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ON FUZZIFIED REPRESENTATION OF PIAGETIAN REVERSIBLE THINKING

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In this paper, we represent the Piagetian reversible thinking by using the concept of fuzzy complements. In this case the turning point to the reversible thinking can be corre-sponded to the equilibrium points of a fuzzy set, which are shown through some examples. On the other hand, Piaget considered disequilibrium in the theory of equi-libration as a stimulus for students' cognitive structure, furthermore as the driving forces behind intellectual growth. But we suggest a different application of Piagetian equi-librium in another point of view as a tool understanding students' psychological stability in problem solving activity.

1. Introduction

Reversibility is the ability to reverse one's thinking to return to the starting point after performing a mental operation. According to the cognitive development of children and adolescents by Piaget, one of the characteristics of concrete operational stage is ability to use operations that are reversible. As the cognitive structures develop, the children's adaptive responses become more covert and mental, while the dependence on the physical environment of their experience is gradually decreasing. The most important characteristic of mental operation is its reversibility, which refers to the process of reversing a thought.

Since reversibility let us be able to recognize problems in various ways, it is a very important factor in creative problem solving, which is also closely relevant to flexible thinking. However, most of students except the few excellent have many troubles for converting into the opposite direction of thought, thus they suffer hardships in problem solving. To develop students' reversible thinking teachers should provide them with a great deal of practices on a wide variety of problem types containing concepts of inverse operations, the converse of sentences, reciprocal relations and etc. in a daily life context.

The theory of fuzzy sets provides a strict mathematical framework in which vague conceptual phenomena can be precisely and rigorously studied. In operations of fuzzy sets, fuzzy complements are in a complementary relation with fuzzy sets. So, if a certain mental operation or a process of thought is represented by a fuzzy set, we can choose an *optimal* fuzzy complement of the fuzzy set, which corresponds to the reversible thinking.

On the otherhand, Piaget explained the concept of equilibration, which is the innate need for keeping balance between organism and its environment, as the driving forces behind intellectual growth. If imbalance occurs between them, the organism tries to do whatever is necessarily in order to recover balance, so that it has motivational properties for learning. Hence equilibrium means the balanced state between organism and its environment, but disequilibrium means the imbalanced state between them.

Even though disequilibrium stimulates student's cognitive structure, many students could feel comfort and ease in the state of Piaget's equilibrium. In problem solving, emotional factors might also contribute to blocks (Slavin 1979), so that it is very important to provide students a relaxed and comfortable atmosphere as possible. Therefore, if we change our viewpoint to Piagetian equilibrium theory into the viewpoint for investigating students' mental state, Piagetian theory could make classrooms better learning atmosphere.

The fuzzy set is a good mathematical tool for modeling of human mind so that equilibrium points of a fuzzy set for an appropriate fuzzy complement indicate the balanced state of emotions. Thus fuzzy sets and fuzzy complements could be used to understand students' psychological state.

In this paper, first we are going to represent the Piagetian reversible thinking using the concept of fuzzy sets and fuzzy complements so that the reversible points coincides with the equilibrium points of some fuzzy sets. Next, we are going to match Piaget's equilibrium to the concept of equilibria of fuzzy complements.

2. Fuzzy Complement and Equilibrium

Fuzzy sets refer to not only *crisp* sets but also objects with imprecise boundaries such as sweet apples, tall men, real numbers close to 10, reliable friends, diligent students, etc. by *membership degrees*. Membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of the degree of membership.

Definition 1. A *fuzzy set* A in a set X is a set defined by the *membership function*

$$\mu_A : X \to [0,1]$$

assigning to each element of X a value representing its grade of membership in the fuzzy set A. For convenience, the fuzzy set A is identified with the function μ_A.

Let A be a fuzzy set in a set X. Then, by definition, $A(x) = \mu_A(x)$ is interpreted as the degree to which x belongs to A. If A is a *crisp* set in X, then its membership function is the characteristic function χ_A of the set A.

We introduce the definition of a fuzzy complement and its examples.

Definition 2. Let A be a fuzzy set in a set X. If cA is a fuzzy set in X such that $cA(x) = \mu_{cA}(x)$ denotes the degree to which x does not belong to A, then cA is called a *fuzzy complement* of A of type c.

Example 1. A fuzzy complement defined by $cA(x) = 1 - A(x)$ for all $x \in X$ is called the *classical* (*or standard*) fuzzy complement.

If A is a *crisp* set in X, then the classical fuzzy complement c of A is given to be $cA(x) = 1 - \chi_A(x)$ for each $x \in X$. It coincides with the characteristic function χ_{A^c} of A^c since $\chi_{A^c}(x) = 1 - \chi_A(x)$, where A^c is the *crisp* complement of A. Therefore the classical fuzzy complement is an extension of the complement of *crisp* sets.

As a notational convention, let the complement cA of A of type c be defined by a function

$$c : [0,1] \to [0,1],$$

which assigns a value $c(A(x))$ to each membership grade $A(x)$ of any given fuzzy set A, that is, $c(A(x)) = cA(x)$ for all $x \in X$.

This notation makes us be able to give a *global* definition of a fuzzy complement c for every fuzzy set in X.

Definition 3. A function $c : [0,1] \to [0,1]$ is called a *fuzzy complement* if it satisfies the following conditions;

(i) $c(0) = 1$ and $c(1) = 0$,

(ii) for all a, $b \in [0,1]$, if $a \leq b$, then $c(a) \geq c(b)$

Example 2. Let $t \in [0,1)$. A function $c_t : [0,1] \to [0,1]$ defined by

$$c_t(a) = \begin{cases} 1 & \text{for } a \leq t \\ 0 & \text{for } a > t, \end{cases} \tag{1}$$

where $a \in [0,1]$, is a fuzzy complement.

Theorem 1. *If a function $c : [0,1] \to [0,1]$ satisfies the following conditions;*

(i) *for all a, b in $[0,1]$, if $a \leq b$, then $c(a) \geq c(b)$* (decreasing property), *and*

(ii) *$c(c(a)) = a$ for each $a \in [0,1]$* (involutive property),

then c is a continuous and bijective function such that $c(0) = 1$ and $c(1) = 0$.

Example 3. For each $\lambda > -1$, the function $c_\lambda : [0,1] \to [0,1]$ defined by

$$c_\lambda(a) = \frac{1-a}{1+\lambda a} \tag{2}$$

where $a \in [0,1]$ is an *involutive* fuzzy complement of the *Sugeno class.* For each value of parameter $\lambda > -1$, we obtain a particular fuzzy complement c_λ (see Figure 1). If $\lambda = 0$, we obtain the classical fuzzy complement (cf. Example 1).

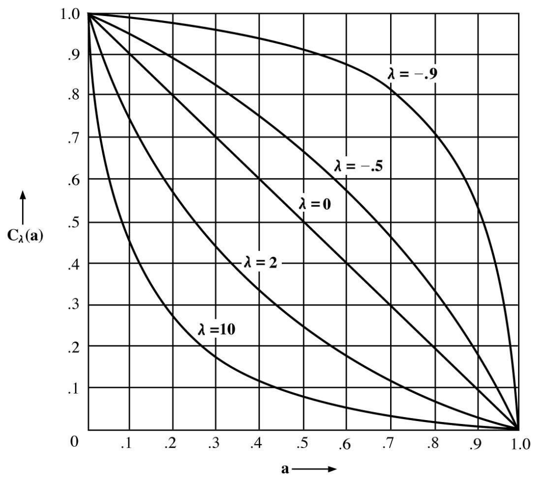

Figure 1. $c_\lambda(a) = (1-a)/(1+\lambda a)$

An *involutive* fuzzy complement plays the important role in practical applications and its inverse is itself. In Example 2, the fuzzy complement c_t for each $t \in [0,1)$ does not satisfy the involutive property. The function c_t is neither continuous nor bijective.

Perhaps the most important property of involutive complements is expressed by the following two theorems. By either of these theorems, we can generate an unlimited number of fuzzy complements or classes of fuzzy complements such as the Sugeno class.

Theorem 2. (Klir & Yuan 1995). *Let $c : [0,1] \to [0,1]$ be a function. Then, c is an involutive fuzzy complement if and only if there exists a continuous function $g : [0,1] \to \mathbb{R}$ such that $g(0) = 0$, g is strictly increasing, and*

$$c(a) = g^{-1}(g(1) - g(a)) \tag{3}$$

for all $a \in [0,1]$.

The function g in Theorem 2 is called an *increasing generator*. For the classical fuzzy complement, the increasing generator is the identity function $g(a) = a$. For fuzzy complements of the Sugeno class (see Example 3), the increasing generators are

$$g_\lambda(a) = (1/\lambda)\ln(1 + \lambda a)$$

for $\lambda > -1$.

Theorem 3. *Let $c : [0,1] \to [0,1]$ be a function. Then, c is an involutive fuzzy complement if and only if there exists a continuous function $f : [0,1] \to \mathbb{R}$ such that $f(1) = 0$, f is strictly decreasing, and*

$$c(a) = f^{-1}(f(0) - f(a))$$

for all $a \in [0,1]$.

The function f defined in Theorem 3 is called a *decreasing generator*.

Definition 4. Let X be a set, A a fuzzy set in X, and $c : [0,1] \to [0,1]$ a fuzzy complement. If $c(a_0) = a_0$ for some $a_0 \in [0,1]$, then a_0 is called the *equilibrium* (or *equilibrium value*) of the fuzzy complement c. Futhermore $x_0 \in X$ is called the *equilibrium point* of A for c if $A(x_0) = a_0$.

The equilibrium value a_0 of a fuzzy complement c is the degree of membership in a fuzzy set A which is equal to the degree of membership in the complement cA. Hence a_0 is the intersection of $y = c(a)$ and $y = a$. Also the equilibrium point x_0 of A for c is the intersection of $y = A(x)$ and $y = cA(x)$ or the intersection of $y = A(x)$ and $y = a_0$.

Example 4. The classical fuzzy complement c has $\frac{1}{2}$ as an equilibrium value since it is the solution of the equation $1 - a = a$ (see Example 1).

If A is a fuzzy set in $\mathbb{R}$ defined by

$$A(x) = \begin{cases} \cos x & \text{when } -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2} \\ 0 & \text{otherwise,} \end{cases}$$

then $-\pi/3$ and $\pi/3$ are equilibrium points of A for the classical complement c (see Figure 2).

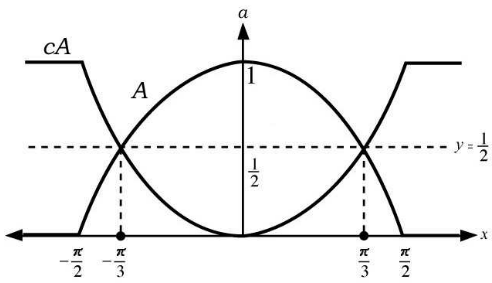

Figure 2. Equilibrium points of A for c

Example 5. The equilibrium value e_{c_λ} for each individual fuzzy complement c_λ of the Sugeno class, defined by equation (2) of Example 3, is given by

$$e_{c_\lambda} = \begin{cases} \dfrac{(1+\lambda)^{\frac{1}{2}} - 1}{\lambda} & \text{for } \lambda \neq 0 \\ 1/2 & \text{for } \lambda = 0 \end{cases} \tag{4}$$

where $\lambda > -1$ (see Figure 3).

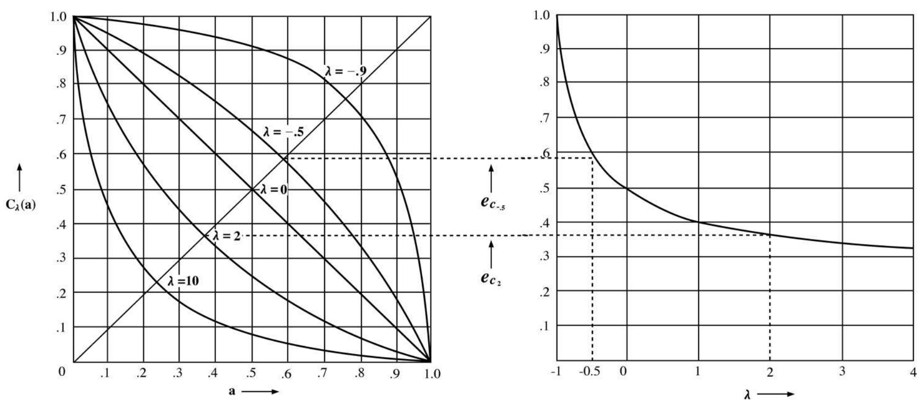

Figure 3. Equilibria for fuzzy complements of the Sugeno class

In Example 2, the function c does not have an equilibrium value and thus there exist no equilibrium points for c.

3. Reversible Thinking and Fuzzy Complements

Many phenomena in daily life can be described by fuzzy sets. From Theorems 2 and 3, we know that decreasing or increasing functions generate involutive fuzzy complements, so that many kinds of situations and their reverse situations are possible to be explained by fuzzy sets $A : X \to [0, 1]$ and fuzzy complements $c : [0, 1] \to [0, 1]$.

The concept of fuzzy complements is a useful tool to explain the reversible thinking. As in the left column of Figure 4, suppose that a certain mental operation (or a situation) is represented by a continuous increasing function $g : [0, 1] \to \mathbb{R}$. Then, by Theorem 2, g generates an *involutive* fuzzy complement c. Since c is involutive, the inverse c^{-1} is the same as c.

Since every involutive fuzzy complement is continuous and bijective by Theorem 1, we can apply the intermediate value theorem to the function $y = c(a)$ on $[0, 1]$. The graph of $y = c(a)$ intersects the identity function $y = a$ at only one point a_0, which is the equilibrium value of c (see Figure 3). From equation (3) of Theorem 2, its reversible point is the point $(a_0, \frac{1}{2}g(1))$, where the reverible point means the turning point to the reverse direction of thought.

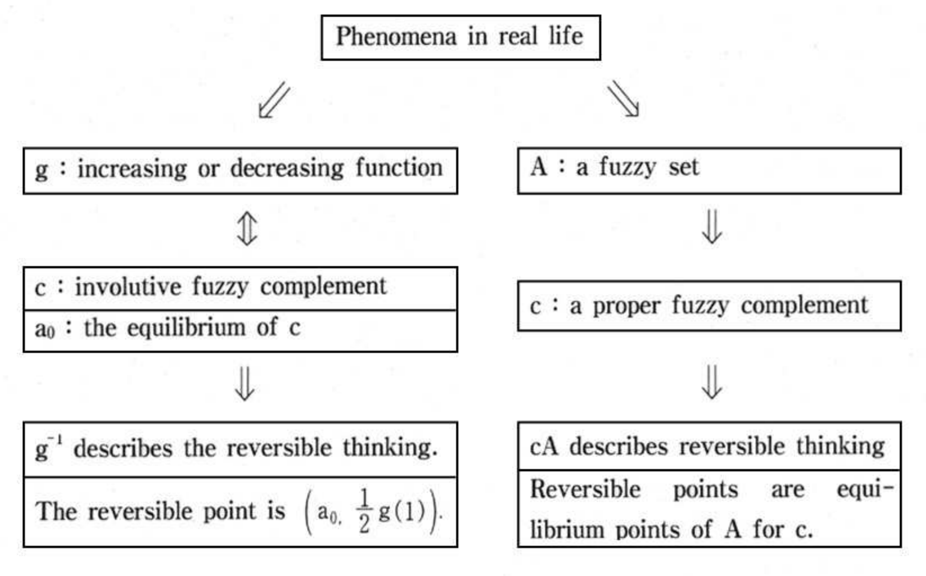

Figure 4. Phenomena in real life

As in the right column of Figure 4, suppose that the same phenomenon is represented by a fuzzy set A. If we choose an appropriate *involutive* fuzzy complement c

for A so that its reversible thinking can be expressed by the fuzzy complement cA, then the reversible point is the intersection of $y = A(x)$ and $y = cA(x)$. It coincides with the equilibrium point of A for c.

Hence the concepts of equilibrium of fuzzy complements and equilibrium points of fuzzy sets determine its reversible point. Actually, since most of useful fuzzy complements are involutive and there exist infinitely many kinds of them, they are capable to provide very important tools in practical applications.

Examples 6 and 7 show that reversible points in many situations of daily life coincide with the equilibrium points of fuzzy complement cA of A of type c, provided with an appropriate fuzzy set A and an appropriate fuzzy complement c.

Example 6. Let us represent our lifetime with a fuzzy set A. If we regard the period of our lifetime as 1, then birth and death are assigned to the values 0 and 1, respectively. The reversible point of our lifetime in this world may be one point of our middle age, which is described to be $\frac{1}{2}$ young and $\frac{1}{2}$ old. Since the fuzzy complement cA of A has to represent the degree of nearness to the other world, the classical fuzzy complement c could explain this situation more pertinently than any other. Thus, the reversible point of our lifetime coincides with the equilibrium point of A for the classical fuzzy complement c introduced in Example 1.

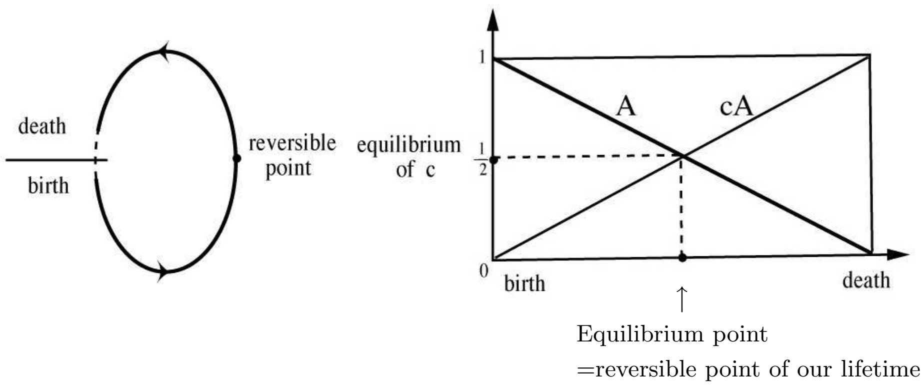

Figure 5. Equilibrium of our lifetime

Example 7. Let us describe a reversible point from hatred to love by the concepts of equilibrium of a fuzzy set.

Let X represent some fixed period of one person P who sometime loved and hated a specified person Q. Let the fuzzy set A be the degree of love of P towards a specified person Q in xy-plane and the fuzzy set B the degree of hatred emotion of P towards Q in xz-plane in Figure 6. Hence the 3-dimensional curve $(x, A(x), B(x))$, where $x \in X$, in xyz-coordinates represent the degrees of love and hatred emotion of P towards Q. Then the relation between hatred and love of P towards Q is illustrated by the graph $z = c(y)$ of the function such that $B = CA$.

In this situation, an appropriate fuzzy complement c could be found in the Sugeno class, say $c_{-0.7}$. Then turning points of emotion from hatred to love are determined by the equilibrium of $c_{-0.7}$ and reversible points of the period X between love and hatred for Q coincide with equilibrium points of A and B for $c_{-0.7}$, hence the equilibrium value of $c_{-0.7}$ is $\frac{1-\sqrt{0.3}}{0.7}$ as shown in the Example 5.

Actually the hatred can be easily transformed into the love if they understand each other, so that it may be analyzed to be a kind of deformed love. Thus the reversible point from hatred to love could be interpreted a mutual understanding.

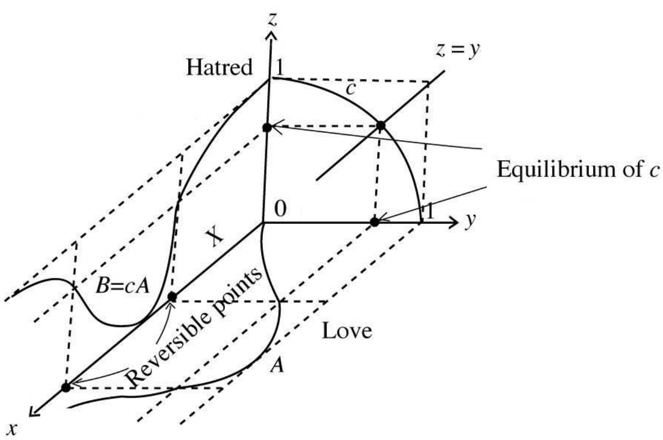

Figure 6. Reversible points of love and hatred

4. Equilibration and Equilibrium of Fuzzy Sets

Piaget believed that all children are born with an innate tendency to interact with and make sense of their environments. In other words, all organisms have an innate tendency to keep a harmonious relationship between cognitive structures and their physical environment, so that their mental structures change in order to incorporate new information or experiences. Since this accommodation modifies cognitive structures, it provides a major vehicle for intellectual development.

Once accommodation occurs, there is a return to equilibrium and this important mechanism of cognitive change is called *equilibration*. Many cycles of equilibrium, disequilibrium, followed by equilibrium at a higher level of competence, provide for slow but steady intellectual growth at all levels of intellectual development, passing through the dual processes of assimilation and accommodation (McCormic & Pressley 1997).

For optimal learning to take place, informations and experiences must be presented so that they can be assimilated into the present cognitive structure but at the same time be different enough to necessitate a change in that structure. Since every experience which a person has involves both assimilation and accommodation, it is very important to let students confront with experiences that do not fit into their current theories of how the world works as a means of advancing their cognitive development.

Piaget regarded disequilibrium in the learning theory as a stimulus for students' cognitive structure (see Figure 7).

Now, let us pay attention to the concept of equilibrium in the equilibration theory. In the learning practice, emotional factors of students have an influence on their learning activity, and so teachers should provide them a relaxed, comfortable learning atmosphere as possible. The more we understand their psychological state, the more we could help their learning.

Most of students feel psychological comfort and peace in the state of balance between their cognitive structure and physical environment. If a specified situation (related to mental state) can be represented by a statement A or a kind of categories, many of them feel mental stability in the condition that 'A and not A', rather than 'A or not A'.

Actually, Piaget's equilibria mainly occur in the neutral state or in the state 'A and not A'. However, since two-valued logic follows the law of excluded the middle that 'A or not A' and a *crisp* set cannot have equilibrium points, it cannot give satisfactory explanation for their mentality and psychological equilibria. Moreover, human mind is essentially vague and it is difficult to characterize someone's personality by definite phrase. By the way, a fuzzy set allows various degrees of membership for the elements of a given set, so that it can be effective tools to explain a certain psychological state as well as Piagetian equilibrium.

In Example 6, we described our life time to be a fuzzy set. But let us consider our life time on the other point of view. In Buddhism, it is a well-known fact that

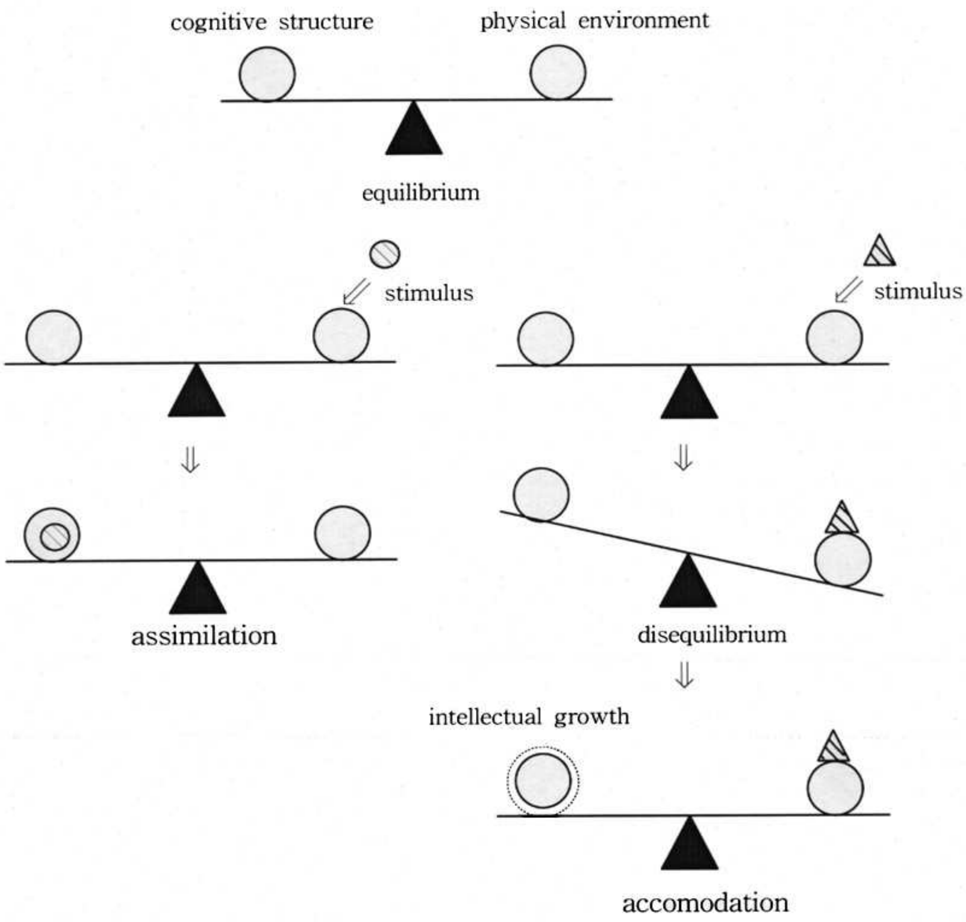

Figure 7. Piagetian equilibration

every organism transmigrates several lives and the death in this world means the birth in the next world. Also we often use the expression 'every human returns to a handful of dust' in the sense that 'a human is mortal'. That is, we unconsciously think death to be a returning point from this world into the next world, a starting point in the cycles of lives. This viewpoint makes us meditate on the real meaning of our life. In this case, the death is a reversible point in this reincarnation.

In Example 7, we mentioned that the emotion love is changable easily into hartred. But both of them, love and hatred, are apt to consume away mutual mental energy, so that an equilibrium in this situation can be said to be each other's thoughtful consideration such as friendship, rather than passionate emotion.

Therefore, we suggest the more active application to Piagetian equilibrium in understanding students of classroom and also insist that a fuzzy set is an effective mathematical tool to explain this concept.

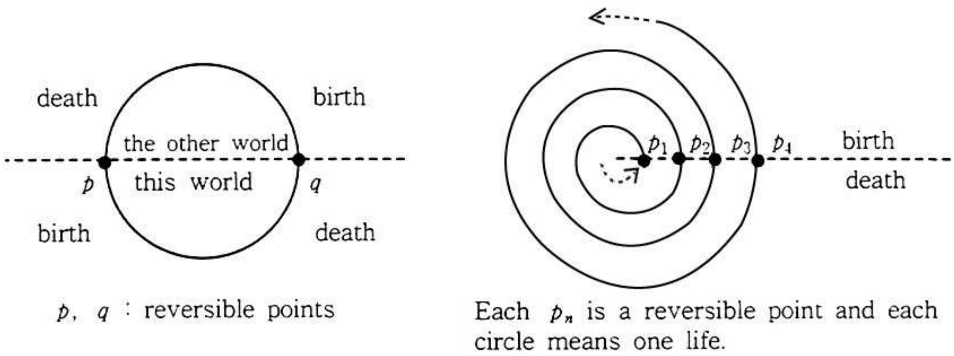

Figure 8. Reincarnation in Buddhism

5. COUNSEL IN MATHEMATICS EDUCATION

According to the fallibilist, mathematics is the outcome of social process and process of human activity relative to culture, values and education (Ernest 1995). Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection (Couran & Robbins 1978).

Also, schools should be a public service that educates students to be critical citizens who can think, challenge and take risks (Skovsmore 1994). That is, public schools should become places that provide the opportunity for students to share their experiences, to work in social relations, and to be introduced to form of knowledge that provide them with the conviction and opportunity to fight for a quality of life in which all human beings benefit.

Therefore, mathematics education must be conducted for students and should help the learners to be independent exercisers of critical judgement. Thus learning should not be a passive receiving or absorption of ready-made knowledge but a process of construction (Anthony 1996) in which the students themselves have to be the primary actors (von Graserfeld 1995).

The difficulty of most application problems in mathematics lies not in the computations, but in knowing how to set the problem up so that it can be solved (Bralsfold

& Stein 1993). In order that mathematics concepts are transformed into mental operations of students in mathematics classroom, students have to be able to manage given informations and the previous knowledge reversibly and combinatorially.

Teachers should admit students the usage of natural language for their mathematical formatting and empirical interpretations in the mathematics classroom. To do that, it is desirable to teach mathematics which is consistent with both natural language and their emotion in the classroom. Since traditional mathematical methods based on the Aristotelian two-valued logic divide the world of yes and no, white and black, true and false, they could not easily describe reality and complex phenomena which do not subject to the two-valued logic. To overcome these inconsistency, current mathematics should now be able to deal with many kinds of sentences occurring in human activities within mathematical category.

Since fuzzy sets and fuzzy logic are deal with objects that are matter-of-degree, with all possible grades of truth between "yes" and "no", they are powerful mathematical tools with multi-valued logic for modeling and controling uncertain systems in industry, humanity and nature (Bojadziev & Bojadziev 1995). They facilitate approximate reasoning with imprecise propositions dealing with natural language and so can be also considered as a modeling language well suited for situations in which fuzzy relation, criteria, and phenomena exist (Zimmermann 1991).

We represented the reversible thinking by using the concepts of fuzzy sets and fuzzy complements. Reversible thinking means not only retracing one's step of previous thinking performance but also a reconstruction of thought that change its direction reversely, freely and rapidly. The latter reversibility is said to be a type of reasoning process since it is relevant only to the reverse direction not previous process of thought. This rather progressive reversibility may abbreviate the middle course of original thought process and not follow the order of previous performed procedure.

If a certain operation or thinking is possible to be described by a fuzzy set, then its reverse or reversible thinking can be represented by an appropriate fuzzy complement. As in Example 6, a simple reversibility is possible to be described by a classical fuzzy complement, an extension of the ordinary complement for crisp sets. For rather progressive reversibility, we can choose an optimal one for each situation since there are infinitely many types of fuzzy complements.

Also, we showed that the concept of equilibria of fuzzy complements can be used in representing the psychological equilibrium of students mathematically. It is expected that more active application of Piagetian equilibrium in a different viewpoint from the previous would develop effective methods for investigating psychological state of them.

As seen in the above, fuzzy sets and fuzzy complements could be useful tools not only to explain reversible thinking but also to describe psychological state of human mind. Thus acceptance and introduction of the concepts of fuzzy sets and fuzzy logic in the practice mathematics education are needed to provide better learning atmosphere to students.

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