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Mathematical and Pedagogical Discussions of the Function Concept

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The evolution of the function concept was delineated in terms of the 17th and 18th Centuries' dependent nature of function, and the 19th and 20th Centuries' arbitrary and univalent nature of function. According to mathematics educators' beliefs about the value of the function concept in school mathematics, certain definitions of the concept tend to be emphasized. This study discusses three types — genetical (dependence), logical (set-theoretical), analogical (machine/equations) — of definition of function and their values.

1. INTRODUCTION

We begin this paper by illustrating several comments secondary pre-service teachers made about the importance of teaching and learning mathematical definitions. These comments are excerpted from the remarks of pre-service teachers who enrolled in the required mathematics methods course "Teaching Secondary School Mathematics" in the Fall Semester 1997 at the University of Michigan (USA).

> Definition is not something that is important to know word by word, but definition should highlight concept what I want my students to take away from the class. [Erin, October 20]

> I think students should know definition so that they can try to decipher what is or is not function. [William, October 20]

> I think that there needs to be some universal terminology so that in going from one class to the next ⋯ like teachers might use a different word for the same thing which really confuses students [Larry, October 15]

> I think that any definition in math ⋯ when I had algebra, we beat the definition of function to death. I mean, if you don't know how to word it exactly right, you failed and I just thought that was so meaningless ⋯ I didn't know what a function was and I couldn't identify what the functions were, but here I am getting A's in math classes at the university level. So it makes me question why is it important to be able to label what a function is ⋯

> I just don't think it's important for a kid to know. [Brown, October 15]

> Because students are not going to have you as a teacher all their life, if they don't learn it with you then they're going to learn something else with another teacher. And maybe the other teacher will introduce something else and they need to know that what you taught them was the same thing as what they're going to expand on with this other teacher [Maya, October 15]

From these statements, three themes emerge:

1) Knowing definition in the form of universal terminology is important,
2) Knowing the underlying concepts of definition is more important than memorizing definition word-by-word, and
3) Knowing definition is important in identifying a function.

Kathy told us that definition is the least important aspect in teaching function because most often pupils do not get much meaning from a definition [Interview, October 27]. But she believed that teacher and pupils should know the definition for the sake of dialogue. She said:

> Because students come from all different places, if you know the definition, that there is some sort of universal aspect to math, to the words of math, to the language, you can express that to your students, no matter where they come from, not necessarily me specifically, but a teacher of math. [Observation, October 15]

Tasha told us that pupils should know the definition of function to understand what the point is and what they are doing — to know a reason, a meaning, and a name for what they are dealing with so that they can say:

> "Oh yeah, this is a function because, you know, it goes along with the definition that we talked about" [Interview: October 27].

She commented that she did not think the definition necessarily had to be the textbook definition:

> I think some form of a definition is helpful. I don't think it necessarily has to be the textbook definition though that has words that the students aren't going to understand and concepts that don't mean anything to them. I think a lot of times textbook definitions are really abstract and they don't help the students understand what the point is. I think the definition that you should give your students is one that should help them understand the concept, not one that's just straight out of the textbook. [Interview, October 27]

As with Kathy, Imani commented that it is important to know definitions just for conversation's sake. She wrote:

> Students should know the definition of mathematical terms in general because not knowing

> the definitions could lead to some pretty unfocused math discussions. Everyone having a feel for the definitions puts everyone on the same page. [Questionnaire, October 20]

Additionally, Imani emphasized students' understanding of the underlying meaning of mathematical terms, definitions, and ideas. She therefore tended to prefer definitions easily understood by the most pupils.

To teach the nature of the concept of function, teachers must posit a definition or

> "A statement conveying fundamental character" (*The American Heritage College Dictionary,* 1993, p. 364).

Many teachers do not perceive the learning and teaching of definition as an exciting activity. But definition is one of the most important concepts in understanding mathematical ideas. Still, "in a large number of cases, the giving of a definition is simply the verbatim repetition of a number of words" (Schultze, 1939, pp. 72-75). Thus, although pupils may know a formal definition of a precise statement, they may not have a clear understanding of the underlying meaning or fundamental character of the definition or concept being defined.

Many teachers tend to believe that definitions should be taught in precise words, and many of them have learned and taught by way of memorizing the precise statement of a given definition. Although definitions need to be worded precisely, they must be taught and learned in ways in which both teacher and pupils can make sense of them. Additionally, teachers can only teach well what they can understand themselves. Some teachers attempt to teach definitions in meaningful ways. For instance, they formulate their definition of function through use of an analogy such as of a machine, black box, or connector. Some teachers may use the definition given by the textbook. Other teachers, however, may chose either others' definitions or formulate their own to assist their pupils' understanding.

In following sections, we delineate the historical development of the function concept and perspectives on the teaching of function in school mathematics, introduce some definitions of function that have been provided in certain secondary mathematics textbooks published in the US between 1905 and 1997. We also discuss three types of definition of function and their value.

2. THE HISTORICAL DEVELOPMENT OF FUNCTION CONCEPT

The function concept was not a new invention of mathematicians of the 17th and 18th Centuries. Rather, the function concept can be traced back to 2000 BC; for example, ancient Babylonians developed tables for finding reciprocals, squares, square roots, cubes, and cube roots (Kline, 1972). General notions of dependence relationships between varying quantities had been expressed by the Middle Ages through the use of geometric terms or verbal descriptions (Youschkevitch, 1976). Mathematicians and scientists such as Galileo, Newton, and Kepler studied physical problems associated with motion during the late 16th and early 17th Centuries (Kline, 1972; Malik, 1980; Youschkevitch, 1976).

The investigation of relationships between varying quantities of natural phenomena and the search for tools to describe and model observed phenomena was fundamental in arriving at the concept of function (Cooney & Wilson, 1993; Kline, 1972).

As can be seen in Table 1.1, most 17th and 18th Century mathematicians defined function as a quantity, operation, formula, expression, or relationship. Nineteenth and early 20th Century mathematicians defined function as rules of correspondence (cf. Table 1.2).

Table 1.1.
Definitions of function – Dependent nature of functions

Year	By Whom	Definition
1665	Newton	Any *relationship* between variables.
1667	Gregory	A *quantity* obtained from other quantities by a succession of algebraic *operations* or by any other operation imaginable.
1673	Leibniz	Any *quantity* varying from point to point of curve.
1697	Bernoulli	*Quantities* formed using algebraic and transcendental *expressions* of variables and of constants.
1714	Leibniz	*Quantities* that *depend on* a variable.
1718	Bernoulli	Function of a certain variable [as] a *quantity* that is composed in some way from that variable and constants.
1748	Euler	*Formula* or *analytic expression* composed in any manner from that variable quantity and numbers or constant quantities representing the relation between variables.
1755	Euler	If x denotes a variable quantity then all the quantities, which *depend on* x in any manner whatever or are determined by it are called its functions. If some *quantities* depend on others in such a way that if the latter are changed the former undergo changes themselves then the former *quantities* are called functions of the latter quantities.
1797	Lagrange	Any *expression* useful for calculation in which these variables enter in any manner whatsoever.
1806	Lagrange	A combination of *operations* that must be performed on known quantities to obtain the values of unknown quantities, and that the latter are properly only the last result of the calculation.

Table 1.2.
Definitions of Function – Arbitrary nature and univalent nature of functions

Year	By Whom	Definition
1829	Dirichlet	y is a function of a variable x, defined on the interval $a < x < b$, if to every value of the variable x in this interval there *corresponds a definite* value y. Also, it is irrelevant in what way this correspondence is established.
1917	Caratherdory	A *rule of correspondence* from a set A to real numbers.
1939	Bourbaki	A *rule of correspondence* between two *sets*.
1939	Bourbaki	Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if, for all x in E, there exists a unique y in F, which is in the given relation with x.
*	Dirichlet/ Bourbaki	Any *correspondence* between two sets which assigns to every element in the *domain* exactly one element in the *range*.

* The end of the first half of the 20th century

The evolution of the function concept can be delineated in terms of (1) the 17th and 18th Centuries' dependent nature of function, and (2) the 19th and 20th Centuries' arbitrary and univalent nature of function. The dependent nature of function is at the origin of the historical evolution of the function concept (Fruedenthal, 1983).

Additionally, the dependent nature of the function concept is one of the important aspects of functional thinking. The geometric image of function for designating a geometric object associated with curves introduced by means of motion was suggested by Torricelli, Descartes, Galileo, Leibniz, Newton, etc., during the 17th Century (Kline, 1972). In 1667 James Gregory defined a function as a quantity obtained from other quantities by a succession of algebraic operations or by any other operation imaginable (Kline, 1972). Gregory's concept of function soon proved too narrow (Kline, 1972).

Newton, from the beginning of his work on the calculus, used the term *fluent* to represent any relationship between variables (Kline, 1972). Newton also wrote about *quantitas correlata* and *quantitas relata* in referring to independent and dependent variables respectively (Youschkevitch, 1976). In 1673 Leibniz used the term *function* to mean any *quantity* varying from point to point of curve such as subtangents and subnormals of a curve (Youschkevitch, 1976). In 1714 Leibniz used the term *function* to mean quantities that depend on a variable (Kline, 1972). John Bernoulli in 1697 spoke of function as *quantities* formed using *algebraic* and *transcendental expressions* of variables and of constants (Kline, 1972).

In 1718 Bernoulli defined function of a certain variable as a quantity that is composed in some way from that variable and constants (Youschkevitch, 1976, p. 60). Euler, a student of Bernoulli, later replaced the term *quantity* with *analytical expression* (Youschkevitch, 1976, p. 61). In 1748 Euler began by defining function as formula or analytic expression composed in any manner from that variable quantity and numbers or constant quantities representing the relation between variables (Kleiner, 1989). The analytic expression involved the four algebraic operations, roots, exponentials, logarithms, trigonometrics, polynomials, power series, derivatives, and integrals. (Kleiner, 1989; Kline, 1972). In 1755, Euler gave the following *dependence* definition of function, in which the terms *formula* and *analytic expression* do not appear:

> If, however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities. This is a very comprehensive notion and comprises in itself all the modes through which one quantity can be determined by others. If, therefore, x denotes a variable quantity then all the quantities which depend on x in any manner whatever or are determined by it are called its functions ··· (Ruthing, 1984, p. 72-73).

In 1797 Lagrange defined a function of one or several variables as any expression useful for calculation in which these variables enter in any manner whatsoever. In 1806 he defined a function as a combination of operations that must be performed on known quantities to obtain the values of unknown quantities, claiming that the latter are properly only the last result of the calculation (Kline, 1972).

Freudenthal (1983, p. 494) stated:

> "The very origin of the function is stating, postulating, producing, and reproducing dependence (or connection) between variables occurring in the physical, social, mental world, that is, in and between these worlds. Particularly important are mathematical variables mutually related or related with the others."

In the early 20th Century, Breslich (1932) considered dependence to be the most important component of the function concept. In addition, Breslich considered recognition of the dependence of one variable quantity on another related variable to be one of the important aspects of functional thinking.

Descartes, Newton and Euler believed that a function could be characterized in terms of (one or more) natural geometric curves — natural continua — in a classical Cartesian plane (Lakoff & Nunez, in press). However, there were pathological cases such as

$$f(x) = \begin{cases} 1 & \text{if x is irrational} \\ 0 & \text{if x is rational} \end{cases} \qquad f(x) = \begin{cases} \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \qquad f(x) = \begin{cases} x\sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

that do not fit the properties defining the prototypical curve (Lakoff and Nunez, in press):

1) It can be generated by the motion of a point,

2) It is continuous,
3) It has a tangent,
4) It has a length,
5) When closed it forms the complete boundary of a region,
6) This region has an area,
7) A curve is not a surface, and
8) It is formed by the intersection of two surfaces.

The 18th Century mathematicians believed that a function has the same analytic expression throughout (Kline, 1972, p. 949):

> Euler and Lagrange allowed functions that have different expressions in different domains. They used the term *continuous* where the same expression held and *discontinuous* at points where the expression changed form (though in the modern sense the entire function could be continuous).

The accepted meaning of function evolved during the 19th Century to include functions that were not necessarily continuous, differentiable, or defined by analytical expressions (Cooney & Wilson, 1993). "Fourier's work on heat conduction and the ensuing debate concerning his writings spurred this evolution" (Cooney & Wilson, 1993, p. 134). When in 1829 Dirichlet studied Fourier's series, he redefined function:

> y is a function of a variable x, defined on the interval $a < x < b$, if to every value of the variable x in this interval there corresponds a definite value y. Also, it is irrelevant in what way this correspondence is established (Luzin, 1940 cited in Kleiner, 1983, p. 291)

Some problems would be difficult to study if mathematicians were to stay with the classical definition of function (Kleiner, 1983). Dirichlet was the first to define function in a way that would accept arbitrary correspondence (Kleiner, 1983).

> "Dirichlet was another prominent mathematician involved in expanding the idea of function to include arbitrary correspondence in addition to those defined by analytical expression." (Cooney & Wilson, 1993, p. 134)

For example, Dirichlet's famous "salt and pepper" function pairing the rational numbers with 0 and the irrational numbers with 1 (Kleiner, 1989; Youschkevitch, 1976).

Dirichlet's definition of function was found at first to be too broad to be the definition of function, though the definition was acceptable to many for being more useful in studying more advanced and modern mathematics such as topology, and metric space (Even, 1989; Malik, 1980).

> "In 1917, Caratheodory defined a function as a rule of correspondence from a set A to real numbers, and in 1939 Bourbaki defined function as a rule of correspondence between two

sets" (Malik, 1980, p. 491).

Bourbaki also proposed the following definition of function in 1939:

> Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if, for all x in E, there exists a unique y in F which is in the given relation with x (Kleiner, 1989, p. 299).

By the mid 20th Century, the modern Dirichlet-Bourbaki definition of function had become the common one in mathematical practice:

> A function is any correspondence between two sets which assigns to every element in the domain exactly one element in the range (Even, 1989).

The term *arbitrary* refers to both the character of the relationships between the two sets on which the function is defined and the sets themselves (Even, 1989). Functions do not have to be represented by a single or any specific expression. Functions also do not have to follow any regularity, nor do they have to be described by a graph with any particular shape. The arbitrary nature of the two sets means that functions do not have to be defined on any specific sets of objects; in particular, the sets do not have to be sets of numbers. The elements of the sets can be numbers, points, curves, coordinates, functions, or permutations (Freudenthal, 1983).

The other feature of the modern definition is univalence (Even, 1989; Freudenthal, 1983). Here are some of the ways univalence can be expressed:

> A function is a relation in which each element of the domain there will be only one element in the range. Each first element is paired with only one second element. If a value of x is given in the domain, there is only one corresponding value of y. For a function represented by a Cartesian graph, any line drawn parallel to the Y-axis crosses the graph of a function at most once. A function is a set of ordered pairs, in which the first member of the pair uniquely determines the second.

> Function is a machine: The operation of the function is the making of a unique output object from each collection of input objects (Lakoff & Nunez, in press).

> Functions are special relations, where a relation from A to B is any subset of the Cartesian product $A \times B$. A relation f is called a function from A to B if for every $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$.

All the above exhibit univalence features. The univalent nature of function is commonly used to help students (Even, 1989; Freudenthal, 1983; Kline, 1972):

1) Identify a function,
2) Construct a function,
3) Keep track of meanings of symbols in the same context, and

4) Keep the process manageable.

Students typically apply the univalence criteria for determining which relations are functions (Even, 1993; May & Van Engen, 1959). In the following section, we delineate perspectives on the teaching of functions in school mathematics.

3. PERSPECTIVES ON THE TEACHING OF FUNCTION IN SCHOOL MATHEMATICS

Many studies (Breslich, 1928; Breslich, 1932; Buck, 1970; Day, 1995; Even, 1989; Even, 1993; Hamley, 1934; Hedrick, 1922; Hight, 1968; Lennes, 1932; Lietzmann, 1932; Lloyd & Wilson, 1998; Lovell, 1971; Malik, 1980; May & Van Engen, 1959; Vinner & Dreyfus, 1989; Willoughby, 1967; Wilson, 1994; Wilson & Shealy, 1995) illustrate the importance of the function concept as well as reasons why certain definitions of function are appropriate in school mathematics.

Since the latter part of the 19th Century and throughout the 20th, mathematics education reform movements have focused attention on the role and importance of the function concept in the mathematics curriculum, in the discipline of mathematics, in its use in fields other than mathematics, and in its use in pupils' daily life. For example, a German Professor Klein addressed the significance of secondary mathematics teachers' attention to the possibility of and need for developing functional thinking in their courses before the International Congress of Mathematicians at its meeting in 1893 (cited in Breslich, 1932).

> "In 1921, the National Committee on Mathematical Requirements of the Mathematical Association of America recommended that the study of functions should be given a central focus in secondary school mathematics" (Cooney & Wilson, 1993, p. 140).

Breslich (1928) and Hamley (1934) pressed for functional thinking to become the unifying principle of school mathematics.

For Breslich (1932), functional thinking consists of the following:

1) Recognizing how a change in one of the related variables affects the values of the others,
2) Recognizing the character of the relationships between variables,
3) Determining the nature of the relationships, and
4) Expressing relationships in algebraic symbols.

Breslich (1928) expressed the view that without functional thinking there can be no real understanding and appreciation of mathematics. To expect real understanding and appreciation of functional thinking and mathematics, Breslich (1928) claimed that the teaching of functional thinking should not be restricted to certain topics, such as equations, graphs, changes in geometric figures, ratios, proportions, and variations. Breslich (1928; 1932) suggested that functional relationships be studied in all mathematical subjects from arithmetic through trigonometry as well as in problems involving pupils' everyday experiences. For example, distance traveled depends on time and rate of travel.

Many laws of physics, such as the laws of uniform falling bodies, are functions, and many business applications (such as interest) are also functions. Breslich (1932) broadened the function concept in this way:

> It is concerned with the relationships, which exist between variables, and with the fact that to a value of one corresponds a definite value of the other (p. 95).

According to mathematics educators' beliefs about the value of the function concept in school mathematics, certain definitions of the concept tend to be emphasized. For example, Lietzmann (1932) stated that Dirichlet's arbitrary nature of function is more appropriate in connecting the function concept with economics, statistics, and graphical representation so useful in students' lives. He also expressed preference for Dirichlet's arbitrary nature of function over Euler's analytical expression definition and claimed that the function concept should not be limited to such special functions as lend themselves to analytical expressions.

Lennes (1932) emphasized the significance of the dependent nature of function in understanding the character of important change in the world in which we live. Even (1993) expressed that the Dirichlet-Bourbaki definition of function helps in understanding current mathematics based on a more modern conception of function.

In contrast, Malik (1980) viewed the modern definition as so static that it could be postponed until the beginning of advanced courses such as topology and analysis at the elementary level because the definition appeals to the discrete faculty of thinking and lacks a feel for smooth change of the variables in phenomena.

May & Van Engen (1959, p. 110) expressed that the ideas of set and univalence are more appropriate than the idea of dependence because they are more understandable, enabling everyone to interpret properly most discussions found in mathematical and scientific writing:

> The definition based on set considerations is precise and clear. A function or relation is a set of ordered pairs. This is a definite entity; one you can almost put your hands on. This being the case, it would seem logical that it be considered as the basis for instruction in elementary mathematics.

In contrast, Willoughby (1967, p. 226) stated:

> The ordered-pair definition of function is correct and convenient to use; however, it has serious defects from a pedagogical point of view. The ordered-pair idea gives a static impression to the pupil, where a dynamic impression is far more appropriate. Even though it may not be as elegant, or as formally simple, a dynamic impression of a function will be far more appealing to children, and will put them in a much better position to use their knowledge about functions.

Buck (1970, p. 255) similarly stated that:

> "Experience seems to show us that the 'a function is a class of ordered pairs' approach is one that imposes severe limitations upon the student and provides a poor preparation for any further work with functions."

Thorpe (1989) points out that the ordered pair definition of function was certainly one of the errors in school mathematics, mentioning that:

> "We should teach the most intuitive and practical definition and not confuse our students with unnecessary abstractions".

Thorpe (1989, p. 13) emphasizes that:

> "A function should be defined as a rule, or perhaps as a certain kind of machine, but certainly not as a set of ordered pairs."

Other researchers (e.g., Breslich, 1928; Day, 1995; Lloyd, 1996; Malik, 1980; Wilson, 1994; Wilson & Shealy, 1995) argue that emphasizing the dependent nature of function will help students understand and interpret behavior, changes, variation and fluctuations of everyday phenomenon as well as the underlying meaning of and connection between mathematical facts or principle

4. DEFINITIONS OF FUNCTION IN SCHOOL MATHEMATICS

We give some definitions of function that have been provided in certain secondary mathematics textbooks published in the US between 1905 and 1997 (see Table 2). We do not claim this list to be completely representative, but the data are sufficient to sensitize the reader to the common definitions of the function concept in secondary school mathematics.

Kennedy and Ragan (1969) found in surveying thirty-five elementary algebra and college algebra textbooks that most textbooks before 1959 used definitions for function that involved rules or correspondences between variables, whereas most of the textbooks from 1959 onward used a definition involving sets of ordered pairs. Cooney and Wilson

(1993, p. 142) stated:

> "Our analysis of sixteen high-school textbooks published between 1958 and 1986 indicated that functions were consistently defined in terms of sets, either as sets of ordered pairs or as correspondences between elements of two sets."

We examined ten high school textbooks published between 1987 and 1997, and of them five define function as sets of ordered pairs, three as correspondence between variables or elements of two sets, and two as dependence relations. Based on sixty-one textbooks (thirty-five of Kennedy & Ragan (1969), sixteen of Cooney & Wilson (1993), and ten of current textbooks), we would state that most commonly function is defined in school mathematics as a set of ordered pairs or as a correspondence between elements of two sets. Table 2 below shows various definitions of function in school curriculum.

Table 2.
Definitions of function in school curriculum

Year	Textbook	Author(s)	Definition of function
1905	Elementary Algebra	Marsh	X
1909	First Course in Algebra	Hawkes, Luby & Touton	An algebraic expression involving one or more letters is a function of the letter or letters involved. Thus $2x + 3$ and $x^2 + 5x - 6$ are functions of one letter, x; $x^2 - 2xy + y^2$ and $x^3 + y^3$ are functions of two letters, x and y. The letters of a function are usually referred to as variables. (p. 259)
1931	Algebra for Today: Second Course	Betz	If two variables, such as x and y, are so related that to each value of x (the independent variable) there corresponds a definite value or set of values of y (the dependent variable), y is called a function of x. (p. 26)
1949	Algebra: Book Two	Welchons & Kricken-berger	If two variables are so related that for any value of one there is a value (or values) of the other, then the second variable is a function of the first variable. (p. 157)
1965	Elementary Functions	SMSG	If with each element of a set A there is associated in some way exactly one element of a set B, then this association is called a function from A to B. (p. 2)
1968	Modern Algebra: Structure and Function	Henderson, Pingry & Klinger	For every (x, y) and $(u, v) \in A \times B$, if when $x = u$, it follows that $y = v$, then $A \times B$ is called a function. (p. 163)

1979	Elementary Algebra	Jacobs	A pairing of two sets of numbers so that to each number in the first set there corresponds exactly one number in the second set. (p. 78)
1983	Algebra 2 and Trigono-metry	Dolciani et al.	A set of ordered pairs in which each first component is paired with exactly one second component. (p. 67)
1984	Algebra I	Foerster	A set of ordered pairs (x, y) for which there is never more than one value of y for any one given value of x. (p. 568)
1984	Algebra and Trigono-metry	Foerster	A relation in which there is exactly one value of the dependent variable for each value of the independent variable in the domain. (p. 35)
1990	Algebra	McConnell et al.	A function is a set of ordered pairs in which each first coordinate appears with exactly one second coordinate. (p. 638)
1992	Algebra 1	Saxon	A relationship between two sets in which: 1. The first set is the domain and the domain is defined. 2. For each member of the domain there is exactly one answer in the second set. (p. 407) A function is a correspondence or mapping between two sets that associates with each element of the first set a unique element of the second set. (p. 408) A function is a set of order pairs such that no two ordered pairs have the same first element and different second elements. (p. 408)
1995	Algebra 1	Cavender & Falgout	A function is a special relation in which every defined x corresponds to one y value. (p. 362) *SIMPLIFIED DEFINITION OF A FUNCTION* *every* x --- *one* y
1996	Foundations of Algebra and Geometry	Seeley & Alcala	In mathematics, we use the word function to indicate that the value of one variable depends on the value of another variable. The value of y in $y = 3x + 2$ depends on the value of x, If we always choose 1 for x, we always get 5 for y. We say that y is a function of x. (p. 622)
1997	Contem-porary Mathematics in Context	Coxford et al.	In mathematical models of situations, the quantities that change are called variables. In many cases, we describe the relation between two variables by saying that one variable is a function of the other, especially if the value of one variable depends on the value of the other. (p. 101)

5. THREE TYPES OF DEFINITION OF FUNCTION

As Table 2 illustrates, function has had many definitions. In this section we describe how some definitions of function are more appropriate for pupils than others.

> "A definition is the designation of the proximate genus and the specific difference." (Old scholastic definition cited in Schultze, 1939, p. 66)

> "A definition is the explaining of a term by means of others which are more easily understood." (De Morgan, cited in Schultze, 1939, p. 66)

> "A definition is a statement conveying fundamental character or a statement of the meaning of a word, phrase, or term." (*The American Heritage College Dictionary*, 1993, p.364)

Definitions have been taught in school mathematics to enable recognition and identification of certain things as such (Schultze, 1939). But "such an identification does not necessarily explain the true nature and the real character of the thing" (Schultze, 1939, p. 73); therefore, "it would not enable us to apply this concept to further work" (Schultze, 1939, p. 74).

Schultze (1939, p. 72-75) stated two aspects (*logical* and *pedagogical*) of definitions.

> The great emphasis put upon the teaching of formal definitions in secondary schools is usually defended, not on account of the importance of knowing these definitions, but on grounds of the "logical training" it is said to give ⋯ In a large number of cases, the giving of a definition is simply the verbatim repetition of a number of words ⋯ Students may know a formal definition of a word, without having a clear notion of its meaning ⋯ Explanations of terms are really more important than definitions, and every new term should be fully explained and its meaning illustrated by concrete example.

By both logical and pedagogical aspects of definitions in teaching for understanding, we assume teachers themselves might need to choose or formulate the definitions for their pupils' understanding. For example, a teacher might use an analogy to explain the concept of function by saying:

> "[A function is] a machine with a little elf inside of it who changes what you input into the machine before he throws it back out of the machine."

We call this type of definition an *analogical* definition. Analogical definitions of function are of two basic types (*expression* and *action*). Expression analogies characterize function as formulas or equations and action analogies as operations or machines. In contrast, a teacher might use logical aspects to define function. One logical definition would state that a function is a correspondence between two sets P and Q in which each element of P corresponds to exactly one element of Q. Another definition would state that a function is a set of ordered pairs (x, y) for which there is never more than one value of y for any one given value of x.

A teacher might also emphasize the genetical aspects of function. One genetical definition would state that a function is a relationship between two variables such that changes in one variable result in changes in the other. Another that a function is a relationship between two variables such that the value of independent variable uniquely determines the value of dependent variable(s). And still another definition would state that a function is a relationship between variables if the value of one variable depends on the value of the other. These definitions we call genetical because they are related to the origin of the function concept.

Teachers should be familiar with and comfortable with all definition types to adapt their presentations to their future pupils' experiences and understandings in various contexts. They should know the advantages and disadvantages of each type of definition. Genetical definitions are usually more applicable than logical or analogical definitions to disciplines such as business, economics, physics, and statistics. They allow pupils to see the connection between the definition and dependent relationships. They also allow pupils to see proving to be of greater practical usefulness in understanding real world events. Analogical definitions are often easier to understand than logical ones. Logical definitions are more appropriate than genetical and analogical definitions in understanding mathematical ideas that are based on modern conceptions of function.

Additionally, a larger number of relations are functions under logical definitions. For example, logical definitions are more appropriate in understanding the functionality of the relationship between the set of counting numbers and the Fibonacci sequence. This is a good example of how some functions are not dependence relationships.

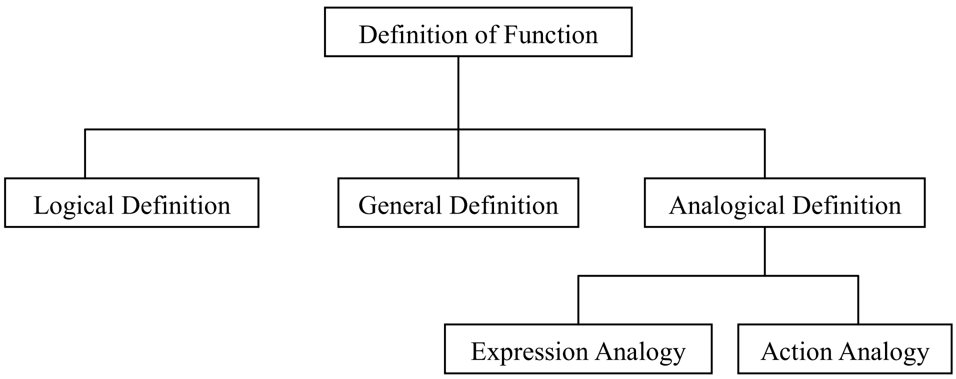

Figure 1. Types of definitions of function.

Table 3.
Types of definitions of function and examples

Logical definitions	Genetical definitions	Action Analogical	Expression Analogical
*A function is a correspondence between two sets P and Q in which each element of P corresponds to exactly one element of Q.	*A function is a relationship between two variables such that changes in one variable result in changes in the other.	*A function is a machine with a little elf inside of it who changes what you input into the machine before he throws it back out of the machine.	*A function is an equation that assigns a value to a variable by using several mathematical properties.
*A function is a set of order pair (x, y) for which there is never more than one value of y for any given value of x.	*A function is a relationship between two variables in which the value of independent variable uniquely determines the value of dependent variable.		
Mapping *Correspondence* *A set of ordered pair* *Rule* *A set of number pair* *A relation of each ordered pair*	*A function is a relationship between variables when the value of one variable depends on the value of the others	*Machine* *Black box* *Operator* *Operation* *Manipulation* *Action* *(Graphing)* *Calculator*	*Formula* *Equation* *Graph* *Vertical line test* *Mathematical statement* *Expression*

Analyses of textbooks indicate that logical definitions of function have been most popular in school mathematics in the past. This fact shows that although many researchers have emphasized the importance of functional thinking and dependence definitions in school mathematics, the emphasis has not reached the secondary classroom. That is, school mathematics have emphasized the importance of “logical training” (Schultze, 1939, p. 73) rather than the importance and meaning of definitions.

We further discuss the advantages and disadvantage of logical, analogical and genetical definitions below.

5.1. Logical Definitions and Their Value

Although logical definitions of the function concept emphasizing sets provide a certain clarity and precision, many mathematics educators question whether this clarity and precision enables pupils to develop a better understanding of function (Cooney & Wilson, 1993). A logical definition of the function concept would have some pedagogical disadvantages. Terms like correspondence used in logical definitions impede pupils' understanding of the definition because of their unfamiliarity with technical terms. Yet the issue of familiarity with technical terms is one of the most essential prerequisites for effective definitions (Schultze, 1939). Teachers need to make sure that all pupils in a class understand the meanings of important terms (Schultze, 1939). Thus the kinds of terms which pupils do not understand and the methods for familiarizing them with technical terms need to be investigated in some future study.

Logical definitions do not allow pupils to connect the concept of function to practical contexts. For example, logical definitions might not be practical for recognizing changes in the surrounding world and identifying the relationships between the changes as tools for dealing with and making sense of them. As a result, the definition would be a hindrance in cultivating the practical values of function. A major goal of reform-based teaching is to provide interesting and useful mathematics to encourage pupils' active involvement in constructing and applying mathematical ideas.

5.2. Analogical Definitions and Their Value

Some teachers might use an analogical definition (e.g., machine, black box, operation, etc.) in addition to a logical definition to help students understand the definition and so assure their interest. One of the purposes of analogy is to liken a less familiar concept to a more familiar one. Analogical definitions have some advantages; however, abuse or misuse of analogical definitions can be harmful for further learning and enriched understanding of the function concept. Some teachers prefer to teach analogical definitions because they bring back good memories and they are easy to understand. Analogies are helpful in explaining and illustrating the concept of function in that they permit pupils to construct ways of leading their minds from the familiar to the unfamiliar (Green, 1971). However, pupils could conceivably be controlled by the analogy itself rather than with use of the analogy as a tool for understanding. For example, one day a student learned that function is a machine with a little elf inside who changes what you input into the machine before he throws it back out of the machine. She liked the definition very much. Later, when she learned a more plausible genetical definition, she could conceivably fail to adopt the genetical definition, possibly not even valuing it. The machine analogy definition might have had some positive aspects and illustrated some of the more important ideas of the function concept; however, the definition illustrates only a part (i.e., univalent nature) of the features of the function concept.

Another problem could be that some pupils might not like the analogical definition (or analogy) provided by their teachers. The definition might even confuse them if they are not familiar with analogy. Therefore, if analogies are to be employed it might be better to let the pupils themselves construct the analogies and formulate their definitions after having been taught about function. Analyzing the definitions, which pupils construct themselves, would helps in understanding whether pupils understand the concept of function. When teaching some idea by using an analogy, the teacher needs to know both the utility and the danger of the analogy in pupils' learning process.

5.3. Genetical Definitions and Their Value

Some mathematics educators (e.g., May & Van Engen, 1959), many pre-service teachers, and some in-service teachers cite disadvantages in employing genetical definitions of function. For example, teachers who like concrete representations or rules might say that genetical definitions are so vague that most pupils cannot easily understand them. Most teachers have themselves been taught the function concept by envisioning concrete objects. For example, many see function as imaginary machine and arrow diagrams used to teach the univalent nature of the function concept. Such teachers might therefore say that the definition is not understandable because it is not easy to represent the definition with concrete objects.

Many mathematics educators have, however, found advantages in genetical definitions of function and their appropriateness in school mathematics. One important value of teaching genetical definitions is the practical use of such definitions in pupils' daily lives as well as in enhancing their functional thinking which is significant component of the mathematical modeling process. Functional thinking is fundamental and useful because it develops pupils' quantitative thinking about the world and analytical thinking about complex situations through examination of the relations between interdependent factors. Students need to analyze functional relationships to explain how a change in one quantity results in a change in another in order to recognize, describe, and generalize the widespread occurrence of regular and chaotic pattern behavior. And then they build mathematical models to predict the behavior of real-world phenomena exhibiting the observed pattern. Therefore, the use of functional relationship is a significant component in the process of mathematical modeling (*mathematizing* unstructured real-world situations). How is this change related to that change? In this context, genetical definitions are more appropriate than logical and analogical definitions.

6. CONCLUSION

When pupils' conceptual understanding, use of knowledge, and establishing connections among mathematical ideas are central goals, a number of studies argue that teachers' enriched and flexible knowledge of a subject is especially important (Borko & Putnam, 1996; Fennema & Franke, 1992). However, a number of studies (e.g., Ball, 1990; Even, 1993; Simon, 1993; Wilson, 1994) show that many teachers do not possess in-depth understanding of their subject. Cha (1999) and Even (1993) show that secondary pre-service teachers have limited understanding of the function concept.

Teachers need to be acquainted with various definitions of the function concept. They also need to possess an understanding of the nature (i.e., dependence, arbitrariness and univalence) of the function concept, and show flexibility in defining the term for their pupils. Hight (1968, p. 579) stated:

> Although the use of the concept has been widespread for over a century, the actual definition of the word "function" has changed in mathematics education during the lifetime of most professional teachers. The transition that has evolved and is presently evolving requires us as teachers to relate various practices and to adapt our presentations to our students' experiences. Because of the importance of the function concept, it is imperative that we have an acquaintance with various definitions and the historical development of common phrases and definitions in addition to an open and searching mind.

A contribution of this study lies in its categorization of various definitions of function into three types (i.e., genetical, logical and analogical) and its discussion of the advantages and disadvantages of each type of definition. These three categories were developed with historical, mathematical, and pedagogical factors in mind. Teachers should be familiar with and comfortable with all definition types. Additionally, they should know the advantages and disadvantages of each type of definition in teaching pupils the concept of function.

Future research would do well to investigate the nature of the advantages and disadvantages not only of each type of definition (logical, genetical and analogical) but also of their respective terms in pupils' *actual learning.* The *actual learning* is across various grades, levels of pupil achievement, and contexts both within and outside mathematics, because that study could give powerful impetus to teachers' pedagogical content preparation. Additionally, the terms that pupils do not understand and the methods of familiarizing them with those terms need to be investigated in future studies.

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