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Note on Stochastic Orders through Length Biased Distributions

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Abstract

We consider $Y = X + \lambda Z$, $\lambda > 0$, where X and Z are independent random variables, and Y is the length biased distribution or the equilibrium distribution of X. The purpose of this paper is to consider the distribution of X or Y when the distribution of Z is given and the distribution of Z when the distribution of X or Y is given, In particular, we obtain that the necessary and sufficient conditions for X to be $\chi^2(v)$ is $Z \sim \chi^2(2)$ and for Z to be $\chi^2(1)$ is $X \sim IG(\mu,\ \mu^2/\lambda)$, where $IG(\mu,\ \mu^2/\lambda)$ is two-parameter inverse Gaussian distribution.

Also we show that X is smaller than Y in the reverse Laplace transform ratio order if and only if X_e is smaller than Y_e in the Laplace transform ratio order. Finally, we can get the results that if X is smaller than Y in the Laplace transform ratio order, then Y_L is smaller than X_L in the Laplace transform order, and that if X is smaller than Y in the reverse Laplace transform ratio order, then μX_L is smaller than νY_L in the Laplace transform order.

Key Words and Phrases: length biased distribution, equilibrium distribution, Laplace transform order, Laplace transform ratio order, reverse Laplace transform ratio order

1. Introduction

The length biased distribution finds various applications in biomedical areas such as family history and disease, early detection of disease, survival and intermediate events, and latency periods of AIDS due to blood transfusion.

Cnaan(1985) described an application of length biased sampling in a cardiology study. Gupta and Keating(1986) introduced relations for reliability measures under length biased sampling.

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The equilibrium distribution was used to describe the concepts of positive aging in Deshpande, Kochar, and Singh(1986), Singh(1989), Deshpande, Singh, Bagai, and Jain(1990)

The length biased distribution and the equilibrium distribution are of great interest on actuarial studies, survival analysis, and reliability. Hence, our objectives are to compare several stochastic order relations of two random variables X and Y with those of these length biased distributions and the equilibrium distributions.

We briefly discuss definitions of the length biased distribution and the equilibrium distribution. For a non-negative random variable X with probability density function $f(x)$ and finite mean μ, the probability density function of length biased distribution X_L of X is given by

$$f_L(x) = \frac{xf(x)}{\mu}, \quad x > 0.$$

For a non-negative random variable X with survival function $\bar{F}$ and finite mean μ, the random variable X_e is called the equilibrium distribution of X or F if

$$P(X_e \le x) = F_e(x) = \int_0^x \frac{\bar{F}(y)}{\mu}\, dy, \quad x \ge 0.$$

The corresponding density function is

$$f_e(x) = F_e{}'(x) = \frac{\bar{F}(x)}{\mu}, \quad x \ge 0.$$

We can find the Laplace-Stieltjes transform of F_e in terms of the Laplace transform of F :

$$\tilde{F}_e(s) = E(e^{-sX_e}) = \int_0^\infty e^{-sx} f_e(x)\, dx = \int_0^\infty e^{-sx}\, \frac{\bar{F}(x)}{\mu}\, dx = \frac{1-\tilde{F}(s)}{\mu s},$$

2. Main Results

We shall consider relations between two random variables X and X_L.

Theorem 1. If Y is the length biased distribution of X, then

$$\phi_Y(t) = \frac{1}{i\mu}\,\phi_X{}'(t), \quad M_Y(t) = \frac{1}{\mu}\, M_X{}'(t),$$

where $\phi_Y(t)$ and $\phi_X(t)$ are the characteristic functions of two random variables Y and X, respectively. $M_Y(t)$ and $M_X(t)$ are the moment generating function of Y and X, respectively.

First, we consider $Y = X + Z$, where X and Z are independent random variables and $Y = X_L$, that is, Y is the length biased distribution of X. If the random variable X has a chi-square distribution with ν degree of freedom, then, using the first equation in Theorem 1, we obtain

$$\frac{1}{i\nu}\,\phi_X{}'(t) = \phi_X(t)\phi_Z(t).$$

Since X is $\chi^2(\nu)$, Y is $\chi^2(\nu+2)$. Therefore

$$\begin{aligned}\phi_Z(t) &= \frac{\phi_X{}'(t)}{i\nu\phi_X(t)} \\ &= \frac{\nu\left(\frac{1}{1-2it}\right)^{\frac{\nu+2}{2}}}{\nu\left(\frac{1}{1-2it}\right)^{\frac{\nu}{2}}}.\end{aligned}$$

Hence, $\phi_Z(t) = (1-2it)^{-1}$ is the characteristic function of $\chi^2(2)$. Therefore Z is $\chi^2(2)$.

Conversely, if random variable Z has $\chi^2(2)$, then X and Y have $\chi^2(\nu)$ and $\chi^2(\nu+2)$, respectively. Hence we have the following theorem.

Theorem 2. Let $Y = X + Z$, where X and Z are independent random variables and $Y = X_L$. Then the necessary and sufficient conditions for $X \sim \chi^2(\nu)$ is $Z \sim \chi^2(2)$.

We shall prove the following theorem by using the moment generating function.

Theorem 3. Let $Y = X + \lambda Z$, $\lambda > 0$, where X with mean μ and Z are independent random variables and $Y = X_L$. Then the necessary and sufficient conditions for Z to be $\chi^2(1)$ is $X \sim IG(\mu,\ \mu^2/\lambda)$.

Proof. Writing $M_Y(t) = \dfrac{1}{\mu}\,M_X{}'(t)$ reduces to

$$\begin{aligned} & \tfrac{1}{\mu}\,M_X{}'(t) = M_X(t)M_Z(\lambda t) \\ \Longleftrightarrow\quad & \tfrac{M_X{}'(t)}{M_X(t)} = \mu M_Z(\lambda t) \\ \Longleftrightarrow\quad & \textstyle\int (\log M_X(t))'\,dt = \int \mu\left(\frac{1}{1-2\lambda t}\right)^{\frac{1}{2}}\,dt + k \\ \Longleftrightarrow\quad & \log \mathrm{M_X(t)} = -\tfrac{\mu}{\lambda}(1-2\lambda t)^{\frac{1}{2}} + k \\ \Longleftrightarrow\quad & M_X(t) = \exp\left[-\tfrac{\mu}{\lambda}(1-2\lambda \mathrm{t})^{\frac{1}{2}} + \mathrm{k}\right].\end{aligned}$$

Since $M_X(0) = 1$, $k = \mu/\lambda$. Then $M_X(t) = \exp\left[\frac{\mu}{\lambda}\left[1-(1-2\lambda t)^{\frac{1}{2}}\right]\right]$ is the moment generating function of $IG(\mu,\ \mu^2/\lambda)$. Therefore, $X \sim IG(\mu,\ \mu^2/\lambda)$.

Similarly, we can show that the converse also holds.

Next, we shall consider $Y = X + \lambda Z$, $\lambda > 0$, where $Y = X_e$ and X with mean μ and Z are independent random variables, the equation may be proven by the Laplace transform, but it is not trivial : Let X be the non-negative absolutely continuous random variable with probability density function $f(x)$ and distribution function $F(x)$, Then, from the Laplace transforms of both sides of $Y = X + \lambda Z$, we obtain the following relation.

$$\frac{1 - \tilde{F}(s)}{\mu s} = \tilde{F}(s)\,\tilde{H}(\lambda s)$$

where $\tilde{H}(s)$ is the Laplace transform of Z. Therefore, we obtain the following

$$\begin{aligned} \tilde{H}(\lambda s) &= \frac{1 - \tilde{F}(s)}{\mu s\,\tilde{F}(s)} \\ &= \frac{(1+2s)^{\frac{v-2}{2}} - 1}{\mu s}. \end{aligned}$$

Now we discuss the stochastic order relations of length biased distributions and those of equilibrium distributions of two random variables X and Y when stochastic *order relations of X and Y are given. First of all, we discuss definitions of the* Laplace–Stieltjes transform and the Laplace transform.

For any non-negative random variable Z with distribuion function F_Z, the Laplace–Stieltjes transform is given by

$$L_Z(s) = \int_0^\infty e^{-st}\, dF_Z(t), \quad s > 0.$$

Note that $L_Z(s)$ is decreasing in s(in this paper 'increasing' and 'decreasing' mean 'nondecreasing' and 'nonincreasing', respectively).

Let $\bar{F}_Z$ be the survival function of Z. The Laplace transform of $\bar{F}_Z$ is defined by

$$L_Z^*(s) = \int_0^\infty e^{-st}\bar{F}_Z(t)\, dt, \quad s > 0.$$

Definition Let X and Y be two non-negative random variables. We say that
1) X is smaller than Y in the Laplace transform ratio order (denoted by $X \le_{Lt-r} Y$) if

$$\frac{L_Y(s)}{L_X(s)} \text{ is decreasing in s} > 0.$$

2) X is smaller than Y in the reverse Laplace transform ratio order (denoted by $X \le_{r-Lt-r} Y$) if

$$\frac{1 - L_Y(s)}{1 - L_X(s)} \text{ is decreasing in s} > 0.$$

3) X is smaller than Y in the Laplace transform order(denoted by $X \leq_{Lt} Y$) if

$$\int_0^\infty e^{-st}\, dF_X(t) = L_X(s) \geq L_Y(s) = \int_0^\infty e^{-st}\, dF_Y(t).$$

From the above is it seen that

$$X \leq_{Lt-r} Y \Longleftrightarrow \frac{1 - sL_Y^*(s)}{1 - sL_X^*(s)} \quad \text{is decreasing in s} > 0,$$

and that

$$X \ \leq_{r-Lt-r} \ Y \Longleftrightarrow \frac{1 - L_Y^*(s)}{1 - L_X^*(s)} \quad \text{is decreasing in s} > 0,$$

$$X \ \leq_{Lt} \ Y \Longleftrightarrow \int_0^\infty e^{-st}\bar{F}_X(t)\, dt \leq \int_0^\infty e^{-st}\bar{F}_Y(t)\, dt\ .$$

Let us now consider the Laplace transform ratio order between X_e and Y_e by considering the reverse Laplace transform ratio order between X and Y.

Theorem 4. $Y \ \leq_{r-Lt-r} \ X$ if and only if $Y_e \leq_{Lt-r} X_e$.

Proof. Let X be a random variable with distribution function F and mean μ, and let Y be a random variable with distribution function G and mean ν, and let $\tilde{F}$ and $\tilde{G}$ be the Laplace transform of F and G, respectively. Then

$$\tilde{F}_e(s) = E(e^{-sX_e}) = \frac{1 - \tilde{F}(s)}{\mu s}, \quad s > 0$$

and

$$\tilde{G}_e(s) \ = \ E(e^{-sY_e}) = \frac{1 - \tilde{G}(s)}{\nu s}, \quad s > 0.$$

Hence

$$\begin{aligned} &\frac{\tilde{F}_e(s)}{\tilde{G}_e(s)} \quad \text{is decreasing in s} > 0 \\ \Longleftrightarrow &\frac{1 - \tilde{F}(s)}{1 - \tilde{G}(s)} \quad \text{is decreasing in } \ \text{s} > 0. \end{aligned}$$

Next, we compare the reverse Laplace transform ratio order and the Laplace transform ratio order for X and Y with the Laplace transform orders for X_L and Y_L.

Let X with mean μ and X_L be the non-negative absolutely continuous random variables with probability density functions $f_X(x)$ and $f_{X_L}(x)$, respectively, and let

$F_X(x)$ and $F_{X_L}(x)$ be the distribrtion functions of X and X_L, respectively. Then we obtain the following results

$$L_X(s) = \int_0^\infty e^{-st}\, dF_X(t) = s\int_0^\infty e^{-st} F_X(t)\, dt,$$
$$L_{X_L}(s) = \int_0^\infty e^{-st}\, dF_{X_L}(t) = -\tfrac{1}{\mu} L_X{}'(s).$$

Moreover, using the above results, we obtain next results

$$\begin{aligned} s^{-1}\int_0^\infty e^{-st}\, dF_{X_L}(t) &= \int_0^\infty e^{-st} F_{X_L}(t)\, dt \\ &= \int_0^\infty e^{-st}\frac{1}{\mu}\left[tF_X(t) - \int_0^t F_X(u)\, du\right] dt \\ &= -\frac{1}{\mu s} L_X{}'(s). \end{aligned}$$

and

$$\begin{aligned} L^*_{X_L}(s) = \int_0^\infty e^{-st}\bar{F}_{X_L}(t)\, dt &= \int_0^\infty e^{-st}(1 - F_{X_L}(t))\, dt \\ &= \frac{1}{s}\left(1 + \frac{1}{\mu} L_X{}'(s)\right). \end{aligned}$$

Similarly, we define $L_Y(s)$, $L_{Y_L}(s)$, and $L^*_{Y_L}(s)$ for the non-negative absolutely continuous random variable Y with mean ν. Thus we obtain the following.

Theorem 5. If $X \leq_{Lt-r} Y$, then $X_L \geq_{Lt} Y_L$.

Proof.

$$\begin{aligned} X \leq_{Lt-r} Y &\iff \frac{L_Y(s)}{L_X(s)} \text{ is decreasing in s} > 0 \\ &\iff L_Y{}'(s)L_X(s) - L_Y(s)L_X{}'(s) \leq 0, \quad s > 0 \\ &\implies \frac{L_Y{}'(s)}{L_X{}'(s)} \leq \frac{L_Y(s)}{L_X(s)} \leq 1, \quad s > 0. \end{aligned}$$

Since $X \leq_{Lt-r} Y$ and hence $\mu = E(X) \leq E(Y) = \nu$,

$$\begin{aligned} &\frac{L_Y{}'(s)}{L_X{}'(s)} \cdot \frac{\mu}{\nu} \leq 1, \quad s > 0 \\ \iff &\frac{L_Y{}'(s)}{\nu} \leq \frac{L_X{}'(s)}{\mu}, \quad s > 0 \\ \iff &X_L \geq_{Lt} Y_L. \end{aligned}$$

Theorem 6. If $X \leq_{r-Lt-r} Y$, then $\mu X_L \leq_{Lt} \nu Y_L$.

Proof.

$$\begin{aligned} X \leq_{r-Lt-r} Y &\iff \frac{1-L_Y(s)}{1-L_X(s)} \text{ is decreasing in } s>0 \\ &\implies \frac{L_Y'(s)}{L_X'(s)} \geq \frac{1-L_Y(s)}{1-L_X(s)} = \frac{\int_0^\infty e^{-st}\bar{F}_Y(t)\,dt}{\int_0^\infty e^{-st}\bar{F}_X(t)\,dt} \geq 1, \quad s>0 \\ &\implies -L_Y'(s) \leq -L_X'(s), \quad s>0. \end{aligned}$$

Therefore $\nu Y_L \geq_{Lt} \mu X_L$.

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