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Optimal Storage Capacity under Random Storage Assignment and Class-based Assignment Storage Policies

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임의 저장 방식과 급별 저장 방식하에서의 최적 저장 규모

이문규

In this paper, we determine the required storage capacity of a unit-load automated storage/retrieval system(AS/RS) under random storage assignment(RAN) and n-class turnover-based storage assignment(CN) policies. For each of the storage policies, an analytic model to determine the optimal storage capacity of the AS/RS is formulated so that the total cost related to storage space and space shortage is minimized while satisfying a desired service level. A closed form of optimal solutions for the RAN policy is derived from the model. For the CN policy, an optimal storage capacity is shown to be determined by applying the existing iterative search algorithm developed for the full turnover-based storage(FULL) policy. Finally, an application of the approach to the standard economic-order-quantity inventory model is provided.

1. Introduction

In this paper, we consider a design issue of a unit-load AS/RS. The required storage capacity for the warehousing system is defined as the amount of storage space needed to accommodate the materials to be stored. The major factors that affect storage sizing include system throughput (alternatively, the reciprocal of the expected time taken to perform an operation) and storage assignment policy. A larger system in storage space will have a lower throughput than a smaller size system if the storage and retrieval behaviors remain the same. Thus, trade-offs between throughput and storage capacity should be considered while designing AS/RSs.

Three kinds of storage assignment policies addressed usually in the literature are RAN, FULL, and CN. Assuming an identical storage size for the three policies, Hausman, Schwarz *et al.*(1976, 1977) derive analytic expressions for the system through- put, and show that significant improvements in throughput are obtainable when FULL and CN are used. An extension to Hausman *et al.*'s study is reported by Ko and Hwang(1992) which takes account of the storage space required for each policy. However, in their study, the storage space is determined without considering the costs involved. Rosenblatt and Roll (1984) presented a search procedure for finding the optimal storage design considering capital investment, space shortage cost, and costs associated with storage policies. Later, for a warehouse in a stochastic environment, the major elements that affect the required storage capacity were examined using a simulation model by the same authors(Rosenblatt and Roll, 1988). Francis *et al.*(1992) presented mathematical models of determining the storage capacities under different storage policies. Other related research on storage sizing includes those done by Mullen(1981) and Bafna(1983) where general procedures that can be used in practice are suggested.

Recently, Lee(1998) presented an approach for determining the required storage capacity under FULL. In the approach, an optimization model is formulated to minimize the total cost related to storage space and space shortage while satisfying a desired service level.

In this paper, Lee's approach is extended to deal with the storage policies, RAN and CN. An application of such an extension to the economic order quantity(EOQ) inventory system is presented.

2. Models for Determining Storage Capacity

If the storage requirement is greater than the storage capacity, a space shortage occurs. Here, we define the shortage probability as the probability of space shortage occurring in the warehouse system. In case of space shortage, the excess space requirement can be met via leased storage space. In this paper, we consider the problem of determining the storage capacity in which the sum of the cost for owning space and that for contracting shortage space is minimized without exceeding a given shortage probability, $\alpha_0 (0 \leq \alpha_0 \leq 1)$.

Let X and X_i , $i=1,\cdots, n$, be random variables which represent the aggregate inventory level of the overall system and the inventory level of item i, respectively. Here, we consider the case where every X_i follows a uniform distribution as follows :

$$X_i \sim U(a_i, b_i), \qquad i=1,\cdots, n.$$

One example of such a case is the system in which the standard EOQ model with a_i being zero is applied to all items stored. The storage capacity at the 1- α service level, $S(\alpha)$, satisfies the following inequality:

$$\Pr(X \leq S(\alpha)) \geq 1 - \alpha$$

Throughout this paper, unless otherwise stated, items are numbered in a decreasing order of b_i values. X is then expressed as a function of X_is and its storage level is obviously influenced by the storage assignment policy used.

We now present a model for determining the economic storage capacity which is large enough to accommodate the incoming full pallet loads of items with a probability being not less than (1- α_0).

2.1 Storage Capacity under RAN

Under RAN, a pallet load of any item is equally likely to be stored in any storage location. Hence, the required storage capacity will be equal to the maximum of aggregate inventory level for all items. However, in real situations, due to the dynamic nature of the replenishment process and retrieval operation of items, it is extremely difficult to exactly perdict the aggregate inventory level. Rosenblatt and Roll(1988) consider the warehousing system operated under a (q, Q) inventory policy where q and Q are the reorder point and the order quantity, respectively. Using a simulation model they show that the capacity required for a service level of 95% is within 1.15 times the average aggregate level.

By the definition of X,

$$X = \sum_{i=1}^{n} X_i$$

and now let

$$Z = (X - \hat{\mu}) / \hat{\sigma}$$

where

$$\hat{\mu} = \sum_{i=1}^{n} (a_i + b_i)/2 \text{ and,}$$

$$\hat{\sigma} = \left(\sum_{i=1}^{n} (b_i - a_i)^2 / 12 \right)^{1/2}.$$

If n is sufficiently large, Z follows approximately the standard normal distribution, $N(0, 1)$, by the Central Limit Theorem. Thus, for a given probability α_0, the storage capacity under RAN can be represented by a function of the unknown variable, α :

$$S_{RAN}(\alpha) = \hat{\mu} + z_\alpha \hat{\sigma}$$

where $\alpha \leq \alpha_0$ and z_α is determined by $\Pr(Z > z_\alpha) = \alpha$ for $0 \leq \alpha \leq 1$. Throughout this paper, we consider that case where $\alpha \leq 0.5$, which may be true for most of warehouses in practice. Since a space shortage occurs when the required inventory level exceeds the storage capacity, the expected amount of space shortage per unit time will be

$E_{RAN}(\alpha)$

= proportion of time in which space shortage occurs* expected amount of shortage space when shortage occurs + proportion of time in which space shortage do not occur* 0

$$= \alpha^* E(X - (\hat{\mu} + z_\alpha \hat{\sigma}) | X \geq \hat{\mu} + z_\alpha \hat{\sigma}) + (1-\alpha)^* 0$$

$$= \int_{\hat{\mu}+z_\alpha\hat{\sigma}}^{\infty} [(X - (\hat{\mu} + z_\alpha \hat{\sigma}))/\sqrt{2\pi}\hat{\sigma}] \exp(-(X-\hat{\mu})^2/2\hat{\sigma}^2) dX$$

$$= \hat{\sigma} \int_{z_\alpha}^{\infty} [(z - z_\alpha) \exp(-z^2/2)/\sqrt{2\pi}] dz$$

$$= \hat{\sigma} [\exp(-z_\alpha^2/2)/\sqrt{2\pi} - \alpha z_\alpha)].$$

Let λ_1 = discount present worth cost per unit storage space owned and operated for a unit period of time and

λ_2 = discount present worth cost per unit space leased or per unit of space shortage for a unit period of time.

Then the total cost per unit time will be

$$TC_{RAN}(\alpha) = \lambda_1 S_{RAN}(\alpha) + \lambda_2 E_{RAN}(\alpha)$$
$$= \lambda_1(\hat{\mu} + z_\alpha \hat{\sigma}) + \lambda_2 \hat{\sigma} r(\alpha)$$

where

$$r(\alpha) = \exp(-z_\alpha^2/2)/(2\pi)^{1/2} - \alpha z_\alpha.$$

Now we want to determine the optimum storage capacity under RAN which minimizes the total cost at the service level of α_0. The problem can be stated as

(P1) Minimize $\lambda_1(\hat{\mu} + z_\alpha \hat{\sigma}) + \lambda_2 \hat{\sigma} r(\alpha)$

subject to

$$\Pr(X \leq S_{RAN}(\alpha)) \geq 1 - \alpha_0 \quad (1)$$
$$0 \leq \alpha \leq 1 \quad (2)$$

Since,

$\Pr(X \leq S_{RAN}(\alpha)) = \Pr(X \leq \hat{\mu} + z_\alpha \hat{\sigma}) = 1 - \alpha$,

the constraints (1) and (2) reduce to $0 \leq \alpha \leq \alpha_0$. Therefore, (P1) can be simply rewritten as an unconstrained optimization problem:

Minimize $0 \leq \alpha \leq \alpha_0$

$$\lambda_1(\hat{\mu} + z_\alpha \hat{\sigma}) + \lambda_2 \hat{\sigma} r(\alpha)$$

Theorem 1.

z_α and $r(\alpha)$ are convex over $0 \leq \alpha \leq 0.5$.

Proof.

Let

$\phi(z) = \exp(-z^2/2)/(2\pi)^{1/2}$, $\xi = z_\alpha$, and

$$\Phi(\xi) = \int_\xi^\infty \phi(z)dz = \alpha.$$

since $\alpha = \Phi(\xi)$, the first and second derivatives of z_α will be respectively

$$z'_\alpha = dz_\alpha/d\alpha = d\xi/d\alpha = (d\Phi(\xi)/d\xi)^{-1}$$
$$= -1/\phi(z_\alpha) < 0 \quad (3)$$

and

$$z''_\alpha = d(-1/\phi(z_\alpha))/d\alpha \equiv \phi'(z_\alpha) z'_\alpha / \phi^2(z_\alpha)$$
$$= z_\alpha / \phi^2(z_\alpha) > 0 \quad (4)$$

for $0 \leq \alpha \leq 0.5$. Consequently z_α is convex.

Next consider the convexity of $r(\alpha)$. Using (3), (4), and $r(\alpha) = \phi(z_\alpha) - \alpha z_\alpha$, we obtain

$$r'(\alpha) = dr(\alpha)/d\alpha = \phi'(z_\alpha) z'_\alpha - z_\alpha - \alpha z'_\alpha$$
$$= -z_\alpha \phi(z_\alpha)(-1/\phi(z_\alpha)) - z_\alpha - \alpha z'_\alpha$$
$$= -\alpha z'_\alpha \geq 0$$

and

$$r''(\alpha) = dr'(\alpha)/d\alpha = -z'_\alpha - \alpha z''_\alpha = r(\alpha)/\phi^2(z_\alpha).$$

Since the amount of space shortage $r(\alpha) \geq 0$ for $0 \leq \alpha \leq 1$, $r''(\alpha) \geq 0$ and hence $r(\alpha)$ is convex. This completes the proof.

It is evident from Theorem 1 that the value of α which minimizes $TC_{RAN}(\alpha)$ can be found by solving the equation

$$\frac{dTC_{RAN}(\alpha)}{d\alpha} = 0$$

Thus,

$$\frac{dTC_{RAN}(\alpha)}{d\alpha} = \lambda_1 \hat{\sigma} z'_\alpha + \lambda_2 \hat{\sigma} r'(\alpha)$$
$$= \hat{\sigma}(\lambda_2 \alpha - \lambda_1)/\phi(z_\alpha) = 0 \quad (5)$$

Solving the equation (5) for α gives $\alpha = \lambda_1/\lambda_2$ Therefore, considering the upper bound of α the optimal solution of (P1), say, α_{RAN}^* is given by $\alpha_{RAN}^* = \min(\lambda, \alpha_0)$ where $\lambda = \lambda_1/\lambda_2$. The corresponding minimum storage capacity will be

$$\lambda_1(\hat{\mu} + z_{\alpha_{Ran}^*} \hat{\sigma}) + \lambda_2 \hat{\sigma} r(\alpha_{RAN}^*)$$

2.2 Storage Capacity under CN

2.2.1 Mathematical Model

In this section, we present a model for the class-based storage policy which appears to be more practical than RAN. Under this policy, items and storage locations are jointly partitioned into a small number of classes based on item turnover distributions and travel times taken by the S/R machine, respectively. Within any class, RAN is assumed to the applied to assign items to storage locations. Suppose that n items are divided into N classes such that class j consists of items $k_{j-1}+1, k_{j-1}+2, \cdots, k_j$, $j=1, \cdots, N$ where $k_0=0$, $k_N=n$, and $N \leq n$.

Denoting by α_j the shortage probability of class j, the storage capacity for each class can be determined by the same technique used for RAN. Hence, the storage capacity for CN will be

$$S_{CN}(\alpha) = \sum_{j=1}^{N} S^j_{CN}(\alpha)$$

$$= \sum_{i=1}^{n}(a_i + b_i)/2 + \sum_{j=1}^{N} z_{\alpha_j}\widehat{\sigma_j}$$

where $\widehat{\sigma_j} = (\sum_{i=k_{j-1}+1}^{k_j} w_i^2/12)^{1/2}$ and $w_i = b_i - a_i$.

Using the similar approach to the one presented in the previous section, we obtain the space shortage for CN as follows:

$$E_{CN}(\alpha) = \sum_{j=1}^{N} \widehat{\sigma_j}(\exp(-z_{\alpha_j}{}^2/2)/(2\pi)^{1/2} - \alpha_j z_{\alpha_j}).$$

Thus, the total cost becomes

$$\begin{aligned} TC_{CN}(\alpha) &= \lambda_1 S_{CN}(\alpha) + \lambda_2 E_{CN}(\alpha) \\ &= \lambda_1(\sum_{i=1}^{n}(a_i + b_i)/2 + \sum_{j=1}^{N} z_{\alpha_j}\widehat{\sigma_j}) \\ &\quad + \lambda_2 \sum_{j=1}^{N} \widehat{\sigma_j}(\exp(-z_{\alpha_j}{}^2/2)/(2\pi)^{1/2} - \alpha_j z_{\alpha_j}) \\ &= \lambda_1 \sum_{i=1}^{n}(a_i + b_i)/2 + \lambda_2 \sum_{j=1}^{N} \widehat{\sigma_j}[(\lambda - \alpha_j)z_{\alpha_j} \\ &\quad + \exp(-z_{\alpha_j}{}^2/2)/(2\pi)^{1/2}] \end{aligned} \quad (6)$$

Since we want to minimize the total cost such that overall shortage probability should not exceed α, the problem, after eliminating the constant terms in (6), can be stated as follows:

(P2) Minimize

$$\sum_{j=1}^{N} \widehat{\sigma_j}[(\lambda - \alpha_j)z_{\alpha_j} + \exp(-z_{\alpha_j}{}^2/2)/(2\pi)^{1/2}$$

subject to

$$\sum_{j=1}^{N}(1-\alpha_j) \geq 1 - \alpha_0$$

$$0 \leq \alpha_j \leq u_\alpha, \qquad \forall j$$

where u_α = a common upper bound of the shortage probabilities.

Notice that if we let $\tau = -\ln(1-\alpha)$, since $z_\alpha = -z_{1-\alpha}$, the variable term in the objective function of (P2) will be

$$\begin{aligned} &(\lambda - \alpha)z_\alpha + \exp(-z_\alpha^2/2)/(2\pi)^{1/2} \\ &= (1 - e^{-\tau} - \lambda)z_{e^{-\tau}} + \exp(-z_{e^{-\tau}}^2/2)/(2\pi)^{1/2}. \end{aligned}$$

The following theorem gives a property of the above expression.

Theorem 2.

Let

$g(\tau) = (1 - e^{-\tau} - \lambda)z_{e^{-\tau}} + \exp(-z_{e^{-\tau}}^2/2)/(2\pi)^{1/2}$.

Then $g(\tau)$ si convex over $0 \leq \tau_j \leq -\ln 0.5$.

Proof

We decompose the function into two distinctive terms:

$$g(\tau) = \lambda g_1(\tau) + g_2(\tau)$$

where

$$g_1(\tau) = -z_{e^{-\tau}} \text{ and}$$

$$g_2(\tau) = (1 - e^{-\tau})z_{e^{-\tau}} + \exp(-z_{e^{-\tau}}^2/2)/(2\pi)^{1/2}.$$

Now, consider the convexity of $g_1(\tau)$ first. The first and second derivatives of $g1(\tau)$ with respect to τ will be

$$\frac{\partial g_1}{\partial \tau} = \frac{\partial z_\alpha}{\partial \alpha}\frac{\partial \alpha}{\partial \tau} \text{ and}$$

$$\frac{\partial^2 g_1}{\partial \tau^2} = \frac{\partial^2 z_\alpha}{\partial \alpha^2}\left(\frac{\partial \alpha}{\partial \tau}\right)^2 + \frac{\partial z_\alpha}{\partial \alpha}\frac{\partial^2 \alpha}{\partial \tau^2}$$

From the proof of Theorem 1, we know that

$$\frac{\partial z_\alpha}{\partial \alpha} = -1/\phi(z_\alpha) < 0 \quad \text{and}$$

$$\frac{\partial^2 z_\alpha}{\partial \alpha^2} = z_\alpha/\phi^2(z_\alpha) > 0.$$

Since $\frac{\partial \alpha}{\partial \tau} = e^{-\tau}$ and $\frac{\partial^2 \alpha}{\partial \tau^2} = -e^{-\tau}$,

$$\frac{\partial g_1}{\partial \tau} < 0 \text{ and } \frac{\partial^2 g_1}{\partial \tau^2} > 0.$$

Consequently, $g_1(\tau)$ is convex.

Next, by definition $g_2(\tau) = r(\alpha)$. Thus, the first and second derivatives of $g_2(\tau)$ will be

$$\frac{\partial g_2}{\partial \tau} = \frac{\partial r(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial \tau} = (\alpha/\phi(z_\alpha))e^{-\tau} > 0, \text{ and}$$

$$\frac{\partial^2 g_2}{\partial \tau^2} = \frac{\partial^2 r(\alpha)}{\partial \alpha^2}\left(\frac{\partial \alpha}{\partial \tau}\right)^2 + \frac{\partial r(\alpha)}{\partial \alpha}\frac{\partial^2 \alpha}{\partial \tau^2},$$

respectively. Through the proof of Theorem 1, we show that

$$\frac{\partial r(\alpha)}{\partial \alpha} = \alpha/\phi(z_\alpha) \text{ and } \frac{\partial^2 r(\alpha)}{\partial \alpha^2} = r(\alpha)/\phi^2(z_\alpha).$$

Therefore,

$$\begin{aligned} \frac{\partial^2 g_2}{\partial \tau^2} &= \frac{r(\alpha)}{\phi^2(z_\alpha)}e^{-2\tau} - \frac{\alpha}{\phi(z_\alpha)}e^{-\tau} \\ &\quad e^{-\tau}[r(\alpha)e^{-\tau} - \alpha\phi(z_\alpha)]/\phi^2(z_\alpha). \end{aligned} \quad (7)$$

Substituting $e^{-\tau} = 1 - \alpha$ into Eq. (7) yields

Table 1. Evaluation of $R(\alpha)$

α	0.0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
$R(\alpha)$	0	0.069	0.108	0.127	0.132	0.126	0.111	0.089	0.062	0.032	0

$$\frac{\partial^2 g_2}{\partial \tau^2} = (1-\alpha)[(1-\alpha)r(\alpha) - \alpha\phi(z_\alpha)]/\phi^2(z_\alpha).$$

Now, we also let $R(\alpha) = (1-\alpha)r(\alpha) - \alpha\phi(z_\alpha)$.
Then, obviously the limits of $R(\alpha)$ are

$$\lim_{\alpha\to 0}[(1-\alpha)r(\alpha) - \alpha\phi(z_\alpha)]$$
$$= \lim_{\alpha\to 0.5}[(1-\alpha)r(\alpha) - \alpha\phi(z_\alpha)] = 0.$$

It seems difficult to show analytically that $R(\alpha)$ is nonnegative for $0 < \alpha < 0.5$. However, since $R(\alpha)$ is a simple function of one variable α, it is very easy to evaluate the function numerically. The numerical results given in <Table 1> show that $R(\alpha) > 0$ for $0 < \alpha < 0.5$. Accordingly,

$$\frac{\partial^2 g_2}{\partial \tau^2} > 0$$

and thus $g_2(\tau)$ are convex. As a result, $g(\tau)$ is convex over $0 \le \tau_j \le -\ln 0.5$.

Following Theorem 2, we can rewrite the problem (P2) as a separable convex program:

(P3) Minimize $\sum_{j=1}^{N} \widehat{\sigma_j}\, g(\tau_j)$

subject to

$$\sum_{j=1}^{N} \tau_j \le -\ln(1-\alpha_0)$$
$$0 \le \tau_j \le u_\tau$$

where $u_\tau = -\ln(1-u_\alpha)$

2.2.2 A Sesarch Procedure for Optimal Solutions

Recently, Lee(1998) suggested a search procedure which generates optimal solutions for the following nonlinear program:

(P4) Minimize $Z = \sum_{i=1}^{n} c_i f(x_i)$ (8)

subject to

$$\sum_{i=1}^{n} x_i \le r_i \qquad (9)$$
$$0 \le x_i \le r_2, \quad \forall i$$

where c_i ($c_i \ge c_j$, i<j), r_1 and r_2 ($r_1 \ge b_2$) are positive constants and $f(x_i)$ is a nonnegative convex function of x_i over $0 \le x_i \le \max(x_0, r_2)$ whose minimum lies at $x_i = x_0 > 0$, $\forall i$.

Also, $f'(x_i)$ is assumed to be nonpositive for $0 \le x_i \le x_0$.

The search procedure which is based on the Fibonacci search method(Jacoby, Kowalik and Pizzo, 1972) is outlined in the following.

Step 0: <Problem Type Check>

If ($r_2 \le \min(x_0, r_1/n)$), then $x_i^* = r_2$ $\forall i$ and stop. If ($x_0 \le r_1/n$ and r_2), then $x_i^* = x_0$ $\forall i$ and stop. Otherwise, proceed to step 1.

Step 1: <Initialization>

Let F_j be the j-th Fibonnacci number which is given by

$$F_j = [((1+\sqrt{5})/2)^{j+1} - (1-\sqrt{5})/2)^{j-1}],$$
$$j = 0, 1, 2, \cdots$$

where $[x]$ means the largest integer which is not greater than x, Denote μ_u^1 and μ_b^1 the initial upper and lower bounds of the Lagrange multiplier for the first constraint (9), μ, which are respectively defined as

$$\mu_b^1 = c_1 f'(0),$$
$$\mu_u^1 = c_n f'(r_2) \text{ if } r_2 \le r_1;$$
$$= 0, \qquad \text{otherwise.}$$

Let J be the smallest integer satisfying $F_J \ge (\mu_u^1 - \mu_b^1)/\varepsilon$ where ε is a required accuracy of the solution.

Step 2: <Initial Solution>

Find the following interior points

$$\mu_1^1 = (F_{J-2}/F_J)(\mu_u^1 - \mu_b^1) + \mu_b^1 \quad \text{and}$$
$$\mu_2^1 = (F_{J-1}/F_J)(\mu_u^1 - \mu_b^1) + \mu_b^1.$$

Determine the row vector

$$\gamma_p^1 = (\gamma_{p1}^1(\mu_p^1), \cdots, \gamma_{pn}^1(\mu_p^1)),\ p=1, 2 \text{ where}$$
$$\gamma_{pi}^1(\mu_p^1) = \min[\max(0, f'^{-1}(\mu_p^1/c_i)), r_2],$$
$$i = 1, \cdots, n \qquad (10)$$

Set the iteration index, $j=1$ and go to step 4.

Step 3: <Iteration J>

Set

$$\mu_1^j = (F_{J-1-j}/F_{J+1-j})(\mu_u^j - \mu_b^j) + \mu_b^j$$

and (for Case 2 and Case 3)/or (for Case 1)

$$\mu_2^j = (F_{J-j}/F_{J+1-j})(\mu_u^j - \mu_b^j) + \mu_b^j.$$

The definition of each case is given in Step 4. Using equation (10), find γ_1^j and/or γ_2^j with $\mu = \mu_p^j$, $p=1,2$, respectively.

Step 4: <Evaluation>

Case 1)

If $\sum_{i=1}^{n}\gamma_{1i}^j(\mu_1^j) < r_1$ and $\sum_{i=1}^{n}\gamma_{2i}^j(\mu_2^j) > r_1$, then set

$$\mu_u^{j+1} = \mu_2^j, \quad \mu_2^{j+1} = \mu_1^j, \quad \mu_b^{j+1} = \mu_b^j$$

or

$$\mu_u^{j+1} = \mu_u^j, \quad \mu_1^{j+1} = \mu_2^j, \quad \mu_b^{j+1} = \mu_1^j$$

Case 2)

If $\sum_{i=1}^{n}\gamma_{1i}^j(\mu_1^j) > r_1$ and $\sum_{i=1}^{n}\gamma_{2i}^j(\mu_2^j) > r_1$, then set $\mu_u^{j+1} = \mu_1^j$ and $\mu_b^{j+1} = \mu_b^j$

Case 3)

If $\sum_{i=1}^{n}\gamma_{1i}^j(\mu_1^j) < r_1$ and $\sum_{i=1}^{n}\gamma_{2i}^j(\mu_2^j) < r_1$, then set $\mu_u^{j+1} = \mu_u^j$ and $\mu_b^{j+1} = \mu_2^j$

Step 5: <Termination Test>

If $\mu_u^{j+1} - \mu_b^{j+1} \le \varepsilon$, then set $\mu^* = (\mu_u^{j+1} - \mu_b^{j+1})/2$,

compute the optimum $\gamma^*(\mu^*)$, and stop; Otherwise, set $j=j+1$ and go to step 3.

Note that the above general procedure guarantees and exact optimum when $r_1/n < x_0$ and $r_1/n < r_2$.

Now, consider our problem (P3). The first derivative of the objective function (8) is obtained by using the chain rule:

$$\begin{aligned}\delta(\tau_j) &= \frac{\partial g(\tau_j)}{\partial \tau_j} = \frac{\lambda z_\alpha + r(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial \tau_j}\\ &= [(\alpha-\lambda)/\phi(z_\alpha)]e^{-\tau_j}\\ &= e^{-\tau_j}(1-e^{-\tau_j}-\lambda)/(\phi(e^{-\tau_j}))\end{aligned} \qquad (11)$$

Note that when $\lambda < 1$,

$$\delta(\tau_j) \le 0 \quad \text{for } \tau_j \le -\ln(1-\lambda)$$
$$\delta(\tau_j) > 0, \ \text{otherwise.}$$

On the other hand, when $\lambda \ge 1$, $\delta(\tau_j) < 0$, $\forall \tau_j$. Therefore it follows from the convexity of $g(\tau_j)$ shown in Theorem 2, when $\lambda < 1$, $g(\tau_j)$ will be nonincreasing over $\tau_j \le -\ln(1-\lambda)$, and nondecreasing over $\tau_j \le -\ln(1-\lambda)$. When $\lambda \ge 1$, $g(\tau_j)$ will be nonincreasing $\forall \tau_j$. By the way, it is easily found from (11) that there is at most only a single value,

$$\tau_j^0 = \tau^0 = -\ln(1-\lambda) \text{ for } \lambda < 1, \quad = \infty, \quad \text{otherwise}$$

of τ_j which satisfies $\delta(\tau_j)=0$ for all j.

From this analysis, we know that the problem (P3) can be solved by the search procedure with the following mapping:

$$c_i = \widehat{\sigma}_i, \quad x_i = \tau_i, \quad f(x_i) = g(\tau_i),$$
$$r_1 = -\ln(1-\alpha_0), \quad r_2 = u_\tau, \quad x^0 = \tau^0, \quad n = N.$$

Finally, it should be noted that since the inverse function $\delta^{-1}(\tau_j)$ can not be expressed in a closed form, we need to use any simple numerical analysis tool to determine $\tau_i = f^{-1}(\mu/c_i)) = \delta^{-1}(\mu/\widehat{\sigma}_i)$ in Step 2 of the procedure.

3. Application to the EOQ Model

In this paper, in order to pave the way to compare computational results for different storage assignment policies, RAN, CN, and FULL, we use the same notation and assumptions for application as in Lee (1998). The storage capacity models developed thus far are applied to the AS/R system in which all items are ordered based on the standard EOQ inventory model. In this case, inventory level is uniform between a_i and b_i such that

$$a_i = 0 \text{ and } b_i = (2\xi d_i)^{1/2}, \quad \forall i$$

where ξ is the ratio of ordering cost to holding cost of item i, which is assumed, for simplicity, to be constant for all items. To represent the demand rate of each item, geometric functions have been often used(Hausman, Schwarz and Graves, 1976; Graves, Hausman and Schwarz, 1977). In this paper, we approximate the demand rate by a discrete geometric probability distribution which is given by:

$$d_i = D_0 f_d(i) = D_0 p(1-p)^{i-1}/(1-(1-p)^n), \quad i = 1, \cdots, n \quad (12)$$

Table 2. Optimal storage capacity and shortage probabilities for different values of λ_2

λ_2	Storage Capacity			Optimal Shortage Probabilities					
	RAN	RAN1	C5	α_{RAN}^*	α_1^*	α_2^*	α_3^*	α_4^*	α_5^*
0.1	649.12	1,014.67	748.31	0.1	0.0455	0.0270	0.0160	0.0095	0.0057
1	649.12	1,014.67	748.32	0.1	0.0448	0.0271	0.0163	0.0098	0.0059
10	649.12	1,014.67	748.76	0.1	0.0381	0.0274	0.0186	0.0121	0.0077
100	691.89	1,014.67	777.77	0.01	0.01	0.01	0.01	0.01	0.01
1000	723.12	1,014.67	837.23	0.001	0.001	0.001	0.001	0.001	0.001

where D_0 and p are the total demand per period measured in full pallet loads and the skewness parameter of the distribution, respectively. Note that $f_d(i)$ in (12) is a truncated geometric probability function. Notice that in this case, the inventory level of item I, X_i, can be considered to follow the uniform distribution, $U(0, b_i)$ where $b_i=[2\xi D_0 p(1-p)^{(i-1)}/(1-(1-p)^n)]^{1/2}$.

Example problems are solved under the following conditions to investigate the effects of change in item demand rate on the solution pattern and the storage capacity:

$n=100$, $D_0=10000$, $\xi=1$, $u_\beta=0.05$, $\alpha=0.1$,
$\lambda_1=1$, $\lambda_2=0.1, 1, 10, 100, 1000$,
$p=0.0075, 0.0448, 0.1088$.

To determine the storage capacity of a warehouse where RAN is used for storage assignment, some companies use a rule of thumb which sets the capacity equal to 85% of that required for FULL with $\alpha=0$(Francis, McGinnis and White, 1992).

The number of different items included in each class for CN are set to be equal in the example problems. The overall results including those obtained by the rule of thumb (here, denoted by RAN1) are summarized in <Table 2> and <Table 3>. In <Table 2>, α_{RAN}* denotes the optimal value of α for RAN and α_i*, $i=1,\cdots,N$, that for class I. From <Table 2>, the following observations are made:

1) In case of CN, higher shortage probabilities are preferentially assigned to highly frequent classes.
2) Since the upper bound of system shortage probability, α_0, is given, α_{RAN}* tends to be equal to λ as the value of λ_2 increases.
3) As the value of λ_2 increases, storage capacities for RAN and CN get bigger, which can be certainly expected in the beginning.
4) The storage capacity for RAN1 came out to be much larger than those for RAN and even C5, which indicates that warehouses designed based on the rule of thumb are too much bigger sized than actually required.

Finally it is observed from <Table 3> that as the skewness of the demand curve increases, the required storage capacities for every storage policy including RAN1 appear to decrease. The reason may be that as p increases higher shortage probabilities are more likely to be assigned to frequent turnover items.

4. Conclusions

In this paper, we consider a storage sizing problem for a unit-load AS/RS under RAN and CN. The objective of the problem is to minimize the overall cost incurred from owning the storage space for the warehouse and that from contracting space outside of the company for shortage space. The problem for each of the storage policy has been formulated as a nonlinear optimization model. Optimal solutions of the model for RAN can be easily obtained by taking advantage of the convexity property for the objective function. The model for CN has been solved by applying the existing search procedure developed for

Table 3. Storage capacity required for different combinations of storage policy and skewness parameter when $\lambda_2 = 1$

p	RAN	RAN1	C2	C3	C4	C5
0.0075	791.51	1,195.06	797.51	830.80	860.58	887.66
0.0448	649.12	1,014.67	681.18	704.20	726.51	748.32
0.1088	467.70	706.24	486.23	494.61	504.03	514.52

FULL by Lee(1998).

Due to the dynamic conditions and statistical dependence among items that typically exist in real situations, it is very difficult to determine exactly the storage requirements. Therefore, where possible, the distribution of the aggregate storage requirement should be developed directly from the historical data. If not, the results obtained using the statistical approach presented here can be used in determining bounds or approximations for the first-cut design of the storage. In addition, since the previous research on storage sizing is very limited, the suggested approach could be a fundamental basis for further studies in this area.

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