순간 발산지수의 카오스계에의 응용, 파트 1: 이론 및 시뮬레이션

신기홍*

Application of the Instantaneous Lyapunov Exponent and Chaotic Systems, Part 1: Theory and Simulation

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ABSTRACT

어느 한 계가 양수의 발산지수(Lyapunov exponent)를 가질 때 이 계는 카오스계로 분류되며 그 동특성은 예측이 불가능해 진다. 감쇠 기계계(소산계)에서는 위상공간(phase space)의 초기 부피가 시간에따라 수축한다. 발산지수들의 합은 음수이며 그 기계계의 감쇠와 관련되며, 따라서 발산지수들의 합은 감쇠의 변화를 감시하는데 사용되어질 수 있다. 그러나 그 감쇠변화를 감시하기 위해서는 발산지수를 계산하는데 사용하는 신호(data) 부분(segment)이 짧아야 한다. 이는 문제점을 야기 시키는데 그 이유는 발산지수가 아주 많은 양의 발산률(divergence rate)의 평균으로서 구해지기 때문이다. 이 문제를 극복하기 위해서, 본 저자는 '순간발산지수 (Instantaneous Lyapunov Exponent)'를 도입하였으며, 이 순간발산지수들의 합이 어떻게 기계계의 감쇠와 관련되어지는 가에 대하여 기술하였다. 미분방정식과 시계열(time series)을 이용한 컴퓨터 시뮬레이션 은 '순간발산지수들의 합'의 중요성을 입증하였다. 그러나 시계열(또는 실험신호)로 부터의 정확한 순간발산지수를 측정하기는 매우 힘들기 때문에 '부분발산지수(Short term averaged Lyapunov Exponent)'를 또한 도입하였다.

Key Words : Lyapunov Exponent(발산지수), Instantaneous Lyapunov Exponents(순간발산지수), Local Lyapunov Exponents(구간발산지수), Chaos(카오스), Phase Space(위상공간)

1. Introduction

In a dynamical system, the spectrum of Lyapunov exponents plays a very important role in the diagnosis of whether the system is chaotic or not. The Lyapunov exponents are a measure of sensitive dependence upon initial conditions and represent the average rate of divergence or convergence of nearby trajectories in phase space. A positive Lyapunov exponent means that the nearby trajectories in phase space will soon diverge and the evolution is sensitive to initial conditions and the system becomes unpredictable. Any system containing at least one positive Lyapunov exponent is defined to be chaotic [1]. One of the important properties is that the sum of Lyapunov exponents is related to the generalized divergence of the flow in phase space of the system, and

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related to the energy dissipation mechanism of a dissipative system, i.e., the energy dissipation means the phase volume contraction. For a dissipative dynamical system, the n-dimensional initial phase volume in an n-dimensional phase space will contract with time, and the sum of Lyapunov exponents is negative. In a mechanical system which possesses some damping the system is categorised as a dissipative system. Thus the sum of Lyapunov exponents must be related to the damping of a mechanical system, and can be utilized to monitor any changes of damping of the system. However, in order to track the changes of damping of a system the data segment used to calculate the Lyapunov exponents must be short. This leads to problems since Lyapunov exponents are calculated from a long term averaged divergence rate as will be discussed later.

To overcome the above problem, we introduce Instantaneous Lyapunov exponents (ILEs) which are the derivative of the logarithm of divergence rate. We describe how the sum of ILEs is related to the generalized divergence of the flow and to the damping of a mechanical system. Computer simulation results from both differential equations and a time series are presented. The algorithm for computing the ILEs from differential equations is based on the use of a phase space plus tangent space approach suggested by Wolf *et al* [2], and on the algorithm developed by Sano *et al* [3] in the case of a time series. These algorithms [2, 3] are modified to compute the ILE, and so the computational aspects are much the same. In practice, it is very difficult to obtain accurate ILEs from a time series due to computational errors, and so short term averaged Lyapunov exponents (SLEs) are introduced. The ILE has been introduced first time in this paper, and is defined for a continuous-time dynamical system. Although it is not the same, a similar concept to the SLE, the 'Local Lyapunov exponent' was introduced by H. D. I. Abarbanel *et al* [4]. However, the 'Local Lyapunov exponent' is defined in a discrete manner, and so, strictly speaking, it is only relevant for discrete-time dynamical systems. For an example of the Local Lyapunov exponents for a discrete-time dynamical system, Lyapunov exponents calculated from n-th iterations of the system refer to the Local Lyapunov exponent of order 'n'. For continuous-time dynamical systems, they defined the Local Lyapunov exponents in the same way using a sampled version of the system. However, they did not discuss any effects of time-discretization of continuous systems. Also the term 'Local' means local to the attractor and not local in time. On the other hand, the SLE is the time averaged version of the ILE, so it is defined in a continuous manner for continuous-time dynamical systems and is the quantity which varies locally in time. Thus the SLE is very different from the Local Lyapunov exponent in this sense.

In this paper, We will discuss the generalized divergence of the flow in phase space and derive the ILEs and SLEs in section 2. The relationship between the damping property of a mechanical system and the generalized divergence of the flow is discussed in section 3. Computer simulation results using simple non-linear differential equations (Duffing equation and Van der Pol equation) are presented in section 4, as well as the results using a time series (displacement signal) obtained from the Duffing equation.

2. Volume in a Phase Space and the Instantaneous Lyapunov Exponents

A continuous-time n-dimensional autonomous dynamical system is modelled by ordinary differential equations of the form

$$\frac{d\mathbf{x}}{dt}=\mathbf{f}(x_1,x_2,\cdots,x_n)=\left[\frac{dx_1}{dt}\frac{dx_2}{dt}\cdots\frac{dx_n}{dt}\right]^T,$$
$$\mathbf{x}(t_0)=\mathbf{x}_0,\ \mathbf{x}(t)\in\Re^n \quad (1)$$

where $\mathbf{x}=[x_1\ x_2\ \cdots\ x_n]^T$ is a vector in n-dimensional phase space and x_i are phase space coordinates. Equation (1) determines the set of solution curves (trajectories or flow) in phase space. The vector function $\mathbf{f}$ is the generalized velocity vector field associated with the flow. Suppose that the long-term evolution of an infinitesimal n-dimensional sphere of initial conditions is monitored, the sphere will become an n-dimensional ellipsoid due to the locally deforming nature of the flow (the flow is a bundle of trajectories). The i-th Lyapunov exponent in the n-dimensional phase space is defined in terms of the length of the i-th principal axis $P_i(t)$ of the ellipsoid [5]

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln \frac{P_i(t)}{P_i(0)} \tag{2}$$

where the λ_i are ordered from the largest to the smallest. The Lyapunov exponents defined in the form of (2) is not very helpful in computation, because it is impossible to average infinitely. Thus the Lyapunov exponent may be described by an estimate which becomes a function of time or a function of the number of iterations. Then equation (2) becomes

$$\hat{\lambda}_i(t) = \frac{1}{t} \ln \frac{P_i(t)}{P_i(0)} \tag{3}$$

Similarly, if the system is described by difference equations or a map, then the Lyapunov exponents are defined in a similar manner to the continuous-time system as

$$\hat{\lambda}_i(k) = \frac{1}{k} \ln \frac{P_i(k)}{P_i(0)} \tag{4}$$

where $P_i(k)$ is the length of the i-th principal axis, and k is the number of iterations of the system. In this paper, assuming the estimates of the Lyapunov exponents are obtained from long-term averaged values, the Lyapunov exponents (λ_i) usually mean the *estimates* of the Lyapunov exponents, so generally we do not use the notation in (3) and (4) explicitly, unless otherwise stated.

If one of the λ_i is positive, nearby trajectories diverge, and the evolution is sensitive to initial conditions and is therefore chaotic. However, the divergence of chaotic trajectories can only be locally exponential, because if the system is bounded $P_i(t)$ cannot go to infinity. Thus to obtain the Lyapunov exponents, one must average the local exponential growth over a long time (infinite in theory). Lyapunov exponents have been defined in terms of the principal axes of an n-dimensional ellipsoid in an n-dimensional phase space. Similarly, the behaviour of the volume of the ellipsoid is related to the sum of Lyapunov exponents. The relative rate of change of a n-dimensional volume 'V' in n-dimensional phase space under the action of flow is given by the 'Lie derivative' (generalized divergence of the flow, and the generalized divergence is the divergence of a vector function in the n-dimensional case) [6, 7].

$$\frac{dV}{dt} = \iint_V \cdots \int \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} dx_1 \cdots dx_n \tag{5}$$

$$\frac{1}{V}\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \tag{6}$$

For a dissipative system, the generalized divergence of the flow must be negative, and the volume will contract with time. The average behaviour of the initial ellipsoidal volume can also be expressed by the Lyapunov exponents (λ_i),

$$V(t) = V(0)e^{(\lambda_1+\cdots+\lambda_n)t} \tag{7}$$

and the relative rate of volume change becomes [8]

$$\frac{1}{V}\frac{dV}{dt} = \sum_{i=1}^{n} \lambda_i \tag{8}$$

Thus, the sum of Lyapunov exponents is equal to the generalized divergence of the flow. However, this is not a rigorous relationship since the Lyapunov exponents are obtained from the long-term averaged divergence rate (see equation (2)), whereas the generalized divergence of the flow may continuously change with the action of the flow (e.g., the Van der Pol equation, which will be shown later). This may be much more rigorously described by introducing the Instantaneous Lyapunov exponents (ILEs). The divergence rate $P_i(t)/P_i(0)$ in equation (3) is continuous in time. Since the trajectories obtained from equation (1) are smooth and bounded, the divergence rate is also smooth and bounded. Thus the divergence rate is differentiable. The Instantaneous Lyapunov Exponent (ILE) for a continuous-time dynamical system is defined as the derivative of the logarithm of the divergence rate

$$\alpha_i(t) = \frac{d}{dt}\left[\ln \frac{P_i(t)}{P_i(0)}\right] \tag{9}$$

Also, the Instantaneous Lyapunov exponent for a discrete-time dynamical system (difference equation) is more easily defined such that

$$\alpha_i(k) = \ln \frac{P_i(k)}{P_i(k-1)} \tag{10}$$

where $\alpha_i(k)$ is the i-th ILE at the k-th iteration of the system, and represents the divergence rates at each iteration. In this paper, continuous-time dynamical systems are considered. Unlike the Lyapunov exponent, the ILE represents the divergence (or convergence) rate at a given time and is a time varying quantity. So the ILE shows how the divergence (or convergence) rate of nearby trajectories changes with time. The Lyapunov exponent becomes the time average of the ILE

$$\hat{\lambda}_i(t) = \frac{1}{t}\int_0^t \alpha_i(t_1)dt_1 \qquad (11)$$

$$\text{or} \quad \hat{\lambda}_i(t) = A[\alpha_i(t)] \qquad (12)$$

where $A[\ ]$ denotes the time average. Thus, the length of the i-th principal axis at time 't' can be described by the ILE,

$$P_i(t) = P_i(0)e^{\int_0^t \alpha_i(t_1)dt_1} \qquad (13)$$

and the n-dimensional phase volume at time 't' becomes

$$V(t) = V(t_0)e^{\sum_{i=1}^{n}\int_0^t \alpha_i(t_1)dt_1} = V(t_0)e^{\int_0^t \sum_{i=1}^{n}\alpha_i(t_1)dt_1}$$
$$= V(t_0)e^{A\left[\sum_{i=1}^{n}\alpha_i(t_1)\right]t} \qquad (14)$$

let $\sum_{i=1}^{n}\alpha_i(t) = \sigma(t)$ for convenience, i.e. $\sigma(t)$ is the sum of ILEs. Then, equation (14) becomes

$$V(t) = V(t_0)e^{\int_0^t \sigma(t_1)dt_1} = V(t_0)e^{A[\sigma(t_1)]t}$$

$$\therefore \frac{1}{V(t)}\frac{dV(t)}{dt} = \sigma(t) \qquad (15)$$

where $\sigma(t) = \frac{d}{dt}A[\sigma(t_1)]\cdot t + A[\sigma(t_1)]$, and $A[\sigma(t)] = \sum_{i=1}^{n}\hat{\lambda}_i(t)$.

Thus, from equation (15), the sum of ILEs is equal to the generalized divergence of the flow. Since the sum of Lyapunov exponents is the time averaged value of the sum of ILEs, the sum of Lyapunov exponents represents the long time (infinite in theory) averaged behaviour of the generalized divergence of the flow while the sum of ILEs represents the instantaneous behaviour of the generalized divergence of the flow. The SLE is defined as the short-term average of the ILE, i.e., ILEs are averaged over a defined window length to calculate the SLEs.

$$S_i(t) = A[\alpha_i(t)]_{t_1}^{t_1+t_w} \qquad (16)$$

where t_w is the window length. The relationship between the ILE and SLE is graphically illustrated in Fig. 1. The SLE is introduced especially for the case of a time series and experimental data, because the ILEs obtained from the algorithm (linear approximation of the flow) based on Sano *et al* [3] are subject to 'numerical' noise.

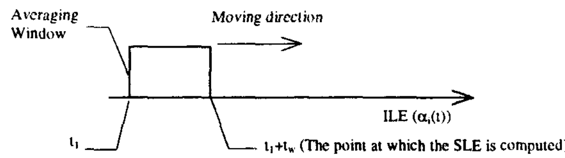


Fig. 1 Calculation of SLE from ILE.

By assumming that the numerical noise is uncorrelated, the noise can be reduced by taking an average (SLEs). The size of window length may depend on the application. For the particular application presented in this paper, the window length is determined by trial and error, i.e., striking a balance between the detectability of changes of damping and numerical errors as will be shown later.

3. Generalized Divergence of the Flow and the Damping Property of a Mechanical System

The equation of motion of a linear single degree of freedom system can be written as

$$m_{eq}\ddot{x}_1 + c_{eq}\dot{x}_1 + k_{eq}x_1 = f_{eq}(t) \qquad (17)$$

where m_{eq}, c_{eq}, k_{eq}, and f_{eq} are the equivalent of mass, viscous damping, spring and forcing respectively. When the forcing term is harmonic ($f\cos\omega t$), the equation of motion can be described in augmented state form (1),

$$\frac{dx_1}{dt} = x_2 = f_1(x_1, x_2, x_3)$$
$$\frac{dx_2}{dt} = \frac{f}{m_{eq}}\cos x_3 - \frac{c_{eq}}{m_{eq}}x_2 - \frac{k_{eq}}{m_{eq}}x_1 = f_2(x_1, x_2, x_3) \qquad (18)$$
$$\frac{dx_3}{dt} = \omega = f_3(x_1, x_2, x_3)$$

where $x_3 = \omega t$, and the generalized divergence of the flow is

$$\sum_{i=1}^{3}\frac{\partial f_i}{\partial x_i} = -\frac{c_{eq}}{m_{eq}} \qquad (19)$$

Thus the sum of Lyapunov exponents and the sum of ILEs are equal to $-\frac{c_{eq}}{m_{eq}}$ which is the same as the generalized divergence of the flow.

The generalized divergence of a multi-degree of freedom system is a little different from that of a single degree of freedom system. For a multi-degree of freedom linear system, the equation of motion can be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \quad (20)$$

where **M**, **C** and **K** are the mass matrix, damping matrix and stiffness matrix, and **x** and **f**(t) are n-dimensional vectors. If the system is described in 2n-dimensional non-autonomous phase space, by letting '**p**=**x**' and '**q**= $\dot{\mathbf{X}}$ ', the equations of motion may be written as

$$\begin{aligned} \frac{d\mathbf{p}}{dt} &= \mathbf{q} = \mathbf{f}_1(p_1,\cdots,p_n,q_1,\cdots,q_n,t) \\ \frac{d\mathbf{q}}{dt} &= \mathbf{M}^{-1}\mathbf{f}(t) - \mathbf{M}^{-1}\mathbf{C}\mathbf{q} - \mathbf{M}^{-1}\mathbf{K}\mathbf{p} = \mathbf{f}_2(p_1,\cdots,p_n,q_1,\cdots,q_n,t) \end{aligned} \quad (21)$$

where, $\mathbf{f}_1=(f_1, \cdots, f_n)$ and $\mathbf{f}_2=(f_{n+1}, \cdots, f_{2n})$. The generalized divergence of the flow becomes

$$\sum_{k=1}^{n}\frac{\partial f_k}{\partial p_k} + \sum_{k=n+1}^{2n}\frac{\partial f_k}{\partial q_k} = -\mathrm{sum}\left\{\mathbf{M}^{-1}\mathbf{C}\right\} \quad (22)$$

where sum{} denotes the sum of all elements of the vector. This can be interpreted as the total damping of the system. As a result, in a linear system, the generalized divergence of the flow is independent of the external force (assuming that the external forces are harmonic) and the restoring force, and thus represents the viscous damping property of the system.

For some non-linear systems, the generalized divergence of the flow may change continuously. Note that equation (22) does not hold for non-linear systems since the damping matrix '**C**' and stiffness matrix '**K**' may be the function of **p** and **q**, and the partial derivatives in (22) depend on the non-linearities of the system. The equation of motion of a class of non-linear multi-degree of freedom system may be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}(\mathbf{x},\dot{\mathbf{x}})\dot{\mathbf{x}} + \mathbf{K}(\mathbf{x},\dot{\mathbf{x}})\mathbf{x} = \mathbf{f}(t) \quad (23)$$

By letting '**p**=**x**' and '**q**= $\dot{\mathbf{X}}$ ', the equations of motion may be written as

$$\begin{aligned} \frac{d\mathbf{p}}{dt} &= \mathbf{q} = \mathbf{f}_1(p_1,\cdots,p_n,q_1,\cdots,q_n,t) \\ \frac{d\mathbf{q}}{dt} &= \mathbf{M}^{-1}\mathbf{f}(t) - \mathbf{M}^{-1}\mathbf{C}(\mathbf{p},\mathbf{q})\mathbf{q} - \mathbf{M}^{-1}\mathbf{K}(\mathbf{p},\mathbf{q})\mathbf{p} = \mathbf{f}_2(p_1,\cdots,p_n,q_1,\cdots,q_n,t) \end{aligned} \quad (24)$$

The generalized divergence of the flow becomes

$$\begin{aligned} &\sum_{k=1}^{n}\frac{\partial f_k}{\partial p_k} + \sum_{k=n+1}^{2n}\frac{\partial f_k}{\partial q_k} = \\ &-\mathrm{sum}\left\{\frac{\partial}{\partial \mathbf{q}}\left(\mathbf{M}^{-1}\mathbf{C}(\mathbf{p},\mathbf{q})\mathbf{q}\right)\right\} - \mathrm{sum}\left\{\frac{\partial}{\partial \mathbf{q}}\left(\mathbf{M}^{-1}\mathbf{K}(\mathbf{p},\mathbf{q})\mathbf{p}\right)\right\} \end{aligned} \quad (25)$$

From (25), it can be shown that the generalized divergence may not only depend on **C** but also depends on **K** when different modes interact with each other. However, if **K** is a pure stiffness matrix (i.e., function of **p** only '**K**(**p**)'), the second term of the right hand side of (25) disappears. In general **K**(**p**,**q**) is not a pure stiffness term but includes damping as well, so we may still say that the generalized divergence represents the total damping property of the system.

4. Numerical Simulations Using Differential Equations and a Time Series

The algorithms in the references [2, 3] are modified to compute the ILE, and the details of the computational aspects of ILE can be found in [9, 10]. The simulation is focused on the possibility of detecting changes of damping of a non-linear (chaotic) mechanical system. Two differential equations are investigated- the Duffing equation and the Van der Pol equation. Additionally a time series obtained from the Duffing equation is considered. Simulations are conducted in which the damping parameter is changed at a certain time. Comparisons are made of changes in the sum of ILEs and the sum of Lyapunov exponents. These results are also compared to the sum of SLEs.

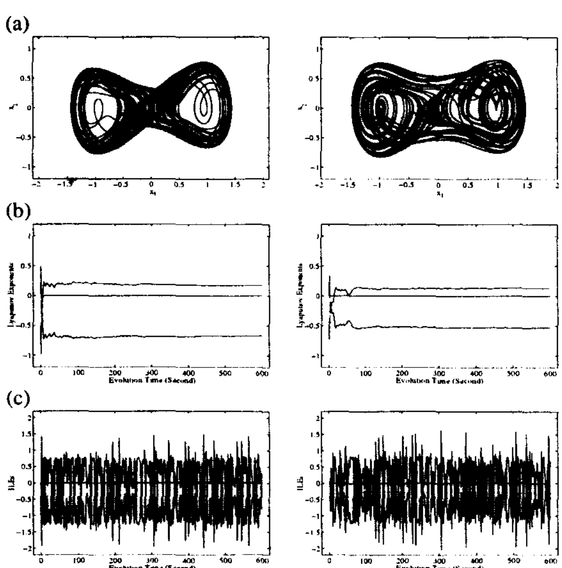
Example 1: The Duffing equation ($\ddot{x}_1 + c\dot{x}_1 - kx_1(1-x_1^2) = A\cos\omega t$) is considered which can be expressed in the form (1).

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 = f_1(x_1,x_2,x_3) \\ \frac{dx_2}{dt} &= -cx_2 + kx_1(1-x_1^2) + A\cos(x_3) = f_2(x_1,x_2,x_3) \\ \frac{dx_3}{dt} &= \omega = f_3(x_1,x_2,x_3) \end{aligned} \quad (26)$$

and the generalized divergence of the flow is equal to the negative of the damping coefficient which is constant,

$$\sum_{i=1}^{3}\frac{\partial f_i}{\partial x_i} = \left[\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right] = -c \quad (27)$$

This implies that the sum of ILEs is equal to the negative of the damping coefficient and does not vary with time. The forcing parameter and stiffness parameter are fixed and two different damping parameters are chosen in the simulation (Fig. 2). The phase portraits for these systems are shown in Fig. 2(a), and the positive Lyapunov exponent shows that both systems are chaotic (Fig. 2(b). The corresponding ILEs are also shown in Fig. 2(c). The ILEs looks apparently random, however the sum of ILEs is always the same and equal to the generalized divergence of the flow, and equal to the negative of the damping coefficient as shown in Fig. 2(d). Clearly the sum of Lyapunov exponents is the same as the sum of ILEs, so it is not shown in the Fig.. As a result, it can be said that the sum of ILEs represents the damping property. Now, we consider the system in which the damping parameter is 0.5 and is changed to 0.4 at a certain time (at 600 sec for this example). The results of this system are shown in Fig. 3. In Fig. 3(a), it is shown that the sum of Lyapunov exponents varies very slowly when the damping parameter is changed, and so it is very difficult to see whether there is any change at this time. On the other hand, in Fig. 3(b), the sum of ILEs is clearly distinguishable at the right point when the damping parameter is changed by changing the value from –0.5 to –0.4. This shows how the sum of ILEs can effectively be used for detecting changes of damping of a non-linear (chaotic) mechanical system. The sum of SLEs is presented in Fig. 3(c), and demonstrates the ability to detect changes of damping. The SLE is obtained by averaging 10 previous forcing periods of ILEs.



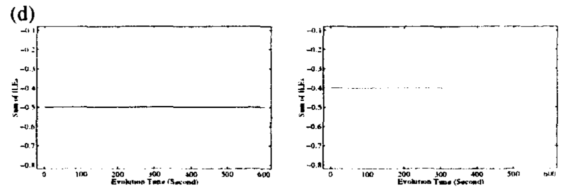


Fig. 2 Results of the simulation for the Duffing equation (left: A = 0.4, k = 1, ω = 1, c = 0.5, and right: A = 0.4, k = 1, ω = 1, c = 0.4)
(a) Phase portraits
(b) Lyapunov exponents for each case: both have a positive Lyapunov exponent
(c) ILEs for each case: ILEs are fluctuating caused by the continuously varying local divergence rate of the nearby trajectrories
(d) Sum of ILEs for each case: they are equal to –0.5(left hand side of the Fig.) and –0.4(right hand side of the Fig.), and represent damping properties, and these show that the rate volume contraction is always the same

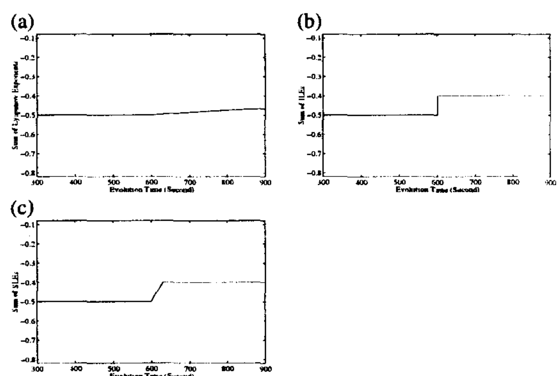


Fig. 3 Results of the simulation for the Duffing equation (A = 0.4, k = 1, ω = 1, the damping parameter is changed from 'c = 0.5' to 'c = 0.4' at time 600 sec)
(a) The change of the sum of Lyapunov exponents is not easily distinguishable
(b) The change of the sum of ILEs is very distinctive
(c) The change of the sum of SLEs is clearly monitored: this is a averaged version of the sum of ILEs at every previous 10 forcing period

<u>Example 2:</u> As another example an autonomous system, the Van der Pol equation ($\ddot{x} + c(x^2 - 1)\dot{x} + kx = 0$) can be

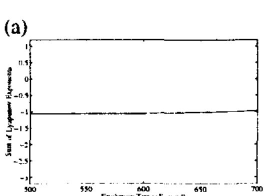
expressed in the form of (1)

$$\frac{dx}{dt} = y = f_1(x, y)$$
$$\frac{dy}{dt} = -c(x^2 - 1)y - kx = f_2(x, y) \qquad (28)$$

(N.B. This system is not chaotic) The generalized divergence of the flow is

$$\sum_{i=1}^{2} \frac{\partial f_i}{\partial x_i} = \left[\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right] = -c(x^2 - 1) \qquad (29)$$

This shows that the generalized divergence of the flow is continuously changing with time, i.e., it is the function of the variable 'x'. Since the generalized divergence of the flow is equal to the sum of ILEs, the sum of ILEs is continuously varying with time and represents the continuously changing damping property of the system. On the other hand, the sum of Lyapunov exponents shows the average behaviour of the damping property of the system, and approaches the negative value of the damping coefficient. The sum of SLEs shows the short term averaged behaviour of the damping property of the system. The stiffness parameter is fixed and two different damping parameters are chosen in the simulation. Now, we consider the system for which the damping parameter is 1 and is changed to 0.5 at certain times (at 600 sec for this example). The results of this system are shown in Fig. 4. In Fig. 4(a), we see that the sum of Lyapunov exponents varies very slowly when the damping parameter is changed, and so it is very difficult to see whether there are any changes at this time. On the other hand, in Fig. 4(b), the sum of ILEs is clearly distinguishable at the right point when the damping parameter is changed. Also the sum of SLEs is presented in 4(c), and clearly demonstrates the ability to detect changes of damping. From the above two non-linear systems, it is shown that the ILEs or the SLEs has clear advantage over the Lyapunov exponents in monitoring the change of the damping property of a system.



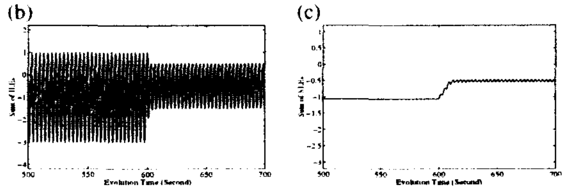


Fig. 4 Results of the simulation for the Van der Pol equation (k = 1, the damping parameter is changed from 'c = 1' to 'c = 0.5' at time 600 sec)
(a) The change of sum of Lyapunov exponents is not easily distinguishable
(b) The change of sum of ILEs are very distinctive
(c) The change of SLEs is clearly monitored: this is a averaged version of the sum of ILEs at every previous 10 forcing period

Example 3: The use of time series data

When we do not have prior knowledge of the differential equations describing a system we need to use measured data only. In order to obtain the Lyapunov exponents from a time series, it is necessary to reconstruct a phase portrait from a time series [11, 12]. From the reconstructed phase portrait, one can estimate the Lyapunov exponents. The various methods of the estimation of Lyapunov exponents can be found in [3, 4, 13-16], as well as ILEs in [9, 10]. In this simulation, a time series (displacement signal: variable 'x_1') of the Duffing equation is used. All the parameters are the same as in the previous simulation in Fig. 3, i.e., A=0.4, k=1, c=0.5 and changed to 0.4, ω=1. As mentioned earlier, the sum of ILEs are noisy. Thus, the sum of SLEs are used for this simulation by assuming that one can reduce noise by taking a time average. The results are similar to the previous simulations. As shown in Fig. 5 (a), the sum of Lyapunov exponents does not reveal the changes due to the nature of estimation of Lyapunov exponents. On the other hand, the change of damping parameter is demonstrably monitored from the sum of SLEs as shown in Fig. 5 (b) and (c). The effect of size of the 'Window length (orbital periods)' for calculating the SLEs is also shown in these Fig., i.e., the smaller window length gives earlier detection while the larger window length gives less variation. A smaller window length with reliable estimation of SLEs would be ideal, however this is mainly depending on the algorithm used for the estimation. Algorithms for calculating the Lyapunov exponents from a time series are still a growing subject as are studies of the SLEs (and ILEs).

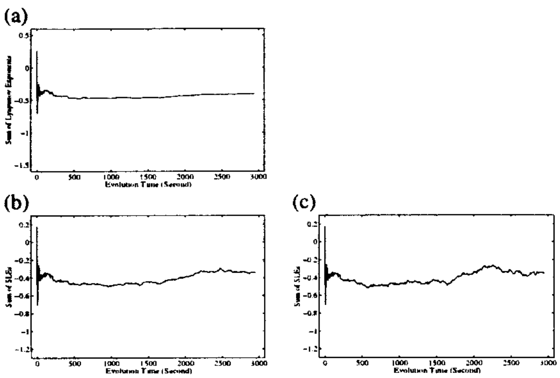


Fig. 5 Results of the simulation from a time series (corresponding to x_1 of the Duffing equation in Fig. 2, and A = 0.4, k = 1, ω = 1, and the damping parameter is changed from 'c = 0.5' to 'c = 0.4' at time 1350 sec)
(a) Sum of Lyapunov exponents: the changes of the sum of Lyapunov exponents are not clear
(b) Sum of SLEs averaged at 100 previous orbital periods: numerical errors are greatly reduced, but after 2000 sec the changes are evident
(c) Sum of SLEs averaged at 60 previous orbital periods: numerical errors are not much reduced, however the changes are evident after 1500 seconds

Conclusion

Lyapunov exponents are very useful for the quantification of chaotic dynamics, but they represent only the average behaviour of the system, i.e., the Lyapunov exponent is a measure of exponential growth rate of nearby trajectories on average. On the other hand, the ILEs describe the instantaneous behaviour of the system, so when the characteristics of a system are subject to change it may be possible to monitor these, i.e., the ILE is a temporal measure and varies with time. As an example of the use of the ILE and the sum of ILEs we use mechanical systems to detect any changes of the damping. The numerical results show the possibility of using ILEs in such variable conditions. The details of this method to a real physical system is given in Part 2.

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