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THE DOMINATION NUMBER OF AN ORIENTED TREE

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ABSTRACT. We study the relations among the domination number, the independent domination number, and the independence number of an oriented tree and establish their bounds. We also do the same for a binary tree.

1. Introduction

Let D be a digraph. A subset S of vertices of D is a *dominating set* of D if for each vertex v not in S there exists a vertex u in S such that (u, v) is an arc of D. Note that the set of all vertices of D is a dominating set of D. A dominating set of D with the smallest cardinality is called a *minimum dominating set* of D and its cardinality is the *domination number* of D. We will reserve $\alpha(D)$ for the domination number of D. A subset I of vertices of D is an *independent set* of D if no two vertices of I are joined by an arc in D. The *independence number* $\beta(D)$ of D is the number of vertices in any largest independent subset of vertices in D. An *independent dominating set* of D is an independent and dominating set of D. The *independent domination number* $\alpha'(D)$ of D is the number of vertices in any smallest independent dominating subset of vertices in D. For definitions and notation not given here see [1].

An *oriented tree* is a tree in which each edge is assigned a unique direction and an *oriented forest* is defined analogously.

A *binary* (*search*) *tree* is an oriented tree which enjoys the following properties (see [2]):

(1) There is a unique vertex v_0 (called the root) such that for any

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vertex v distinct from v_0 there is one and only one path starting at v_0 and ending at v.

(2) For each vertex v the number of arcs beginning with v is zero or two. In the former case v is called a *leaf* while in the latter case it is called an *interior vertex.*

(3) The set of arcs is partitioned into two sets L and R (the left and right arcs, respectively). For each interior vertex there is precisely one left arc and one right arc starting with this vertex.

Equivalently (see [3]), a binary (search) tree may be defined as an oriented rooted tree that consists either of a single vertex or is constructed from an ordered pair of smaller binary trees by joining their roots from a new vertex that serves as the root in the tree thus formed. The vertices are not labeled, although the root is distinguished from the remaining vertices, and two such trees are regarded as the same if and only if they have the same ordered pair of branches with respect to their roots. Notice that every vertex is incident with either zero or two arcs that lead away from the root; this fact implies that such trees must have an odd number of vertices.

Let T be a binary tree on $2n+1$ vertices. Then T has n interior vertices and $n+1$ leaves. Let I_0, I_1, I_2 be the sets of interior vertices with zero leaves, only one leaf, two leaves, respectively. It is of interest to observe that $|I_2| = |I_0|+1$ since $|I_0|+|I_1|+|I_2| = n$ and $|I_1|+2|I_2| = n+1$.

Let T be a binary tree. The *level number* of a vertex v in T is the length of the unique path from the root to v in T and the *height* of T is the maximum of the level numbers of the vertices of T. A binary tree of height h is *balanced* if every leaf has distance h or $h-1$ from the root, while it is *fully balanced* if every leaf has distance h from the root.

In section 2 we show that

$$1 \leq \alpha(T) \leq \alpha'(T) \leq \beta(T) \leq n-1,$$

$$\beta(T) \geq n/2$$

for any oriented tree T of order n. In section 3 we show that

$$\alpha(T) \leq \alpha'(T) \leq \beta(T),$$
$$\left\lceil \frac{2n+1}{3} \right\rceil \leq \alpha(T) \leq n,$$
$$n+1 \leq \beta(T) \leq \left\lfloor \frac{2(2n+1)+1}{3} \right\rfloor$$

for any binary tree T on $2n+1$ vertices.

2. Oriented trees

In this section we study the relations among the domination number, the independent domination number, and the independence number of an oriented tree and establish their bounds.

It is easy to see that a 3-cycle has no independent dominating set and a 4-cycle has two independent dominating sets. But J. von Neumann and O. Morgenstern showed [4] that every digraph without cycles has a unique independent dominating set, and M. Richardson showed [5] that every digraph without odd cycles has an independent dominating set. The proofs were long and involved. However, for oriented forests (and hence oriented trees), we have the following short algorithmic proof.

THEOREM 1. *Every oriented tree T has a unique independent dominating set.*

Proof. It is sufficient to prove this theorem for oriented forests and so we shall state an algorithm which finds an independent dominating set for an oriented forest T. The algorithm begins by putting vertices with indegree zero into an independent dominating set. Next we remove the vertices that are already in the independent dominating set together with their out-neighbors to get a new oriented forest and repeat this process for the new oriented forest.

Algorithm: Let $T_1 = T$ be the given oriented forest and let $K_0 = \emptyset$. Put $i = 1$ and go to (1).

(1) Choose the set S_i of all vertices with indegree zero in the oriented forest T_i and let $K_i = K_{i-1} \cup S_i$.

(2) Let T_{i+1} be the oriented subforest of T_i induced by $V - N^+[K_i]$, where $N^+[K_i]$ denotes the union of the out-neighbors of K_i and K_i itself. If T_{i+1} is an empty digraph, let $K = K_i$ and stop. Otherwise, return to (1) putting $i = i + 1$.

Let T' be an oriented tree with n vertices. Then the average indegree of T' is

$$\Big(\sum_{v \in T'} indeg(v)\Big)/n = \frac{n-1}{n} < 1.$$

Thus there is a vertex v of T' with indegree zero. This implies that the algorithm terminates after finitely many steps.

First we want to prove that K is an independent dominating set of T. It is obvious that K is a dominating set of T. To show that K is an independent set, we let u and v be in K. Assume there is an arc between u and v, say, uv in T. Then, by (1), u and v cannot be chosen for K in the same step. If u were chosen for K in an earlier step than the step in which v was chosen, then v would not be in K. Therefore v must be chosen for K in an earlier step i than the step in which u is chosen for K. For this, u should have been deleted in an earlier step than step i. Thus u is not in K, which contradicts the fact that u is in K.

Next we want to show that T has a unique independent dominating set. Suppose that T has two distinct independent dominating sets K and L. Then any one of K and L cannot be a proper subset of the other. Otherwise, one of them contains an arc and cannot be independent. Let v_1 be a vertex in $K - L$. Then there is a vertex v_2 in $L - K$ that dominates v_1 and next there is a vertex $v_3 \neq v_1$ in $K - L$ that dominates v_2. Repeat this argument. Then we have a sequence $\{v_i\}$ of vertices such that $v_i \neq v_{i+2}$. Let j be the smallest integer such that $v_j = v_k$ for some $k < j$. Then $v_k = v_j, v_{j-1}, \cdots, v_k$ is a cycle of length at least 3 in the underlying tree of T. This contradicts that T is an oriented tree. □

It is easily seen that $\alpha(G) \leq \beta(G)$ for undirected graphs G. But it does not hold for directed graphs as we have already seen in a directed 3-cycle. However, for oriented trees, it still is true.

COROLLARY 1. *Let T be an oriented tree of order n. Then we have*

$$1 \leq \alpha(T) \leq \alpha'(T) \leq \beta(T) \leq n - 1$$

and

$$\beta(T) \geq n/2.$$

Proof. The first part is immediate from the definitions. For the second part, observe that the independence number of an oriented tree is the same as that of the underlying unoriented tree. □

Here is an example that shows that the three invariants need not be equal. Let $n \geq 4$ be an integer and let T be an oriented tree with $V = \{u_1, \cdots, u_n, v_1, \cdots, v_n\}$ and $E = \{(u_1, u_j) \mid j = 2, \cdots, n\} \cup \{(v_1, v_j) \mid j = 2, \cdots, n\} \cup \{(u_1, v_1)\}$. Then it is easy to see that $\alpha(T) = 2$, $\alpha'(T) = n + 1$, and $\beta(T) = 2n - 2$. Therefore we have $\alpha(T) < \alpha'(T) < \beta(T)$.

Let α, α', β, and n be positive integers satisfying $1 \leq \alpha \leq \alpha' \leq \beta \leq n - 1$ and $\beta \geq n/2$. Then can we construct an oriented tree T of order n having $\alpha(T) = \alpha$, $\alpha'(T) = \alpha'$, and $\beta(T) = \beta$? By checking all oriented trees with four vertices, we know that all possible outcomes of (α, α', β) are (1, 1, 3), (2, 2, 2), (2, 3, 3), and (3, 3, 3). Thus there are no oriented trees of order 4 having, for example, the outcome $(1, 2, 3)$.

THEOREM 2. *Let $n \geq 2$ be an integer. Then for any α such that $1 \leq \alpha \leq n-1$, there is an oriented tree T of order n whose domination number is α.*

Proof. We construct T as follows. The vertex set of T is $V = [n]$ and the arcs consist of (i, n) for $i = 1, 2, \ldots, \alpha - 1$ and (n, j) for $j = \alpha, \alpha+1, \ldots, n-1$. Then T is an oriented tree and $\{1, 2, \ldots, \alpha-1, n\}$ is a minimum dominating set of T. Therefore T has domination number α. □

THEOREM 3. *Let $n \geq 2$ be an integer. Then for any α' such that $1 \leq \alpha' \leq n-1$, there is an oriented tree T of order n whose independent domination number is α'.*

Proof. We construct T as follows. The vertex set of T is $V = [n]$. If $\alpha' \geq (n - 1)/2$, then the arcs consist of (i, n) for $i = 1, 2, \ldots, \alpha'$ and $(j, j + \alpha')$ for $j = 1, 2, \ldots, n - \alpha' - 1$. If $\alpha' < (n - 1)/2$, then the arcs consist of (i, n) for $i = 1, 2, \ldots, \alpha'$, $(j, j + \alpha')$ for $j = 1, 2, \ldots, \alpha'$, and (α, k) for $k = 2\alpha + 1, \ldots, n - 1$. Then T is an oriented tree and

$\{1, 2, \ldots, \alpha'\}$ is the independent dominating set of T. Therefore T has independent domination number α'. $\square$

3. Binary trees

In this section we study relations among the domination number, the independent domination number, and the independence number of a binary tree and establish their bounds.

THEOREM 4. *Let T be a binary tree on $2n+1$ vertices. Then we have*

$$\alpha(T) \le \alpha'(T) \le \beta(T), \tag{1}$$

$$\left\lceil \frac{2n+1}{3} \right\rceil \le \alpha(T) \le n, \tag{2}$$

$$n+1 \le \beta(T) \le \left\lfloor \frac{2(2n+1)+1}{3} \right\rfloor. \tag{3}$$

Proof. Corollary 1 implies (1). To prove (2), observe that every vertex in T dominates at most three vertices and that the set of all interior vertices of T is a dominating set for T. This establishes (2). The set of all leaves of T forms an independent set of cardinality $n+1$ and hence $n+1 \le \beta(T)$.

Now we want to prove the last inequality. Let $|T|$ be the underlying tree of the binary tree T. Suppose $S = \{u_1, u_2, \cdots, u_k\}$ is any independent set in $|T|$. For each $i = 2, \cdots, k$, there is a unique $u_1 - u_i$ path in $|T|$. Let R be the set of all predecessors of u_i in the paths for $i = 2, \cdots, k$. Since the set R is disjoint from the set S, we have $|R| \le (2n+1) - k$. In addition, since every vertex in $|T|$ has degree at most 3, we have $(k-1)/2 \le |R|$. Therefore we have $(k-1)/2 \le (2n+1) - k$ and hence $k \le [2(2n+1)+1]/3$. $\square$

Here is an example that shows the three invariants in (1) need not be equal. Let n be an odd integer. Consider any binary tree of order $2n+1$ and height n. Such a tree always has a leaf adjacent from the root. Now attach two new vertices to this leaf. The resulting oriented tree T is a binary tree of order $2n+3$. It is easily seen that $\alpha(T) = n+1$, $\alpha'(T) = n + 2$, and $\beta(T) = n + 3$.

Now let us consider the sharpness of the bounds of (2) and (3) in Theorem 4 and let T_3 denote the binary tree of order 3.

The bounds in (2) are sharp. Let T be any binary tree of height n. Then the set of all interior vertices of T is a minimum dominating set for T and so $\alpha(T) = n$. Hence the upper bound in (2) is sharp.

To see the sharpness of the lower bound of (2), there are three cases to consider.

Case 1: $2n + 1 = 3k$. Consider k copies of T_3. Put one of these copies with the root at the bottom and stack the remaining $k - 1$ copies one by one from left to right by joining the leaf of the bottom copy to the roots of two stacking copies. Observe that $k - 1$ is even in this case and hence this stacking is always possible. It is easy to see that the resulting binary tree has order $2n + 1$ and domination number $k = (2n + 1)/3$.

Case 2: $2n + 1 = 3k + 1$. Consider k copies of T_3 and a single vertex. Put the single vertex at the bottom, which will serve as a root, and stack two copies by joining the root at the bottom to the roots of two stacking copies. Next stack the remaining $k - 2$ copies one by one from left to right by joining the leaf of the bottom to the roots of two stacking copies. Observe that k is even in this case and hence this stacking is always possible. It is easy to see that the resulting binary tree has order $2n + 1$ and domination number $k + 1 = \lceil (2n + 1)/3 \rceil$.

Case 3: $2n + 1 = 3k + 2$. Consider k copies of T_3 and two vertices. Put one of these copies with the root at the bottom and stack the remaining $k - 1$ copies one by one from left to right by joining the leaf of the bottom to the roots of two stacking copies. Now join the remaining two vertices from any one of the leaves of the binary tree already constructed. Observe that $k - 1$ is even in this case and hence this stacking is always possible. It is easy to see that the resulting binary tree has order $2n + 1$ and domination number $k + 1 = \lceil (2n + 1)/3 \rceil$.

The lower bound in (3) is sharp. A binary tree of order $2n + 1$ and height n has independence number $n + 1$.

There is a binary tree whose independence number attains the upper bound in (3) for infinitely many n. For example, a fully balanced binary tree of even height will do.

References

[1] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Wadsworth & Brooks, Monterey, 1986.
[2] A. G. Konheim and D. J. Newman, *A note on growing binary trees*, J. of Graph Theory **4** (1973), 57–63.
[3] A. Meir and J. W. Moon, *Packing and covering constants for certain families of trees. I*, J. of Graph Theory **1** (1977), 157–174.
[4] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton Univ. Press, Princeton, 1944.
[5] M. Richardson, *Solutions of irreflexive relations*, Ann. Math. **58** (1953), 573–590.

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