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Estimation for Autoregressive Models with GARCH(1,1) Error via Optimal Estimating Functions.

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Abstract

Optimal estimating functions for a class of autoregressive models with GARCH(1,1) error are discussed. The asymptotic properties of the estimator as the solution of the optimal estimating equation are investigated for the models. We have also some simulation results which suggest that the proposed optimal estimators have smaller sample variances than those of the conditional least-squares estimators under the heavy-tailed error distributions.

Key Words and Phrases: GARCH model, Optimal Estimating Functions Consistency, Asymptotic Normality, Conditional Least-squares Estimators, Asymptotic Relative Efficiency.

1. Introduction

Autoregressive conditional heteroscedasticity (ARCH) models were originally developed by Engle(1982). ARCH models can be specified by the conditional variance as a function of past squared errors and have been estimated by the maximum likelihood method. Bollerslev(1986) extended the ARCH models to a more general specifications, that is, the generalized ARCH(GARCH) models.

The idea of optimal estimating functions was proposed by Godambe (1960) in iid set-ups. Godambe(1985) has further developed the theory of optimal estimating functions for discrete time stochastic processes. Tavaneswaran and Abraham(1988) established the optimal estimating functions for a class of non-linear time series models based on finite samples. Godambe's(1960, 1985) optimal estimating function is to establish an optimatity in a class of the unbiased estimation functions based on finite samples. McCullagh(1983) studied the consistency and asymptotic normality of the solutions of the optimating estimating functions in independent

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case. Pantula(1988) studied the asymptotic properties of the maximum likelihood estimators for a p-th order autoregressive process with ARCH errors.

In this paper, we will present the optimal estimating function for a p-th order autoregressive model with GARCH(1,1) errors and investigate the consistency and the asymptotic normality of the solution of the optimal estimating function. Also some simulation results which compare the sample variances of the proposed estimators and the conditional least-squares estimators.

2. The Optimal Estimating Functions

Consider a stochastic process $\{X_t\}, t \geq 1$, satisfying the following equation

$$X_t = \theta^T \underline{X}_{t-1} + \epsilon_t \ , \tag{2.1}$$

where $\theta^T = (\theta_1, \cdots, \theta_p)$, $\underline{X}_{t-1} = (X_{t-1}, \cdots, X_{t-p})$, $p \geq 1$. Conventional autoregressive processes assume that $\{\epsilon_t\}$ is a sequence of iid random variables with zero mean and constant variance. There are several processes in the literature that consider the conditional variance which depend on the past data. Engle (1982) established autoregressive conditionally heteroscedastic (ARCH) models which considered the conditional variance allowing the change over time as a linear function of past errors. Therefore, in this time series set-up, we define the conditional mean and the conditional variance as $E(\epsilon_t|\mathcal{F}_{t-1}) = 0$ and $Var(\epsilon_t|\mathcal{F}_{t-1}) = h_t$, which is called a General ARCH (GARCH) model, where $h_t = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + \beta_1 h_{t-1}, \mathcal{F}_{t-1}$ is a σ-field generated by $\underline{X}_{t-1}$, and $\alpha_0 > 0, \alpha_1 \geq 0, \beta_1 \geq 0$. The estimating function approach has been used to solve various problems of finite sample estimation. In this section, we review Godambe's(1985) criterion on stochastic processes and apply it to obtain optimal estimator for the model in (2.1). First, we define a regular unbiased estimation function.

Definition 2.1

A regular unbiased estimating function defined as

$$E_\theta[g(X_1, \cdots, X_n)] = 0 \ ,$$

where $g = \sum_{t=1}^n a_{t-1}k_t$, the function k_t is such that $E[k_t|\mathcal{F}_{t-1}] = 0$ and a_{t-1} is a function of $X_1, \cdots, X_{t-1}$ for $t = 1, \cdots, n$.

Theorem 2.1

In the class of unbiased estimating function g, the optimal estimating function g^* is the one which minimizes

$$E(g^2)/E(\frac{\partial g}{\partial \theta})^2$$

and this is give by

$$g^* = \sum_{t=1}^{n} a_{t-1}^* k_t \;, \tag{2.2}$$

where $a_{t-1}^* = \frac{E[\frac{\partial k_t}{\partial \theta}|\mathcal{F}_{t-1}]}{E[k_t^2|\mathcal{F}_{t-1}]}$.

Proof. See Godambe(1985)

Note that the optimal estimating function for the model in (2.1) is given by

$$S_n(\theta) = \sum_{t=1}^{n} \underline{X}_{t-1}(X_t - \theta^T \underline{X}_{t-1})/h_t \;, \tag{2.3}$$

where $h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 h_{t-1}$.

3. Estimation of the GARCH(1,1) model

We will estimate under the model in (2.1). For the simplicity of the estimation, we assume that α_0, α_1, and β_1 are known. By using the optimal estimating function in (2.3), we establish the optimal estimator for $\theta = (\theta_1, \cdots, \theta_p)$ as the solution of the estimating equation in (2.3). This leads to

$$\hat{\theta}_n = (\sum_{t=1}^{n} \underline{X}_{t-1}\underline{X}_{t-1}^T/h_t)^{-1}(\sum_{t=1}^{n} \underline{X}_{t-1}\underline{X}_t/h_t), \tag{3.1}$$

Where $\underline{X}_{t-1}^T = (X_{t-1}, \cdots, X_{t-p})$ and $h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 h_{t-1}$.

Note that we assume $\{X_t\}$ to be stationary and ergodic. The necessary and sufficient condition for the stationarity of the GARCH(1,1) model is $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and $\alpha_1 + \beta_1 < 1$. (See Bollerslev(1986))

Consider the following regularity conditions:

1. $\{X_t\}$ is a stationary and ergodic.
2. $F(\theta) = E[h_t^{-1} \underline{X}_{t-1} \underline{X}_{t-1}^T]$ and $F(\theta)$ is assumed to be a $p \times p$ positive definite matrix.

Theorem 3.1

Under the regularity condition 1 and 2, we have

i. $\hat{\theta}_n \longrightarrow \theta$ a.s.,

ii. $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, F^{-1}(\theta))$

where $\hat{\theta}_n$ is defined in (3.1) and $F(\theta) = E[h_t^{-1}\underline{X}_{t-1}\underline{X}_{t-1}^T]$.

Proof.

(i) This result can be established by the ergodic theorem.

(ii) Note that

$$\sqrt{n}(\hat{\theta}_n - \theta) = [\frac{\sum_{t=1}^{n}(\underline{X}_{t-1}\underline{X}_{t-1}^T)/h_t}{n}]^{-1}\frac{1}{\sqrt{n}}[\sum_{t=1}^{n}\frac{\underline{X}_{t-1}(X_t - \theta^T\underline{X}_{t-1})}{h_t}] \quad (3.2)$$

First, it can be shown by the ergodic theorem that

$$\frac{1}{n}\sum_{t=1}^{n}\{\underline{X}_{t-1}\underline{X}_{t-1}^T)/h_t\} \longrightarrow F(\theta) \ , \ a.s. \quad (3.3)$$

Second, since $\sum_{t=1}^{n}\{\frac{\underline{X}_{t-1}(X_t-\theta^T\underline{X}_{t-1})}{h_t}\}$ is a sum of stationary and ergodic martingale differences, therefore it can be shown that

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\{\frac{\underline{X}_{t-1}(X_t - \theta^T\underline{X}_{t-1})}{h_t}\} \xrightarrow{d} N(0, F(\theta)) \ , \quad (3.4)$$

by using a martingale cential limit theorem (See Hall and Heyde(1980)). The result in Theorem 3.1(ii) can be proved by using (3.3), (3.4) and Slustky's theorem.

4. Simulation Results

A simulation study was carried out to compare the sample variances of the conditional least-squares(CLS) estimator and the proposed optimal estimator of the AR(1) model with GARCH(1,1) error. We consider 3 different error distribution, i.e., standard normal, maxture of two normal, i.e., $\alpha N(0,1) + (1-\alpha)N(0,\sigma^2)$, where $\sigma^2 > 1$ and $0 < \alpha < 1$ and the double exponential distribution. The sample sizes of n=200 and 400 are used. The CLS estimator of θ in the AR(1) model is defined as

$$\hat{\theta}_{CLS} = \sum_{t=1}^{n} X_t X_{t-1} / \sum_{t=1}^{n} X_{t-1}^2 \quad (4.1)$$

Let $\hat{\theta}_{OE}$ be the solution of the estimating equation in (2.2). We first show the biases of the CLS and the optimal estimators. We compare the limiting variances of the optimal estimators with the CLS estimators of θ by simulation. The biases and the asymptotic relative efficiency (ARE) of $\hat{\theta}_{OE}$ with respect to $\hat{\theta}_{CLS}$ are given by

$$Bias(\hat{\theta}_{OE}) = \hat{\theta}_{OE} - \theta \ , \quad (4.2)$$

$$Bias(\hat{\theta}_{CLS}) = \hat{\theta}_{CLS} - \theta \ , \tag{4.3}$$

$$ARE = Var(\hat{\theta}_{CLS})/Var(\hat{\theta}_{OE}) \tag{4.4}$$

In the tables 1,3,5 and 7, the biases of $\hat{\theta}_{CLS}$ and $\hat{\theta}_{OE}$ are not much different from θ under the standard normal error distribution. But the biaes of $\hat{\theta}_{CLS}$ are fairly bigger than those of $\hat{\theta}_{OE}$ under the heavy-tailed error distributions regardless of sample size, that is, n=200 and 400.

In the tables 2,4,6 and 6, the simulation results show that the AREs are little bit bigger than 1 under the standard normal error distribution. This means that the CLS estimators perform better than the optimal estimators under the normal assumption. Under the heavy-tailed error distributions, the AREs are smaller than 1 especially when σ^2 is getting larger and α is also getting bigger in case of the contaminated normal cases. This implies that the optimal estimators perform quite well under the heavy-tailed cases and work better than the CLS estimators as well.

Table 1 : Biases for a GARCH(1,1) Model with
$\theta = 0.1, \alpha_0 = 0.3, \alpha_1 = 0.2, \beta_1 = 0.2$

			Standard normal	Standard Double exponential (1)	Contiminated normal α =10 (%) Variance		
					5	10	20
n	200	OE	0.0615	0.0682	0.0691	0.0678	0.0683
		CLS	0.0641	0.1236	0.0920	0.1014	0.1046
	400	OE	0.0310	0.0406	0.0333	0.0390	0.0355
		CLS	0.0323	0.0654	0.0485	0.0560	0.0591

Table 2 : AREs of the optimal Estimators for a GARCH(1,1) model with
$\theta = 0.1, \alpha_0 = 0.3, \alpha_1 = 0.2, \beta_1 = 0.2$

		Standard normal	Standard Double exponential (1)	Contiminated normal α =10 (%) Variance		
				5	10	20
n	200	1.0203	0.9279	0.7736	0.7474	0.7282
	400	1.0143	0.9299	0.7551	0.7405	0.7186

Table 3 : Biases for a GARCH(1,1) Model with
$\theta = 0.3, \alpha_0 = 0.3, \alpha_1 = 0.2, \beta_1 = 0.2$

			Standard normal	Standard Double exponential (1)	Contiminated normal α =10 (%) Variance		
					5	10	20
n	200	OE	0.0532	0.0574	0.0599	0.0636	0.0612
		CLS	0.0553	0.0799	0.0806	0.0905	0.0995
	400	OE	0.0396	0.0517	0.0409	0.0407	0.0448
		CLS	0.0421	0.0784	0.0577	0.0133	0.0703

Table 4 : AREs of the optimal Estimators for a GARCH(1,1) model with
$\theta = 0.3, \alpha_0 = 0.3, \alpha_1 = 0.2, \beta_1 = 0.2$

		Standard normal	Standard Double exponential (1)	Contiminated normal α =10 (%) Variance		
				5	10	20
n	200	1.0265	0.9377	0.7701	0.7475	0.7447
	400	1.0183	0.9034	0.7682	0.7338	0.7235

Table 5 : Biases for a GARCH(1,1) Model with
$\theta = 0.5, \alpha_0 = 0.3, \alpha_1 = 0.2, \beta_1 = 0.2$

			Standard normal	Standard Double exponential (1)	Contiminated normal α =10 (%) Variance		
					5	10	20
n	200	OE	0.0555	0.0613	0.0567	0.0597	0.0648
		CLS	0.0570	0.0897	0.0788	0.0954	0.0985
	400	OE	0.0378	0.0563	0.0418	0.0418	0.0443
		CLS	0.0402	0.0716	0.0564	0.0665	0.0711

Table 6 : AREs of the optimal Estimators for a GARCH(1,1) model with $\theta = 0.5, \alpha_0 = 0.3, \alpha_1 = 0.2, \beta_1 = 0.2$

		Standard normal	Standard Double exponential (1)	Contiminated normal α =10 (%) Variance 5	10	20
n	200	1.0254	0.9243	0.7762	0.7747	0.7402
	400	1.0200	0.9208	0.7661	0.7378	0.7181

Table 7 : Biases for a GARCH(1,1) Model with $\theta = 0.7, \alpha_0 = 0.3, \alpha_1 = 0.2, \beta_1 = 0.2$

			Standard normal	Standard Double exponential (1)	Contiminated normal α =10 (%) Variance 5	10	20
n	200	OE	0.0448	0.0524	0.0480	0.0503	0.0519
		CLS	0.0458	0.0896	0.0650	0.0764	0.0882
	400	OE	0.0314	0.0373	0.0333	0.0390	0.0355
		CLS	0.0321	0.0704	0.0485	0.0560	0.0591

Table 8 : AREs of the optimal Estimators for a GARCH(1,1) model with $\theta = 0.7, \alpha_0 = 0.3, \alpha_1 = 0.2, \beta_1 = 0.2$

		Standard normal	Standard Double exponential (1)	Contiminated normal α =10 (%) Variance 5	10	20
n	200	1.0232	0.9335	0.7672	0.7324	0.7211
	400	1.0103	0.9327	0.7553	0.7405	0.7109

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