SPARSE ORTHOGONAL MATRICES BY WEAVING

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ABSTRACT. We determine sparse orthogonal matrices of order n which is fully indecomposable by weaving.

1. Introduction

By a *pattern* we simply mean the arrangement of zero and nonzero (denoted $*$) entries in a matrix. An $n \times n$ pattern P is called *orthogonal* if there is a (real) orthogonal matrix U whose pattern is P. By $\#(U)$ or $\#(P)$ we mean the number of nonzero entries in the matrix U or pattern P. An $n \times n$ pattern (or matrix) P is called *fully indecomposable* if it has no $r \times q$ zero submatrix such that $r + q = n$; equivalently, there do not exist permutation matrices Q_1 and Q_2 such that

$$Q_1 P Q_2 = \begin{bmatrix} P_{11} & O \\ P_{21} & P_{22} \end{bmatrix},$$

in which P_{11} and P_{22} are square and nonempty (or, equivalently the bipartite graph of P is connected). If P were an orthogonal pattern and there were such reducing blocks, then an elementary calculation shows that $P_{21} = O$ also.

In 1991, M. Fiedler conjectured that for $n \geq 2$ an $n \times n$ orthogonal matrix which is fully indecomposable has at least $4n-4$ nonzero entries. In [BBS], this conjecture was shown in the affirmative, and also, the zero patterns of the $n \times n$ orthogonal matrices with exactly $4n - 4$ nonzero entries were determined. B. L. Shader [S] gave a simpler proof of this result, and recently this result was extended in [CS1, CS2].

First, we describe a few results from [BBS]. Recursively define a family of $(0,1)$-matrices of order $n \geq 2$ as follows. Let

$$\mathcal{B}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

If n is odd, define

$$(1) \qquad \mathcal{B}_n = \left[\begin{array}{ccccc|c} & & & & & 0 \\ & & & & & \vdots \\ & & \mathcal{B}_{n-1} & & & 0 \\ & & & & & 1 \\ & & & & & 1 \\ \hline 0 & \cdots & 0 & & 1 & 1 \end{array}\right].$$

If $n \geq 4$ is even, define

$$(2) \qquad \mathcal{B}_n = \left[\begin{array}{cccc|c} & & & & 0 \\ & \mathcal{B}_{n-1} & & & \vdots \\ & & & & 0 \\ & & & & 1 \\ \hline 0 & \cdots & 0 & 1 \quad 1 & 1 \end{array}\right].$$

For example,

$$\mathcal{B}_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathcal{B}_6 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

As noted in [BBS], each of the matrices $\mathcal{B}_n$ $(n \geq 2)$ is the zero pattern of an $n \times n$ orthogonal matrix which is fully indecomposable and has exactly $4n - 4$ nonzero entries.

By *the sparse orthogonal matrix* of order n we mean the orthogonal matrix with the same zero pattern as $\mathcal{B}_n$.

We can ask "what's the $n \times n$ sparse orthogonal matrix?". In this paper, we construct such matrices. This construction is also true for complex unitary matrices.

2. Weaving and sparse orthogonal matrices

To construct $n \times n$ fully indecomposable orthogonal matrices (or patterns) with exactly $4n - 4$ nonzero entries, we apply a method called *weaving* described in [C]. It produces a large matrix from a list of row matrices R_i's and column matrices C_j's with appropriate sizes, via a $(0, 1)$-matrix.

To every $m \times n$ $(0, 1)$-matrix $A = [a_{ij}]$ having row sums $r_1, r_2, \ldots, r_m$ and column sums $c_1, c_2, \ldots, c_n$, we associate a *woven product* of matrices as follows. We adopt the following notations.

$s = s(i, j) :=$ the number of nonzero positions in row i of A up to column j,

$t = t(i, j) :=$ the number of nonzero positions in column j of A up to row i.

Now let R_i $(i = 1, 2, \ldots, m)$ have r_i columns $\mathbf{u}_s$, and C_j $(j = 1, 2, \ldots, n)$ have c_j rows $\mathbf{v}_t^T$ using the indices s and t introduced above. We define the *woven product*

$$M(A) = (R_1 \cdots R_m) \circledast (C_1 \cdots C_n) = [M_{ij}]$$

block entrywise by

$$M_{ij} = \begin{cases} \mathbf{u}_s \mathbf{v}_t^T & \text{if } a_{ij} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix A is called the *lattice* of the weaving, and the matrix $M(A)$ obtained by weaving from the lattice A is called the *woven* matrix. Note that the resulting woven matrix depends on the lattice and matrices R_i's and C_j's.

LEMMA 2.1. *Let A be an $m \times n$ $(0,1)$-matrix whose bipartite graph is connected. If the R_i's and C_j's corresponding to the lattice A are all square and fully indecomposable, then the woven product $M(A) = (R_1 \cdots R_m) \circledast (C_1 \cdots C_n)$ is square and fully indecomposable as well.*

Proof. It is clear from the definition of woven product that if the R_i's and C_j's are square then the woven product $M(A) = (R_1 \cdots R_m) \circledast (C_1 \cdots C_n)$ is also a square matrix. Since the R_i's and C_j's are also all fully indecomposable, each of their bipartite graphs is connected. Since A has a connected bipartite graph as well, any vertex in the bipartite graph of $M(A)$ may be reached from any other via a sequence of paths guaranteed either by the connectivity of the bipartite graphs of the R_i's and C_j's or of A. Thus, the bipartite graph of $M(A)$ is connected, and hence $M(A)$ is fully indecomposable. $\square$

There is a canonical way of factorizing woven matrices based on the lattice A. The following lemma is due to R. Craigen [C].

LEMMA 2.2. *Let A be an $m \times n$ $(0,1)$-matrix having row sums $r_1, r_2, \ldots, r_m$ and column sums $c_1, c_2, \ldots, c_n$. If R_i $(i = 1, 2, \ldots, m)$ is an $r_i \times r_i$ matrix and C_j $(j = 1, 2, \ldots, n)$ is a $c_j \times c_j$ matrix, then there exists a permutation matrix P_A such that*

$$(R_1 \cdots R_m) \circledast (C_1 \cdots C_n) = [R_1 \oplus \cdots \oplus R_m] P_A [C_1 \oplus \cdots \oplus C_n],$$

where $\oplus$ denotes a direct sum.

Throughout the rest of this paper, we let N denote the number of nonzero entries in a given lattice.

THEOREM 2.3. *Let A be an $m \times n$ $(0,1)$-matrix having row sums $r_1, r_2, \ldots, r_m$ and column sums $c_1, c_2, \ldots, c_n$. If R_i $(i = 1, 2, \ldots, m)$ is an $r_i \times r_i$ orthogonal matrix and C_j $(j = 1, 2, \ldots, n)$ is a $c_j \times c_j$ orthogonal matrix, then the woven product $M(A) = (R_1 \cdots R_m) \circledast (C_1 \cdots C_n)$ is an $N \times N$ orthogonal matrix.*

Proof. It is clear from Lemma 2.1 that the woven product $M(A)$ is square. By Lemma 2.2, there exists a permutation matrix P_A such that

$$M(A)(M(A))^T = [R_1 \oplus \cdots \oplus R_m] P_A [C_1 \oplus \cdots \oplus C_n] \cdot$$
$$[C_1 \oplus \cdots \oplus C_n]^T P_A^T [R_1 \oplus \cdots \oplus R_m]^T.$$

Since $r_1 + \cdots + r_m = c_1 + \cdots + c_n = N$, it follows that $M(A)(M(A))^T = I_N$. Thus $M(A)$ is an $N \times N$ orthogonal matrix, which completes the proof. □

Now, we are ready to construct the sparse orthogonal matrix by weaving. The pattern of a woven product is the woven product of the patterns of the R_i's and C_j's. It follows from Lemma 2.1 and Theorem 2.3 that the woven product of fully indecomposable orthogonal patterns through a lattice whose bipartite graph is connected will be a (larger) fully indecomposable orthogonal pattern.

For $m \geq 2$, we define the $m \times m$ lattice A_o and $m \times (m+1)$ lattice A_e as follows:

$$
(3) \qquad A_o = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & O \\ & & \ddots & \ddots & \\ & O & & 1 & 1 \\ & & & & 1 \end{bmatrix},
$$

and

$$
(4) \qquad A_e = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & O \\ & & \ddots & \ddots & \\ & O & & 1 & 1 \\ & & & & 1 & 1 \end{bmatrix}.
$$

Then $\#(A_o) = 2m - 1$ and $\#(A_e) = 2m$. From Theorem 2.3, the woven products obtained by the lattices A_o and A_e produce $(2m-1) \times (2m-1)$ woven matrix and $2m \times 2m$ woven matrix, respectively.

Note that row sums r_i's and column sums c_j' of A_o and A_e are 1 or 2. To construct fully indecomposable orthogonal patterns, we shall only use the 1×1 orthogonal pattern $[*]$ and the 2×2 orthogonal pattern $\begin{bmatrix} * & * \\ * & * \end{bmatrix}$ for the R_i's and C_j's.

It is clear from the definition of woven product that the orthogonal patterns of a woven product of the orthogonal patterns R_i's and C_j's

Gi-Sang Cheon

of $M(A_o)$ and $M(A_e)$ have the following forms, respectively:

$$M(A_o) = (R_1 \cdots R_m) \circledast (C_1 \cdots C_m)$$

$$(5) \qquad = \begin{bmatrix}
* & & * & * & & & & & & \\
* & & * & * & & & & & & \\
& & * & * & & * & & * & & \\
& & * & * & & * & & * & & \\
& & & & \ddots & & \ddots & & & \\
& & & & & \ddots & & \ddots & & \\
& & & & & & * & * & & * & * \\
& & & & & & * & * & & * & * \\
& & & & & & & & & * & * \\
\end{bmatrix},$$

$$M(A_e) = (R_1 \cdots R_m) \circledast (C_1 \cdots C_{m+1})$$

$$(6) \qquad = \begin{bmatrix}
* & & * & * & & & & & \\
* & & * & * & & & & & \\
& & * & * & & * & * & & \\
& & * & * & & * & * & & \\
& & & \ddots & & \ddots & & & \\
& & & & \ddots & & \ddots & & \\
& & & & & * & * & & * & * \\
& & & & & * & * & & * & * \\
& & & & & & & * & * & & * \\
& & & & & & & * & * & & * \\
\end{bmatrix}.$$

Then $M(A_o)$ is a $(2m - 1) \times (2m - 1)$ orthogonal pattern which is fully indecomposable, and we get

$$\#(M(A_o)) = 4(2m - 1 - 1) = 4(2m - 1) - 4.$$

Similarly, $M(A_e)$ is a $2m \times 2m$ orthogonal pattern which is fully indecomposable, and we get

$$\#(M(A_e)) = 4(2m - 1) = 4(2m) - 4.$$

Note that if n is an odd then the zero pattern of $M(A_o)$ is precisely coincide with (1), and if n is an even then the zero pattern of $M(A_e)$ is precisely coincide with (2).

It is clear that the 1×1 fully indecomposable orthogonal matrix U whose pattern is $[\,*\,]$ is $[\,1\,]$, and the 2×2 fully indecomposable orthogonal matrix U whose pattern is $\begin{bmatrix} * & * \\ * & * \end{bmatrix}$ has the form

$$(7) \qquad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

where $0 < \theta < 2\pi$, $\theta \neq \frac{\pi}{2}, \pi, \frac{3}{2}\pi$.

Thus we have the following theorem.

THEOREM 2.4. *Let A_o and A_e be the lattices defined in (3) and (4) respectively. Then an $n \times n$ fully indecomposable sparse orthogonal pattern is*

$$\begin{cases} M(A_o) \ \text{ in } (5) & \text{if } n = 2m - 1 \ \text{ for } m \geq 2, \\ M(A_e) \ \text{ in } (6) & \text{if } n = 2m \ \text{ for } m \geq 2. \end{cases}$$

Note that we can get many sparse orthogonal matrices with the same pattern $M(A_o)$ or $M(A_e)$, since we can arbitrarily choose θ in (7).

For example, let

$$A_o = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\#(A_o) = N = 5$, and $r_1 = r_2 = 2 = c_2 = c_3$, $r_3 = 1 = c_1$. Take

$$R_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \quad R_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix},$$

$$C_2 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad C_3 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix},$$

and

$$R_3 = C_1 = [1].$$

Then

$$M(A_o) = (R_1 \ R_2 \ R_3) \circledast (C_1 \ C_2 \ C_3)$$

$$= \begin{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}[1] & \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} & O \\ O & \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} & \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \\ O & O & [1]\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{4} & \frac{3}{4} & 0 & 0 \\ 0 & -\frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ 0 & \frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

Thus $M(A_o)$ is a 5×5 orthogonal matrix with exactly $4n - 4 = 16$ nonzero entries which is fully indecomposable.

References

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