⊠연구논문

NBU에 대한 지수성 검정법에 관한 연구*

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A New Test for New Better than Used Class

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Abstract

In this thesis, a new test statistic is proposed for testing exponentiality against New Better than Used (NBU) alternatives. Our test statistic, which is based on a quadratic function of the order statistics from the sample, is readily applied in the case of small sample. Also, Our test statistic is more simple than the test statistic of Hollander and Proschan(1972).

keyword: exponential distribution, order statistic, new better than used, test statistic, survival function, failure distribution.

1. Introduction

Statisticians find it useful to categorize life distributions F(i.e., distributions such that $F(t)=0$ for $t<0$) according to monotonicity properties of the failure rate, the average failure rate and the mean residual life. See Barlow and Proschan (1965), Barlow, Marshall and Proschan(1963), Birnbaum, Esary and Marshall(1966) and Esary, Marshall and Proschan(1970, 1973). The categories are useful for

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modeling situations where items improve or deteriorate with age.

Let X be a nonnegative continuous random variable denoting the life time of a system and have an absolutely continuous distribution $F(x)$, survival function $\overline{F}(x)$, probability density function (p.d.f.) $f(x)$ and finite mean μ, i.e., $E[X]=\mu$.

A distribution F is New Better than Used (NBU) if

$$\overline{F}(x+y) \le \overline{F}(x)\,\overline{F}(y) \quad \text{for all} \quad x,\ y \ge 0. \tag{1.1}$$

The corresponding concept of a New Worse than Used (NWU) distribution is defined by reversing the inequality in Equation (1.1).

The NBU property has also been referred to as "positive aging" by Bryson and Siddiqui(1969). The NBU property may be interpreted as stating that the chance $\overline{F}(x)$ that a new system will survive to age x is greater than the chance $\overline{F}(x+y)/\overline{F}(y)$ that an unfailed system of age y will survive an additional time x. That is, a new system has stochastically greater life than a used system of any age.

The boundary members of the NBU class, obtained by insisting on equality in Equation (1.1), are of course the exponential distributions, for which used systems are no worse (and better) than new systems, i.e., exponential distributions have the memoryless property.

When testing for $H_0: F$ is exponential versus $H_1: F$ is NBU, Hollander and Proschan(1972) obtained the test statistic, J_n, for the NBU class based on an U-statistic and discussed its properties. In this thesis, we propose a test statistic for testing exponentiality against NBU alternatives. Our test statistic, which is based on the order statistic from the sample, is readily applied in the case of small sample. Also, our test statistic is simpler than the test statistic of Hollander and Proschan.

In section 2, new test statistic for testing of $H_0: F$ is exponential against $H_1: F$ is NBU is proposed and this test is implemented for a sample size ranging from $n=4$ to $n=50$.

Finally in section 3, Monte Carlo simulations are conducted to evaluate the performance of the test for small sample size and to compare the powers of the our test and Hollander and Proschan's test and we give some conclusions and remarks for further researches.

2. The Proposed Test Statistic

We consider the problem of testing

$$H_0 : F(x) = 1 - \exp(-\lambda x) \quad (x \geq 0,\ \lambda > 0) \text{ for } \lambda \text{ unspecified}$$

versus

$$H_1 : F(x) \text{ is in NBU class but not exponential}$$

based on a random sample $X_1, X_2, \dots, X_n$ from the failure distribution F.

Recall the NBU test statistic of Hollander and Proschan(1972). Their test statistic is motivated by consideration of

$$\gamma(F) = \int_0^\infty \int_0^\infty \{\overline{F}(x)\overline{F}(y) - \overline{F}(x+y)\}\, dF(x)\, dF(y)$$

$$= 1/4 - \int_0^\infty \int_0^\infty \overline{F}(x+y)\, dF(x)\, dF(y).$$

Viewing the parameter $\gamma(F)$ as a measure of the deviation of F from H_0, the classical nonparametric approach of replacing F by the empirical distribution function F_n suggests rejecting H_0 in favor of H_1 if $\int_0^\infty \int_0^\infty \overline{F}_n(x+y)\ dF_n(y)$ $dF_n(y)$ is too small. They find it more convenient to reject for small values of the asymptotically equivalent U-statistic

$$J_n = 2[n(n-1)(n-2)]^{-1} \sum \psi(X_{\alpha_1}, X_{\alpha_2} + X_{\alpha_3}),$$

where

$$\psi(a, b) = 1 \qquad \text{if } a > b$$
$$= 0 \qquad \text{if } a \leq b,$$

and the $\sum$ is over all $n(n-1)(n-2)/2$ triples $(\alpha_1, \alpha_2, \alpha_3)$ of three integers such that $1 \leq \alpha_i \leq n$, $\alpha_1 \neq \alpha_2$, $\alpha_1 \neq \alpha_3$, and $\alpha_2 < \alpha_3$. In the sequel, the test which reject for small J_n values is referred to as the NBU test.

Motivated by Hollander and Proschan, we propose the following parameter

$$T(F) = \int_0^\infty \int_0^\infty \{\overline{F}(x)\overline{F}(y) - \overline{F}(x+y)\}\,dx\,dy/\mu^2$$

$$= 1 - \int_0^\infty \overline{G}(y)\,dy/\mu$$

where

$\overline{G}(x) = \int_x^\infty \overline{F}(u)\,du/\mu$ and $G(x)$ is called as the renewal or equilibrium distribution corresponding to $F(x)$. Under H_0, $T(F)=0$ and under H_1, $T(F) > 0$, since F is continuous. Thus $T(F)$ provides a measure of deviation from exponentiality toward nonexponential NBU distributions. Now, let $H(F) = \int_0^\infty \overline{G}(y)\,dy/\mu$, then $H(F) < 1$ favors H_1

In lifetime data, the random variables $X_1 < X_2 < \ldots < X_n$ are naturally ordered. The empirical distribution $F_n(X_i) = i/n,\ i = 0,1,\ldots,n$ where $X_0 = 0$. The empirical survival function is $\overline{F}_n(X_i) = (n-i)/n,\ i = 0,1,\ldots,n$. Let $D_j = (n-j+1)(X_j - X_{j-1}),\ j=1,2,\ldots,n$. Now, $\overline{G}_n(X_i) = \sum_{j=i+1}^{n} D_j / \sum_{j=1}^{n} D_j,\ i=1,\ldots,n-1$ and $\overline{G}_n(X_n) = 0$.

The sample analogue of $H(F)$ will be taken as following test statistic; thus the test statistic is

$$H_n^* = n\sum_{i=1}^{n-1} X_i D_{i+1} / T_n^2\ ,$$

where $T_n = \sum_{i=1}^{n} X_i/n$. If F is in the NBU class, but not exponential, we expect H_n^* to be less than 1.

The small sample null distribution of the statistic H_n^* is given in <Table 2.1>. For $n=4(1)20(5)50$, <Table 2.1> contains lower and upper percentile points, based on Monte Carlo sampling with 10000 replications each, in the 0.01, 0.05 and 0.10 regions.

< Table 2.1 > Critical values of the new better than used statistic H_n^*

	Lower Tail			Upper Tail		
n	$\alpha=0.01$	$\alpha=0.05$	$\alpha=0.10$	$\alpha=0.10$	$\alpha=0.05$	$\alpha=0.01$
4	0.14600	0.22884	0.27066	0.48747	0.50344	0.52176
5	0.22121	0.30231	0.34129	0.56536	0.59107	0.62579
6	0.28413	0.35755	0.39356	0.62468	0.65767	0.71095
7	0.33536	0.40168	0.43518	0.67428	0.71256	0.78013
8	0.37470	0.43706	0.46961	0.71452	0.75749	0.83578
9	0.40861	0.46741	0.49826	0.74727	0.79209	0.87488
10	0.43572	0.49228	0.52157	0.77836	0.82618	0.92027
11	0.45811	0.51418	0.54321	0.80218	0.85150	0.94811
12	0.47798	0.53209	0.56111	0.82392	0.87268	0.97678
13	0.49685	0.57846	0.57750	0.84379	0.89516	1.00520
14	0.51276	0.56277	0.59147	0.85945	0.91119	1.02044
15	0.52743	0.57709	0.60507	0.87408	0.92684	1.04142
16	0.54029	0.58917	0.61687	0.88483	0.93749	1.05407
17	0.55024	0.59954	0.62724	0.89857	0.95009	1.06281
18	0.56347	0.61091	0.63777	0.90703	0.96158	1.07880
19	0.57291	0.62021	0.64691	0.91530	0.97062	1.08703
20	0.58016	0.62885	0.65485	0.92557	0.98077	1.09818
25	0.61784	0.66397	0.69086	0.95764	1.01137	1.12874
30	0.64716	0.69117	0.72691	0.97884	1.03124	1.14728
35	0.66832	0.71244	0.73771	0.99288	1.04623	1.15884
40	0.68721	0.72997	0.75478	1.00402	1.05431	1.16067
45	0.70172	0.74379	0.76836	1.01286	1.06241	1.17094
50	0.71361	0.75675	0.78010	1.01904	1.06715	1.17207

3. Powers of The New Test Statistic

Now, we carry out to estimate the empirical powers of the proposed test H_n^*

by comparing with Hollander and Proschan's test J_n at the significance levels $\alpha=0.05$ and $\alpha=0.10$. for linear failure rate, Makeham, Weibull and gamma alternatives given by

(a) Linear Failure Rate distribution

$$F_1(x) = 1-\exp[-(x+\theta(x+e^{-x}-1))], \quad x \geq 0, \ \theta \geq 0$$

(b) Makeham distribution

$$F_2(x) = 1-\exp[-(x+(\theta x^2)/2))], \quad x \geq 0, \ \theta \geq 0$$

(c) Weibull distribution

$$F_3(x) = 1-\exp[-x^{(1+\theta)}], \quad x \geq 0, \ \theta \geq 0$$

(d) Gamma distribution

$$F_4(x) = \int_0^x (1/\Gamma(1+\theta))e^{-t}t^{\theta}dt, \quad x \geq 0, \ \theta \geq 0$$

The random numbers for the four alternatives are generated from the IMSL subroutines. This study is done for $n=7(2)15(5)50$. In this simulation, 1000 replications are performed for each value of design constants.

The following Tables contain a comparative study of the sample powers for J_n and H_n^* for NBU.

<Table 3.1> shows that H_n^* performs better than J_n in the cases of $n \geq 15$ in Linear Failure Rate distributions. Also <Table 3.2> shows that H_n^* performs better than J_n in the cases of $n \geq 15$ in Makeham distributions. <Table 3.3> shows that H_n^* performs similarly with J_n in the cases of $n \geq 20$ but worse than J_n in the cases of $n < 20$ in Weibull distributions. Finally, <Table 3.4> shows that H_n^* performs worse than J_n in the most cases in gamma distributions. Therefore we recommend our proposed statistic H_n^* as test statistic for NBU class in the case of $n \geq 15$ in Linear Failure Rate distributions and Makeham distributions.

We will drive the limiting distribution and consistency of H_n^* which is based on a quadratic function of the order statistics from the sample in further studies.

< Table 3.1 > Comparisons of small sample powers in L.F.R.

	$\theta=1$				$\theta=3$			
	$\alpha=0.10$		$\alpha=0.05$		$\alpha=0.10$		$\alpha=0.05$	
n	J_n	H_n^*	J_n	H_n^*	J_n	H_n^*	J_n	H_n^*
7	.227	.195	.118	.101	.345	.269	.179	.154
9	.255	.243	.146	.135	.391	.347	.240	.220
11	.290	.268	.165	.169	.423	.423	.297	.263
13	.300	.307	.194	.172	.484	.472	.326	.307
15	.328	.337	.210	.213	.504	.525	.361	.356
20	.376	.409	.259	.281	.604	.642	.457	.493
25	.441	.507	.286	.350	.703	.762	.556	.601
30	.491	.557	.346	.419	.763	.848	.638	.701
35	.533	.635	.377	.455	.824	.901	.708	.787
40	.585	.714	.423	.527	.878	.933	.757	.859
45	.615	.732	.458	.575	.894	.953	.812	.876
50	.662	.776	.504	.642	.919	.966	.848	.924

< Table 3.2 > Comparisons of small sample powers in Makeham.

	$\theta=1$				$\theta=3$			
	$\alpha=0.10$		$\alpha=0.05$		$\alpha=0.10$		$\alpha=0.05$	
n	J_n	H_n^*	J_n	H_n^*	J_n	H_n^*	J_n	H_n^*
7	.181	.165	.088	.081	.282	.232	.145	.124
9	.204	.188	.111	.098	.313	.287	.195	.172
11	.215	.207	.122	.114	.343	.327	.224	.203
13	.227	.204	.127	.123	.377	.369	.241	.223
15	.234	.246	.133	.130	.391	.404	.271	.267
20	.274	.269	.159	.169	.478	.491	.334	.348
25	.297	.319	.172	.192	.560	.584	.400	.430
30	.334	.365	.206	.235	.626	.655	.468	.498
35	.348	.384	.232	.265	.679	.719	.518	.572
40	.392	.416	.245	.283	.731	.789	.570	.640
45	.407	.455	.281	.308	.776	.806	.618	.673
50	.436	.487	.287	.327	.793	.851	.682	.728

< Table 3.3 > Comparisons of small sample powers in Weibull

	$\theta=0.5$				$\theta=1.0$			
	$\alpha=0.10$		$\alpha=0.05$		$\alpha=0.10$		$\alpha=0.05$	
n	J_n	H_n^*	J_n	H_n^*	J_n	H_n^*	J_n	H_n^*
7	.384	.310	.217	.181	.699	.581	.507	.404
9	.462	.402	.301	.254	.805	.743	.661	.558
11	.541	.497	.375	.341	.866	.841	.761	.705
13	.586	.570	.408	.383	.927	.918	.844	.814
15	.611	.608	.480	.440	.958	.963	.896	.870
20	.750	.734	.595	.601	.989	.990	.976	.974
25	.839	.839	.728	.705	.999	.998	.996	.997
30	.905	.904	.809	.804	.999	.999	.997	.999
35	.936	.937	.953	.861	.999	1.000	.999	.999
40	.955	.958	.898	.913	1.000	1.000	1.000	1.000
45	.972	.974	.932	.937	1.000	1.000	.999	1.000
50	.989	.990	.963	.963	1.000	1.000	1.000	1.000

< Table 3.4 > Comparisons of small sample powers in gamma

	$\theta=1.0$				$\theta=2.0$			
	$\alpha=0.10$		$\alpha=0.05$		$\alpha=0.10$		$\alpha=0.05$	
n	J_n	H_n^*	J_n	H_n^*	J_n	H_n^*	J_n	H_n^*
7	.399	.286	.213	.169	.642	.494	.422	.321
9	.467	.387	.295	.242	.759	.681	.583	.497
11	.537	.476	.373	.325	.838	.775	.738	.696
13	.606	.546	.415	.379	.917	.876	.795	.741
15	.652	.614	.474	.426	.938	.907	.849	.809
20	.769	.729	.617	.574	.983	.968	.959	.933
25	.850	.809	.731	.691	.993	.987	.984	.968
30	.901	.857	.794	.746	1.000	.995	.998	.985
35	.937	.891	.862	.799	1.000	.997	.999	.991
40	.951	.912	.893	.849	1.000	.998	1.000	.997
45	.970	.934	.921	.873	1.000	.999	1.000	.997
50	.982	.954	.951	.904	1.000	.998	1.000	.998

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