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AMLE for the Rayleigh Distribution with Type-II Censoring

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Abstract

By assuming a type-II censoring, we propose the approximate maximum likelihood estimators (AMLEs) of the location and the scale parameters of the two-parameter Rayleigh distribution and calculate the asymptotic variances and covariance of the AMLEs.

Key Words and Phrases: Approximate maximum likelihood estimator, Raylegh distribution, Type-II censoring.

1. Introduction

The random variable X has the Rayleigh distribution if it has a probability density function (pdf) of the forms;

$$f(x;\theta,\sigma) = \frac{(x-\theta)}{\sigma^2}\exp\Big(-\frac{(x-\theta)^2}{2\sigma^2}\Big), \qquad x \geq \theta,\ \sigma > 0, \tag{1}$$

where θ and σ are the location and the scale parameters, respectively.

The Rayleigh distribution is very useful in communication engineering and is a special case of a two-parameter Weibull distribution. The Rayleigh distribution has a linearly increasing failure rate, and this property, as rightly pointed out by Polovko (1968), makes it quite suitable as a model for a component that possibly has no manufacturing defects but that ages rapidly.

Cohen (1991) obtained the maximum likelihood estimators of the scale parameter in singly right censored and truncated samples from the Rayleigh distribution.

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In most cases of censored and truncated sample, the explicit estimators may not be obtained by the maximum likelihood method. So we need another method that generates the explicit estimator. The approximate maximum likelihood estimation method was first developed by Balakrishnan (1989a, b) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution and the mean and standard deviation in the normal distribution with censoring. Some historical remarks and a good summary of the approximate maximum likelihood estimation may be found in Balakrishnan and Cohen (1991). Kang and Kim (1994a) provided the approximate maximum likelihood estimators (AMLE) of the location parameter in the Rayleigh and the Pareto distribution based on Type-II censored samples. Kang and Kim (1994b) also provided the AMLE of the scale parameter of the Weibull distribution with Type-II censoring. Kang (1996) obtained the AMLE for the scale parameter of the double exponential distribution based on Type-II censored samples and showed that the proposed estimator is generally more efficient than the best linear unbiased estimator and the optimum unbiased absolute estimator. Recently Woo et al. (1998) obtained the AMLE of the scale parameter of the p-dimensional Rayleigh distribution with singly right censored samples.

In this paper, we obtain the AMLEs of the location parameter and the scale parameter in the two-parameter Rayleigh distribution with Type-II censoring when two parameters are unknown by the approximate maximum likelihood estimation method and the asymptotic variances of AMLEs.

2. Estimation for parameter

Consider the two-parameter Rayleigh distribution with the density function (1) and the cumulative distribution function (cdf)

$$F(x;\theta,\sigma) = 1 - \exp\Big\{-\frac{(x-\theta)^2}{2\sigma^2}\Big\}. \tag{2}$$

Let us consider an experiment in which n Rayleigh components are put to test simultaneously at time $x = 0$, and the failure times of these components are recorded. Suppose some initial observations are censored (possibly because of some failures during the time when some checks and adjustmemts are being made on the devices) and some final observations are also censored (possibly because the experimenter terminates the experiment before all components have failed). Then let

$$X_{r+1:n} \le X_{r+2:n} \le \cdots \le X_{n-s:n} \tag{3}$$

be the available Type-II censored sample from the Rayleigh distribution with pdf (1), where the first r and the last s observations are censored.

The likelihood function based on the Type-II censored sample in (3) is given by

$$L = \frac{n!}{r!s!}[F(X_{r+1:n};\theta,\sigma)]^r[1-F(X_{n-s:n};\theta,\sigma)]^s \prod_{i=r+1}^{n-s} f(X_{i:n};\theta,\sigma), \tag{4}$$

which upon denoting $Z_{i:n} = (X_{i:n}-\theta)/\sigma$, can be written as

$$L = \frac{n!}{r!s!}\sigma^{-A}[F(Z_{r+1:n})]^r[1-F(Z_{n-s:n})]^s \prod_{i=r+1}^{n-s} f(Z_{i:n}), \tag{5}$$

where $A = n-r-s$ is the size of the censored sample (3), and $f(z) = ze^{-z^2/2}$ and $F(z) = 1-e^{-z^2/2}$ are the pdf and the cdf of the standard Rayleigh distribution, respectively.

Now, we will obtain the AMLEs of the location parameter and the scale parameter. First, we differentiate the log likelihood function for θ and σ as follows;

$$\frac{\partial \ln L}{\partial \theta} = -\frac{1}{\sigma}\Big[r\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})} - sZ_{n-s:n} + \sum_{i=r+1}^{n-s}\frac{1}{Z_{i:n}} - \sum_{i=r+1}^{n-s} Z_{i:n}\Big] = 0 \tag{6}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma}\Big[2A + rZ_{r+1:n}\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})} - sZ_{n-s:n}^2 - \sum_{i=r+1}^{n-s} Z_{i:n}^2\Big] = 0. \tag{7}$$

Equations (6) and (7) do not admit the explicit solutions for θ and σ, so we will expand the functions $f(Z_{r+1:n})/F(Z_{r+1:n})$ and $1/Z_{i:n}$ in Taylor series around the points $F^{-1}(p_{r+1}) = (-2\ln q_{r+1})^{1/2}$ and $F^{-1}(p_i) = (-2\ln q_i)^{1/2}$, respectively and approximate them by

$$\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})} \simeq a_1 + a_2 Z_{r+1:n} \tag{8}$$

and

$$\frac{1}{Z_{i:n}} \simeq c_{1i} + c_{2i}Z_{i:n}, \tag{9}$$

where $p_i = i/(n+1)$, $q_i = 1-p_i$,

$$a_1 = q_{r+1}(-2\ln q_{r+1})^{3/2}/p_{r+1}^2,$$

$$a_2 = q_{r+1}\Big(1+\frac{2}{p_{r+1}}\ln q_{r+1}\Big)/p_{r+1},$$

$$c_{1i} = 2/(-2\ln q_i)^{1/2},$$

and

$$c_{2i} = 1/2\ln q_i.$$

Now making use of the approximate expressions in (8) and (9), we may approximate the likelihood equations of (6) and (7) as follows;

$$\begin{aligned} \frac{\partial \ln L}{\partial \theta} &\simeq \frac{\partial \ln L^*}{\partial \theta} \\ &= -\frac{1}{\sigma}\Big[r(a_1 + a_2 Z_{r+1:n}) - sZ_{n-s:n} + \sum_{i=r+1}^{n-s}(c_{1i} + c_{2i}Z_{i:n}) - \sum_{i=r+1}^{n-s} Z_{i:n}\Big] \\ &= 0 \end{aligned} \tag{10}$$

and

$$\begin{aligned} \frac{\partial \ln L}{\partial \sigma} &\simeq \frac{\partial \ln L^*}{\partial \sigma} \\ &= -\frac{1}{\sigma}\Big[2A + rZ_{r+1:n}(a_1 + a_2 Z_{r+1:n}) - sZ^2_{n-s:n} - \sum_{i=r+1}^{n-s} Z^2_{i:n}\Big] = 0. \end{aligned} \tag{11}$$

Upon solving equations (10) and (11) for θ and σ, we derive the AMLEs of θ and σ as follows;

$$\hat{\theta}_{AMLE} = B\hat{\sigma}_{AMLE} + C \tag{12}$$

and

$$\hat{\sigma}_{AMLE} = \frac{-B_1 + \sqrt{B_1^2 - 4A_1 \times C_1}}{2A_1}, \tag{13}$$

where

$$B = -(ra_1 + \sum_{i=r+1}^{n-s} c_{1i})/D_1,$$

$$C = (-ra_2X_{r+1:n} + sX_{n-s} - \sum_{i=r+1}^{n-s} c_{2i}X_{i:n} + \sum_{i=r+1}^{n-s} X_{i:n})/D_1,$$

$$D_1 = A - ra_2 + s - \sum_{i=r+1}^{n-s} c_{2i},$$

$$A_1 = 2A - ra_1B + ra_2B^2 - sB^2 - AB^2,$$

$$B_2 = ra_1X_{r+1:n} - ra_1C - 2ra_2CX_{r+1:n} + 2ra_2BC + 2sBX_{n-s}$$

$$- 2sBC + 2B \sum_{i=r+1}^{n-s} X_{i:n} - 2ABC,$$

and

$$\begin{aligned} C_1 &= ra_2 X_{r+1:n}^2 - 2ra_2 C X_{r+1:n} + ra_2 C^2 - sX_{n-s}^2 + sC^2 - \sum_{i=r+1}^{n-s} X_{i:n}^2 \\ &\quad + 2s\, CX_{n-s:n} + 2C \sum_{i=r+1}^{n-s} X_{i:n} - AC^2. \end{aligned}$$

Since the AMLEs $\widehat{\theta}_{AMLE}$ and $\widehat{\sigma}_{AMLE}$ in (12) and (13) are the solutions of the approximate likelihood equations in (10) and (11), the asymptotic variance-covariance matrix of $\widehat{\theta}_{AMLE}$ and $\widehat{\sigma}_{AMLE}$ is given by

$$\begin{bmatrix} E\Big(- \frac{\partial^2 \ln L^*}{\partial \theta^2}\Big) & E\Big(- \frac{\partial^2 \ln L^*}{\partial \theta \partial \sigma}\Big) \\ E\Big(- \frac{\partial^2 \ln L^*}{\partial \theta \partial \sigma}\Big) & E\Big(- \frac{\partial^2 \ln L^*}{\partial \sigma^2}\Big) \end{bmatrix}^{-1}.$$

Now from equations (10) and (11), we can obtain

$$\begin{aligned} E\Big(- \frac{\partial^2 \ln L^*}{\partial \theta^2}\Big) &= D_1/\sigma^2, \\ E\Big(- \frac{\partial^2 \ln L^*}{\partial \sigma^2}\Big) &= D_2/\sigma^2, \\ E\Big(- \frac{\partial^2 \ln L^*}{\partial \theta \partial \sigma}\Big) &= D_3/\sigma^2, \end{aligned} \tag{14}$$

where

$$D_1 = A - ra_2 + s - \sum_{i=r+1}^{n-s} c_{2i},$$

$$D_2 = 3\Big(\sum_{i=r+1}^{n-s} E(Z_{i:n}^2) + sE(Z_{n-s:n}^2) - ra_2 E(Z_{r+1:n}^2)\Big) - 2A - 2ra_1 E(Z_{r+1:n}),$$

$$D_3 = 2sE(Z_{n-s:n}) + 2 \sum_{i=r+1}^{n-s} E(Z_{i:n}) - r(a_1 + a_2 E(Z_{r+1:n})) - ra2E(Z_{r+1:n}),$$

$$E(Z_{i:n}) = \sqrt{\frac{\pi}{2}} \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^{i-1-j} \binom{i-1}{j} / \ (n-j)^{3/2},$$

and

$$E(Z_{i:n}^2) = 2 \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^{i-1-j} \binom{i-1}{j} / \ (n-j)^2.$$

¿From the above expressions, we obtain the asymptotic variances and the asymptotic covariance as follows;

$$\mathrm{Var}(\widehat{\theta}_{AMLE}) \simeq \frac{D_2}{(D_1 D_2 - D_3^2)}\sigma^2,$$

$$\mathrm{Var}(\widehat{\sigma}_{AMLE}) \simeq \frac{D_1}{(D_1 D_2 - D_3^2)}\sigma^2,$$

and

$$\mathrm{Cov}(\widehat{\theta}_{AMLE}, \widehat{\sigma}_{AMLE}) \simeq -\frac{D_3}{(D_1 D_2 - D_3^2)}\sigma^2.$$

3. The simulated result

Random numbers of the two-parameter Rayleigh distribution were generated by IMSL subroutine RNUN and transformed $\theta + \sqrt{(-2\sigma^2 \ln(1 - \mathrm{RNUN}))}$. We investigate the mean squared errors (MSE) of $\widehat{\theta}_{AMLE}$ and $\widehat{\sigma}_{AMLE}$ for $\theta = 1.0$, $\sigma = 1.0$. The simulation procedure is repeated 2,000 times for each sample size $n = 10(10)30$, $r = 0(1)4$, and $s = 0(1)4$. These values are given in Table 1.

¿From Table 1, we observe that the MSEs of the AMLEs decrease as n increases for fixed r and s and the MSEs increase as s increase for fixed n.

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Table 1. The MSEs for the location parameter and the scale parameter when two parameters are unknown.

$\theta = 1.0,\ \sigma = 1.0$

n	r	s	MSE($\hat{\theta}_{AMLE}$)	MSE($\hat{\sigma}_{AMLE}$)
10	0	0	.10413	.06417
	0	1	.10974	.07552
	0	2	.11685	.08983
	1	0	.10875	.06632
	1	1	.11609	.07891
	1	2	.12605	.09520
	2	0	.14336	.07816
	2	1	.15603	.09460
	2	2	.17316	.11628
20	0	0	.04422	.02805
	0	1	.04550	.03083
	0	2	.04670	.03299
	0	3	.04771	.03518
	1	0	.04236	.02740
	1	1	.04385	.03033
	1	2	.04515	.03249
	1	3	.04612	.03458
	2	0	.04876	.03007
	2	1	.05079	.03345
	2	2	.05271	.03615
	2	3	.05418	.03871
	3	0	.05469	.03249
	3	1	.05729	.03633
	3	2	.06018	.03978
	3	3	.06234	.04296

Table 1. (continued)

n	r	s	MSE($\widehat{\theta}_{AMLE}$)	MSE($\widehat{\sigma}_{AMLE}$)
30	0	0	.02583	.01783
	0	1	.02622	.01865
	0	2	.02659	.01946
	0	3	.02700	.02038
	0	4	.02752	.02157
	1	0	.02412	.01708
	1	1	.02449	.01787
	1	2	.02483	.01861
	1	3	.02521	.01948
	1	4	.02577	.02067
	2	0	.02660	.01851
	2	1	.02717	.01950
	2	2	.02762	.02036
	2	3	.02817	.02141
	2	4	.02884	.02273
	3	0	.02982	.01990
	3	1	.03060	.02105
	3	2	.03128	.02212
	3	3	.03209	.02340
	3	4	.03297	.02493
	4	0	.03347	.02131
	4	1	.03450	.02264
	4	2	.03541	.02389
	4	3	.03670	.02556
	4	4	.03790	.02737