to compute the eigenvalues of the equations of motion governing the free in-plane and out-of-plane vibrations of curved beams. The present method gives results which agree very well (less than 0.3%) with the exact ones and with numerical solutions by other methods for the cases treated while requiring only a limited number of grid points.

The study reported herein is sponsored by KRF (Korea Research Foundation) 97 research funds.

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