Journal of the Korean
Data & Information Science Society
1999, Vol. 10, No. 1, pp. 155 ~ 162

Empirical Bayes Confidence Intervals of the Burr Type XII Failure Model

Dal-Woo Choi [1]

Abstract

This paper is concerned with the empirical Bayes estimation of one of the two shape parameters(θ) in the Burr(β, θ) type XII failure model based on type-II censored data. We obtain the bootstrap empirical Bayes confidence intervals of θ by the parametric bootstrap introduced by Laird and Louis(1987). The comparisons among the bootstrap and the naive empirical Bayes confidence intervals through Monte Carlo study are also presented.

Key Words and Phrases: Burr type XII distribution, Moment prior, Parametric empirical Bayes estimators, Type-III parametric bootstrap, Type-II censored data.

1. Introduction

The Burr type XII(Burr(β, θ)) distribution is used as a lifetime model by several authors. Burr and Cislak(1968) have shown that if the parameters are appropriately chosen, the Burr(β, θ) covers a large portion of the curve shape characteristics of type I, IV, VI in the Pearson family. Thus the use of the Burr(β, θ) as a failure model is appropriate and use to applied statisticians. Lewis(1981) noted that the Weibull and exponential distributions are special limiting cases of the parameter values of the Burr(β, θ). The usefulness and properties of this distribution as a failure model were discussed by Papadopoulos(1978), Al-Hussaini, Ali Mousa and Jaheen(1992), Al-Hussaini and Jaheen(1992) and Ali Mousa(1995). Empirical Bayes(EB) methods effectively incorporate information from past data(or other components in simultaneous estimation) by means of analyzing the marginal density of all the data present and past given the prior parameters. We consider the familiar exchangeable Bayesian model. That is, we are simultaneously testing k populations. For the i-th population, $i = 1, \cdots, k$, we test n_i devices until the number of failures are r_i. At

[1]Lecturer, Department of Software Engineering, Kaya University, Kyungbuk, 717-800, Korea

first stage, the independent lifetime t_{ij} for each device tested in the i-th population is assumed to be Burr(β, θ) with known parameter β and unknown parameter θ_i, i.e.,

$$f(t_{ij}|\theta_i, \beta) = \beta\theta_i {t_{ij}}^{\beta-1}(1 + {t_{ij}}^{\beta})^{-(\theta_i+1)}. \tag{1}$$

Let $t_i = (t_{i1}, t_{i2}, \cdots, t_{ir_i})$ denote the ordered lifetimes of the r_i devices that failed in the i-th population, where $t_{i1} < \cdots < t_{ir_i}$. Then

$$T_i = \sum_{j=1}^{r_i} \ln(1 + {t_{ij}}^{\beta}) + (n_i - r_i)\ln(1 + {t_{ir_i}}^{\beta}) \tag{2}$$

is the sufficient statistic for θ_i and has gamma distribution $G(r_i, \theta_i)$ [see Al-Hussaini, Ali Mousa and Jaheen(1992)], i.e.,

$$f(T_i|\theta_i) = \frac{\theta_i^{r_i}}{\Gamma(r_i)} T_i^{r_i-1}\exp(-\theta_i T_i),\ r_i,\ \theta_i > 0 \tag{3}$$

At the second stage, the θ_i's are supposed independently and identically distributed the gamma distribution $G(u+1,\ v)$, as used by Papadopoulos(1978) and Evans and Regab(1983), which is given by

$$\pi(\theta_i) = \frac{v^{u+1}}{\Gamma(u+1)}\theta_i^u\exp(-\theta_i v),\ u > -1,\ v > 0, \tag{4}$$

Then the posterior distribution of θ_i given T_i is $G(r_i + u + 1,\ v + T_i)$, i.e.,

$$f(\theta_i|T_i, u, v) = \frac{(v + T_i)^{r_i+u+1}}{\Gamma(r_i + u + 1)} {\theta_i}^{r_i+u}\exp(-(v + T_i)\theta_i). \tag{5}$$

Based on a squared error loss function, the Bayes estimator of the "current" values θ_i are their posterior mean, [see for instance Maritz(1989)], is given by

$$\mu_B(i) = E(\theta_i|T_i) = \frac{r_i + u + 1}{(v + T_i)},\ u > -1. \tag{6}$$

Also, the marginal density of T_i is given by

$$h(T_i|u, v) = \frac{v^{u+1}}{B(r_i, u+1)} \frac{T_i^{r_i-1}}{(v + T_i)^{r_i+u+1}}. \tag{7}$$

Thus the joint marginal density of $\mathbf{T}$ is given by

$$h(\mathbf{T}|u, v) = \prod_{i=1}^{k} \frac{v^{u+1}}{B(r_i, u+1)} \frac{T_i^{r_i-1}}{(v + T_i)^{r_i+u+1}}, \tag{8}$$

where $\mathbf{T} = (T_1, T_2, \cdots, T_k)$.

2. Moment(Hyperparameter) estimators to prior selection

Lemma(Berger (1985)) Let $\mu_f(\theta)$ and $\sigma_f^2(\theta)$ denote the conditional mean and variance of T (i.e, the mean and variance with respect to the density $f(t|\theta)$). Let μ_m and σ_m^2 denote the marginal mean and variance of T with respect to $h(T)$. Assuming these quantities exist, then $\mu_m = \mathrm{E}^{\pi}[\mu_f(\theta)]$ and $\sigma_m^2 = \mathrm{E}^{\pi}[\sigma_f^2(\theta)] + \mathrm{E}^{\pi}[(\mu_f(\theta) - \mu_m)^2]$.

We will assume $r_i = r$ for all i. Using Lemma which relates the marginal mean and variance to prior moments, we have for all i,

$$\mu_m = \mathrm{E}^m[T_i] = \int_0^{\infty} T_i h(T_i|u,v)dT_i = \frac{r}{r-1}\frac{v+1}{u} \tag{9}$$

and

$$\begin{aligned}\sigma_m^2 &= \mathrm{E}^{\pi}(\sigma_f^2) + \mathrm{E}^{\pi}[(\mu_f - \mu_m)^2] \\ &= \frac{r^2}{(r-1)^2}\Big[\frac{1}{r-2}\frac{(v+1)(v+2)}{u^2} + \frac{(v+1)(v+2)}{u^2} - \frac{(v+1)^2}{u^2}\Big] .\end{aligned} \tag{10}$$

Since μ_m and σ_m^2 are the marginal mean and variance for $T_i, i = 1, \cdots, k$, we can estimate them as follows : $\widehat{\mu}_m = \sum_{i=1}^{k} T_i \ / \ k$ and $\widehat{\sigma}_m^2 = \sum_{i=1}^{k}(T_i - \widehat{\mu_m})^2 \ / \ (k-1)$. We can solve for u and v from (9) and (10) using $\widehat{\mu}_m$ and $\widehat{\sigma}_m^2$ instead of μ_m and σ_m^2.

It follows that the moment prior estimates of u and v are

$$\widehat{v} = \max\Big[\frac{S_1^2}{S_2 - S_1^2} - 1, \ 1\Big] \ \text{ and } \ \widehat{u} = \max\Big[\frac{\widehat{v}+1}{S_1}, \ 0\Big] \tag{11}$$

where

$$S_1 = \frac{r+1}{r}\widehat{\mu_m} \ \text{ and } \ S_2 = \frac{(r-1)(r-2)}{kr^2}\sum_{i=1}^{k} T_i^2 = S_1^2 + \frac{S_1}{u} . \tag{12}$$

The truncated version is obtained because without the condition $v > -1$ we do not have finite variance in the prior distribution. Moreover, u must be positive.

3. Naive empirical Bayes confidence interval

Let $\widehat{\delta} = (\ \widehat{u}, \ \widehat{v}\ )$ be the moment prior estimator of $\delta = (u, v)$ computed from the marginal distribution of T_i. Then the estimated posterior distribution of θ_i given T_i is $G(\widehat{u} + r_i + 1, \ \widehat{v} + T_i\ )$, that is,

$$f(\theta_i|T_i, \widehat{u}, \ \widehat{v}\ ) = \frac{(\widehat{v}+T_i)^{r_i+\widehat{u}+1}}{\Gamma(r_i + \widehat{u} + 1)}\theta_i^{r_i+\widehat{u}}\exp(-(\widehat{v}+T_i)\theta_i) \tag{13}$$

Now, we construct the equal-tailed $100(1-2\alpha)\%$ EB naive confidence interval for θ_i based upon $f(\theta_i|T_i, \widehat{u}, \widehat{v})$ as follows:

$$\left(\frac{\mathcal{F}^{-1}_{2(\widehat{v}+r_i+1)}(\alpha)}{2(T_i+\widehat{u})}, \ \frac{\mathcal{F}^{-1}_{2(\widehat{v}+r_i+1)}(1-\alpha)}{2(T_i+\widehat{u})} \right), \tag{14}$$

where $\mathcal{F}_k$ denotes the cumulative distribution function of chi-square distribution with k (not necessarily integer) degrees of freedom. This interval is called "naive" because they ignore randomness in $\widehat{\delta} = (\widehat{u}, \widehat{v})$. Though relatively easy to compute, they are often too short, inappropriately centered, or both, and hence fail to attain the nominal coverage probability.

4. Bootstrapped empirical Bayes confidence intervals

We will suggest several bootstrap methods in order to correct the bias in the naive confidence interval based on $f(\theta_i|T_i, \widehat{u}, \widehat{v})$ and show how they may be used to compute confidence intervals.

Marginal posterior method

Laird and Louis(1987) suggested approximating the marginal posterior of θ_i given T_i by type III parametric bootstrap. That is, given $\widehat{\delta} = (\widehat{u}, \widehat{v})$, drew θ_i^* from $\pi(\theta|\widehat{\delta})$. Then drew t_{ij}^* from $f(t_i|\theta_i^*)$, and finally calculate $\delta^* = (u^*, v^*)$ using t_{ij}^*. Repeating this process N times, they obtained $\delta_j^* = (u_j^*, v_j^*)$, $j = 1, \cdots, N$ distributed as $g(\cdot|\widehat{\delta})$. By type III parametric bootstrap, we obtain the discrete mixture distribution mimicking the hyperprior calculation given by

$$H^*_{\mathrm{T}}(\theta_i|T_i) = \frac{1}{N}\sum_{j=1}^{N} \mathcal{F}_{2(v_j^*+r_i+1)}\left(\frac{\theta_i \, v_j^*}{2(T_i+u_j^*)}\right). \tag{15}$$

As defined, $H^*_{\mathrm{T}}(\theta_i|T_i)$ can be used directly to produce equal-tailed confidence intervals for θ_i by solving

$$\frac{\alpha}{2} = \int_{-\infty}^{C_L} dH^*_{\mathrm{T}}(\theta_i|T_i) = \int_{C_U}^{\infty} dH^*_{\mathrm{T}}(\theta_i|T_i). \tag{16}$$

Therefore, the $100(1-2\alpha)\%$ EB marginal confidence interval for θ_i is given by $(C_L, \ C_U)$.

Bias-corrected naive method

In the exchangeable case, Carlin and Gelfand(1991) developed a direct conditional bias-corrected naive method. Let $q_\alpha(T_i, \delta = (u, v))$ is such that $\Pr(\theta_i \le q_\alpha(T_i, \delta) \mid \theta_i \sim f(\theta_i \mid T_i, \delta)) = \alpha$. Define $r(\widehat{\delta}, \delta, T_i, \alpha) = \Pr(\theta_i \le q_\alpha(T_i, \widehat{\delta}) \mid \theta_i \sim f(\theta_i \mid T_i, \delta))$ and $R(\delta, T_i, \alpha) = E_{\widehat{\delta}|T_i,\delta}\{r(\widehat{\delta}, \delta, T_i, \alpha)\}$, where the expectation is taken over $g(\widehat{\delta}|T_i, \delta)$ which is a density with respect to Lebesgue measure. We omit the detail. See Choi(1996, 1997). By unconditional EB correction method, we obtain the type III parametric bootstrap estimate of $R(\delta, T_i, \alpha'_{(1)})$ given by

$$\frac{1}{N}\sum_{j=1}^{N} \mathcal{F}_{2(\widehat{v}+r_i+1)}\left(\frac{\widehat{u}+T_i}{u_j^*+T_i}\mathcal{F}^{-1}_{2(v_j^*+r_i+1)}(\alpha'_{(1)})\right) = \alpha \tag{17}$$

where we equate to α and solve for $\alpha'_{(1)}$. Then the $100(1-2\alpha)\%$ unconditional EB bias-corrected(I) naive interval for θ_i is given by

$$\left(\frac{\mathcal{F}^{-1}_{2(\hat{v}+r_i+1)}(\alpha'_{(1)})}{2(T_i+\widehat{u})},\ \frac{\mathcal{F}^{-1}_{2(\hat{v}+r_i+1)}(1-\alpha'_{(1)})}{2(T_i+\widehat{u})}\right). \tag{18}$$

If we desire interval corrected only for unconditional EB coverage, the bootstrap equation becomes

$$\frac{1}{N}\sum_{j=1}^{N} \mathcal{F}_{2(\widehat{v}+r_i+1)}\left(\frac{\widehat{u}+T_{ij}}{u_j^*+T_{ij}}\mathcal{F}^{-1}_{2(v_j^*+r_i+1)}(\alpha'_{(2)})\right) = \alpha \tag{19}$$

where we equate to α and solve for $\alpha'_{(2)}$. Analogous to expression (18), we obtain the $100(1-2\alpha)\%$ unconditional EB bias-corrected(II) naive intervals for θ_i given by

$$\left(\frac{\mathcal{F}^{-1}_{2(\hat{v}+r_i+1)}(\alpha'_{(2)})}{2(T_i+\widehat{u})},\ \frac{\mathcal{F}^{-1}_{2(\hat{v}+r_i+1)}(1-\alpha'_{(2)})}{2(T_i+\widehat{u})}\right). \tag{20}$$

For correction conditional on T_i, Carlin and Gelfand(1990) modified the Laird and Louis(1987) procedure. By conditional EB bias-corrected method on $T_i = t_i$, we obtain the type III parametric bootstrap estimate of $R(\delta, T_i, \alpha'_{(3)})$ given by

$$\frac{1}{N}\sum_{j=1}^{N} \mathcal{F}_{2(\widehat{v}+r_i+1)}\left(\frac{\widehat{u}+T_i}{u_j^*+T_i}\mathcal{F}^{-1}_{2(v_j^*+r_i+1)}(\alpha'_{(3)})\right) = \alpha \tag{21}$$

where we equate to α and solve for $\alpha'_{(3)}$. Therefore, the $100(1-2\alpha)\%$ conditional EB bias-corrected(III) naive interval for θ_i is given by

$$\left(\frac{\mathcal{F}^{-1}_{2(\hat{v}+r_i+1)}(\alpha'_{(3)})}{2(T_i+\widehat{u})},\ \frac{\mathcal{F}^{-1}_{2(\hat{v}+r_i+1)}(1-\alpha'_{(3)})}{2(T_i+\widehat{u})}\right). \tag{22}$$

5. Comparisons and conclusions

Random samples of different sizes are generated from the Burr(β, θ) distribution (see AL-Hussaini and Jaheen(1992)) and the EB confidence intervals are approximated by Monte Carlo method. We consider the censoring rate (CR) defined by $100(1-r/n)\%$ of 0%(=complete case), 10%, 50%. For given independent random samples a EB confidence intervals are computed by each methods with bootstrap replications $B = 1000$ times. And the Monte Carlo samplings are repeated 500 times. We define the mean length by $(\sum_j^R(\widehat{\theta_{j,up}} - \widehat{\theta_{j,lo}}))/R$, where R is the number of Monte Carlo simulation replication. The comparisons of the naive and the bootstrap intervals against the moment prior estimator when the hyperparameters u, v are unknown are presented in Table. We can observe the following results:

1. Bias-corrected naive bootstrap interval(I) obtained using type III parametric bootstrap has more accurate than those of the other bootstrap intervals in the desired nominal coverage.
2. The coverage probabilities of bootstrap intervals obtained using given type III parametric bootstrap are better than that of the naive interval.
3. The mean lengths of bootstrap intervals obtained using given type III parametric bootstrap are no longer than that of the naive interval.
4. The coverage probabilities of all the intervals are linearly down as the censoring rate increases.

Table. Comparison of Empirical Bayes Confidence Intervals in Burr($\beta = 2$, θ).

1. Sample size: $n = 10$, $\alpha = 0.05$

Interval method	$CR =$ 0 %		$CR =$ 10 %		$CR =$ 50 %	
	Coverage	Length	Coverage	Length	Coverage	Length
EB Naive	0.716	1.096	0.712	1.318	0.669	1.864
Bias-correct(I)	0.920	0.958	0.901	1.121	0.880	1.419
Bias-correct(II)	0.888	0.906	0.863	0.982	0.803	1.125
Bias-correct(III)	0.903	0.898	0.891	0.951	0.856	1.001
EB Marginal	0.782	0.910	0.756	1.002	0.694	1.437

2. Sample size: $n = 20$, $\alpha = 0.05$

Interval method	$CR =$ 0 %		$CR =$ 10 %		$CR =$ 50 %	
	Coverage	Length	Coverage	Length	Coverage	Length
EB Naive	0.812	0.892	0.768	0.978	0.717	1.065
Bias-correct(I)	0.926	0.899	0.911	0.918	0.898	0.995
Bias-correct(II)	0.908	0.754	0.893	0.806	0.850	0.916
Bias-correct(III)	0.918	0.739	0.900	0.816	0.889	0.895
EB Marginal	0.895	0.883	0.881	0.914	0.786	0.986

3. Sample size: $n = 50$, $\alpha = 0.05$

Interval method	$CR =$ 0 %		$CR =$ 10 %		$CR =$ 50 %	
	Coverage	Length	Coverage	Length	Coverage	Length
EB Naive	0.876	0.752	0.850	0.825	0.823	1.093
Bias-correct(I)	0.943	0.778	0.938	0.806	0.905	0.987
Bias-correct(II)	0.920	0.694	0.901	0.758	0.873	0.926
Bias-correct(III)	0.936	0.582	0.919	0.673	0.896	0.725
EB Marginal	0.915	0.668	0.903	0.700	0.853	0.847

References

1. AL-Hussaini, E. K., Ali Mousa, M. A. M., and Jaheen, Z. F. (1992). Estimation under the Burr type XII failure model based on censored data: A comparative study. *Test*, 1, 47-60

2. AL-Hussaini, E. K. and Jaheen, Z. F. (1992). Bayesian estimation of the parameters, reliability and failure function of the Burr type XII failure model. *Journal of Statistical Computation and Simulation*, 41, 31-40.

3. Ali Mousa, M. A. M. (1995). Empirical Bayes estimators for the Burr type XII accelerated life testing model based on type-2 censored data. *Journal of Statistical Computation and Simulation*, 52, 95-103.

4. Berger, G. O. (1985). *Statistical Decision Theory and Bayesion Analysis*, Springer-Verlag, New York.

5. Burr, I. W. and Cislak, P. J. (1968). On a genernal system of distribution: I. Its curve-shaped characteristics; II. The sample-median. *Journal of American Statistical Association*. 63, 627-635.

6. Carlin, B. P. and Gelfand, A. E. (1990). Approaches for empirical Bayes confidence intervals, *Journal of American Statistical Association.* 85, 105-114.

7. Carlin, B. P. and Gelfand, A. E. (1991). A sample reuse method for accurate parametric empirical Bayes confidence intervals. *Journal of the Royal Statistical Society*, B, 53, 189-200.

8. Choi, D. W. (1996). *Bootstrapping empirical Bayes confidence intervals in the Weibull distribution.* Ph. D dissertation, Kyungpook National University.

9. Choi, D. W. (1997). Parametric empirical Bayes estimators with item-censored data. *Journal of Statistical Theory & Methods*, 8, 263-272.

10. Even, I. G. and Regab, A. S. (1983). Bayesian inference given a type-2 censored sample from Burr distribution. *Communications in Statistics, Theory and Methods*, 12, 1569-1580.

11. Laird, N. M. and Louis, T. A. (1987). Empirical Bayes confidence intervals based on bootstrap samples(with comments). *Journal of American Statistical Association*, 82, 739-757.

12. Lewis, A. W. (1981). *The Burr distribution as a general parametric family in survivorship and reliability theory applications.* Ph. D. Dissertation. University of North Carolina.

13. Papadopoulos, A. S. (1978). The Burr distribution as a failure model from a Bayesian approach. *IEEE Transactions on Reliability*, R-5, 369-371.

14. Maritz, J. S. and Lwin, T. (1989). *Empirical Bayes Methods*, Chapman & Hall, Lendon.