Kangweon-Kyungki Math. Jour. **7** (1999), No. 1, pp. 123–130

# WEAK COMPACTNESS AND EXTREMAL STRUCTURE IN $L^P(\mu, X)$

Chun-Kee Park

Abstract. We characterize the compactness, weak precompactness and weak compactness in $L^p(\mu, X)$ and in more general space $P_c(\mu, X)$. Moreover, we present this characterization in terms of extremal structure in $X$.

## 1. Introduction

Let $(\Omega, \Sigma, \mu)$ be a finite measure space, $X$ a real Banach space, $X^*$ the dual space of $X$ and $B_X$ the unit ball of $X$.

Denote by $L^p(\mu, X)(1 \leq p < \infty)$ the Banach space of all equivalence classes of $X$-valued Bochner integrable functions $f$ defined on $\Omega$ with $\int_\Omega \|f\|^P d\mu < \infty$ . The norm $\|\cdot\|_p$ is defined by

$$\|f\|_p = (\int_\Omega \|f\|^p d\mu)^{\frac{1}{p}}, f \in L^p(\mu, X)$$

Denote by $\mathcal{L}^1(\mu, X)$ (resp. $P_c(\mu, X)$ ) the space of all strongly measurable Pettis integrable (resp. Pettis integrable)functions $f : \Omega \to X$ (resp. having an indefinite integral with relatively compact range)with the Pettis norm $\|f\|_{p_1} = \sup_{x^* \in B_{X^*}} \int_\Omega |x^* f| d\mu$.

Denote by $K(\mu, X)$ the space of all $\mu$ -continuous vector measures $G : \Sigma \to X$ whose range is relatively compact with the semivariation norm. Notice that $L_1(\mu, X) \subseteq \mathcal{L}^1(\mu, X) \subseteq P_c(\mu, X) \subseteq K(\mu, X)$. Diestel, Ruess and Schactermayer [5] and Diaz [2] presented characterizations of weakly compact subsets of $L^1(\mu, X)$. Brooks and Dinculeanu

Received January 14, 1999.

1991 Mathematics Subject Classification: 46B25, 46E40, 28A20.

Key words and phrases: weak compactness, Bochner integrable functions, Pettis integrable functions, extremal structure.

This paper was supported by the Research Foundation of Kangwon National University 1997.

[1] and Emmanuele [6] characterized weak compactness (resp. precompactness) of subsets of $\mathcal{L}^1(\mu, X)$(resp. $P_c(\mu, X)$ ).

THEOREM 1.1([7] COROLLARY 3.4). *A sequence $(f_n) \subset \mathcal{L}^1(\mu, X)$ converges weakly to zero if and only if $\int_A f_n d\mu \to 0$ weakly for each $A \in \Sigma$.*

If $\pi = (A_i)_{i \in I}$ be a finite partition of $\Omega$ , we define the conditional expectation $E_\pi f$ of $f$ by

$$E_\pi f = \sum_{i \in I} [\mu(A_i)]^{-1} \left( \int_{A_i} f d\mu \right) \chi_{A_i}.$$

It is well known that the family of finite partitions is directed by refinement, that $\|E_\pi f\| \leq \|f\|$ and that $\|E_\pi f - f\| = 0$ for all $f \in P_c(\mu, X)$.

THEOREM 1.2([6] THEOREM 1). *Let $H$ be a bounded subset of $P_c(\mu, X)$. Then the following facts are equivalent :*

(1) *$H$ is precompact,*
(2) *(i) $\{\int_A f d\mu : f \in H\}$ is relatively compact in $X$ for all $A \in \Sigma$,*
*(ii) $\lim_\pi E_\pi f = f$ uniformly on $f \in H$.*

We shall denote by $\text{ext} B_{X^*}$ the set of all extreme points of the dull ball $B_{X^*}$ and we shall denote by $\sigma_e$ the weak topology on $X$ generated by ext $B_{X^*}$. Observe that the $\sigma_e$ -topology on $X$ is Hausdorff and that the closed unit ball $B_X$ is also closed in the $\sigma_e$ -topology. Moors [8] presented a characterization of weak compactness in Banach spaces in terms of $\sigma_e$ -topology.

THEOREM 1.3([3]. *Let $X$ be a Banach space and $(x_n)$ be a bounded sequence in $X$. Then $(x_n)$ converges weakly to $x \in X$ if and only if $(x_n)$ converges to $x$ in the $\sigma_e$ -topology.*

In this paper, we characterize compactness of subsets of $L^1(\mu, X)$ and weak compactness of subsets of $P_c(\mu, X)$ in terms of conditional expectation and $\sigma_e$ -topology.

## 2. Results

Notice that a subset $K$ of a Banach space is relatively norm compact if and only if it is totally norm-bounded.

THEOREM 2.1. *Let $K$ be a bounded subset of $L^1(\mu, X)$. Then $K$ is relatively $L^1(\mu, X)$ -norm compact in $L^1(\mu, X)$ if and only if*

(1) *$\{\int_A f d\mu : f \in K\}$ is relatively norm compact in $X$ for all $A \in \Sigma$,*

(2) *for each $\epsilon > 0$, there is a finite partition $\pi$ of $\Omega$ such that $\|E_\pi f - f\|_1 < f$ uniformly on $f \in K$.*

*Proof.* Suppose that $K$ is relatively $L^1(\mu, K)$ -norm compact in $L^1(\mu, X)$. Then (1) follows from the continuity of the mapping $f \to \int_A f d\mu$ of $L^1(\mu, X)$ into $X$. Let $\epsilon > 0$ be given. Then since $K$ is totally bounded in $L^1(\mu, X)$ there are $f_1, f_2, \cdots, f_n \in K$ such that $K \subset \bigcup_{i=1}^{n} N_{\frac{\epsilon}{3}}(f_i)$ , where $N_\epsilon(f_i)$ is a $\epsilon$- neighborhood of $f_i$ .

Notice that $\lim_\pi \|E_\pi f - f\|_1 = 0$ for each $f \in L^1(\mu, X)$, and so we can find a finite partition $\pi$ of $\Omega$ such that $\|E_\pi f_i - f_i\|_1 < \frac{\epsilon}{3}$ for $i = 1, 2, \cdots, n$.

Fix $f \in K$. Then there is $i (1 \leq i \leq n)$ such that $f \in N_{\frac{\epsilon}{3}}(f_i)$. Hence we have

$$\begin{aligned}\|E_\pi f - f\|_1 &\leq \|E_\pi f - E_\pi f_i\|_1 + \|E_\pi f_i - f_i\|_1 + \|f_i - f\|_1 \\ &\leq 2\|f - f_i\|_1 + \|E_\pi f_i - f_i\|_1 < \epsilon.\end{aligned}$$

Conversely, suppose that the conditions (1) and (2) hold. Let $\epsilon > 0$ be given. Then there is a finite partition $\pi = (A_i)_{1 \leq i \leq k}$ of $\Omega$ such that $\|E_\pi f - f\|_1 < \frac{\epsilon}{3}$ uniformly on $f \in K$.

We will first show that $\{E_\pi f : f \in K\}$ is relatively $L^1(\mu, X)$ - norm compact in $L^1(\mu, X)$. Consider a sequence $(E_\pi f_n)$ in $\{E_\pi f : f \in K\}$ . From (1) we can obtain a subsequence $(f_{n_j})$ of $(f_n)$ and $x_{A_1}, \cdots, x_{A_k} \in X$ such that $\lim_{j \to \infty} \| \int_{A_i} f_{n_j} d\mu - x_{A_i}\| = 0$ for $i = 1, 2, \cdots, k$. So we have

$$\begin{aligned}\| \sum_{i=1}^{k} \frac{x_{A_i}}{\mu(A_i)} \chi_{A_i} - E_\pi f_{n_j} \|_1 &= \sum_{i=1}^{k} \int_{A_i} \| \frac{x_{A_i}}{\mu(A_i)} - \frac{\int_{A_i} f_{n_j} d\mu}{\mu(A_i)} \| d\mu \\ &= \sum_{i=1}^{k} \|x_{A_i} - \int_{A_i} f_{n_j} d\mu\| \to 0 \text{ as } j \to \infty.\end{aligned}$$

This implies that $\{E_\pi f : f \in K\}$ is relatively $L^1(\mu, X)$ -norm compact in $L^1(\mu, X)$. Hence there are $f_1, \cdots, f_n \in K$ such that $\{E_\pi f : f \in K\} \subset \bigcup_{i=1}^{n} N_{\frac{\epsilon}{3}}(E_\pi f_i)$. Now fix $f \in K$. Then there is an $i (1 \leq i \leq n)$ such that $E_\pi f \in N_{\frac{\epsilon}{3}}(E_\pi f_i)$. We have

$$\|f - f_i\|_1 \leq \|f - E_\pi f\|_1 + \|E_\pi f - E_\pi f_i\|_1 + \|E_\pi f_i - f_i\|_1 < \epsilon.$$

Thus $K \subset \bigcup_{i=1}^{n} N_\epsilon(f_i)$. Hence $K$ is totally bounded in $L^1(\mu, X)$. This completes the proof. □

The next theorem is a generalization of Theorem 1.2

THEOREM 2.2. *Let $K$ be a bounded subset of $P_c(\mu, X)$. Then $K$ is weakly precompact in $P_c(\mu, X)$ if and only if*

(1) *$\{\int_A f d\mu : f \in K\}$ is weakly precompact in $X$ for all $A \in \Sigma$ ,*
(2) *$\lim_\pi E_\pi f = f$ weakly in $P_c(\mu, X)$ uniformly on $f \in K$.*

*Proof.* Assume that $K$ is weakly precompact in $P_c(\mu, X)$. Condition (1) follows from the fact that the mapping $f \to \int_A f d\mu$ of $(P_c(\mu, X), \Im^\omega)$ into $(X, \Im^\omega)$ is continuous.

Now we note that $K$ is totally bounded in $(P_c(\mu, X), \Im^\omega)$. For each $g \in P_c(\mu, X)^*, g \circ E_\pi$ is also in $P_c(\mu, X)^*$. Let $\epsilon > 0$ be given. Then the set $N(O; g, g \circ E_\pi; \frac{\epsilon}{3}) = \{f \in P_c(\mu, X) : |g(f)| < \frac{\epsilon}{3}$ and $|(g \circ E_\pi)(f)| < \frac{\epsilon}{3}\}$ is an open neighborhood of O in $(P_c(\mu, X), \Im^\omega)$. Because $K$ is totally bounded in $(P_c(\mu, X), \Im^\omega)$, there are $f_1, f_2, \cdots, f_m \in P_c(\mu, X)$ such that $K \subset \bigcup_{i=1}^{m} [f_i + N(O; g, g \circ E_\pi; \frac{\epsilon}{3})]$. For any $f \in K$, there is an $i$ $(1 \leq i \leq m)$ such that $f \in f_i + N(O; g, g \circ E_\pi; \frac{\epsilon}{3})$. Since $\lim_\pi E_\pi f = f$ in norm for all $f \in P_c(\mu, X)$, we have $\lim_\pi g(E_\pi f) = g(f)$ for all $f \in P_c(\mu, X)$. Hence there is a $\pi'$ such that

$$\pi > \pi' \Rightarrow |g(E_\pi f_i) - g(f_i)| < \frac{\epsilon}{3} \text{ for } i = 1, 2, \cdots, m.$$

Hence

$$\pi > \pi' \Rightarrow |g(E_\pi f) - g(f)| \leq |g(E_\pi f) - g(E_\pi f_i)| + |g(E_\pi f_i) - g(f_i)| + |g(f_i) - g(f)| < \epsilon.$$

Thus $\lim_{\pi} E_\pi f = f$ weakly in $P_c(\mu, X)$ uniformly on $f \in K$.

Conversely, assume that conditions (1) and (2) hold. Let $(f_n)$ be any sequence in $K$ and let $g \in P_c(\mu, X)^*$ be arbitrary. Then given $\epsilon > 0$ , by (2) there is a finite partition $\pi' = (A_i)_{i \in I}$ of $\Omega$ such that $|g(E_{\pi'} f) - g(f)| < \frac{\epsilon}{3}$ uniformly on $f \in K$. So $E_{\pi'}(K)$ is contained in the set $\sum_{i \in I} \mu(A_i)^{-1}\{\int_{A_i} f d\mu : f \in H\}\chi_{A_i}$, which is weakly precompact by (1). Hence $(E_{\pi'} f_n)$ has a weak Cauchy subsequence , say $(E_{\pi'} f_{n_k})$. Therefore there is a $N \in \mathbf{N}$ such that

$$k, k' > N \Rightarrow |g(E_{\pi'} f_{n_k}) - g(E_{\pi'} f_{n_{k'}})| < \frac{\epsilon}{3}.$$

Hence we have

$$\begin{aligned} k, k' > N \Rightarrow & |g(f_{n_k}) - g(f_{n_{k'}})| \\ & \leq |g(f_{n_k}) - g(E_{\pi'} f_{n_k})| + |g(E_{\pi'} f_{n_k}) - g(E_{\pi'} f_{n_{k'}})| \\ & + |g(E_{\pi'} f_{n_{k'}}) - g(f_{n_{k'}})| \\ & < \epsilon. \end{aligned}$$

Thus $K$ is weakly precompact in $P_c(\mu, X)$. □

THEOREM 2.3. *Let $K$ be a bounded subset of $P_c(\mu, X)$. Then $K$ is weakly precompact in $P_c(\mu, X)$ if and only if*

(1) *$\{\int_A f d\mu : f \in K\}$ is weakly precompact in $X$ for all $A \in \Sigma$,*

(2) *for any sequence $(f_k) \subset K$ there is a sequence $(\pi_n)$ of finite partitions, cofinal to the net $(\pi)$, such that $\lim_n E_{\pi_n} f_k = f_k$ weakly in $P_c(\mu, X)$ uniformly on $k \in \mathbf{N}$.*

*Proof.* Assume that $K$ is weakly precompact in $P_c(\mu, X)$. Then it is clear that condition (1) holds.

Now let $(f_k)$ be any sequence in $K$. It is well known that there is a sequence $(\pi_n)$ of finite partitions cofinal to the net $(\pi)$ such that $\lim_n \|E_{\pi_n} f_k - f_k\|_{p_1} = 0$ for all $k \in \mathbf{N}$. Notice that the set $\{f_k : k \in \mathbf{N}\}$ is totally bounded in $(P_c(\mu, X), \Im^\omega)$. Using the similar method in the proof of Theorem 2.2, we have $\lim_n E_{\pi_n} f_k = f_k$ weakly in $P_c(\mu, X)$ uniformly on $k \in \mathbf{N}$.

Conversely, assume that conditions(1) and (2) hold. Using the similar method in the proof of Theorem 2.2, we can show that $K$ is weakly precompact in $P_c(\mu, X)$. □

Notice that the mapping $T : \mathcal{L}^1(\mu, X) \to K(\mu, X), T(f)(A) = \int_A f d\mu, f \in \mathcal{L}^1(\mu, X), A \in \Sigma$ , is linear and isometry [4].

Define $\tilde{T} : P_c(\mu, X) \to K(\mu, X)$ by $\tilde{T}(f)(A) = \int_A f d\mu$ for each $f \in P_c(\mu, X)$ and $A \in \Sigma$. Then $\tilde{T}$ is also well-defined, linear and isometry.

The next lemma is an extended version of Theorem 1.1.

LEMMA 2.4. *Let $(f_n)$ be a sequence in $P_c(\mu, X)$. Then $(f_n)$ converges to 0 weakly in $P_c(\mu, X)$ if and only if $\int_A f_n d\mu \to 0$ weakly in $X$ for each $A \in \Sigma$.*

*Proof.* Suppose that $(f_n)$ converges to 0 weakly in $P_c(\mu, X)$, and let $A \in \Sigma$. Then $T_A : P_c(\mu, X) \to X, T_A(f) = \int_A f d\mu$ , is a bounded linear operator. For each $x^* \in X^*$ , we have

$$< x^*, \int_A f_n d\mu >=< x^*,\ T_A(f_n) >= (x^* \circ T_A)(f_n) \to 0 \text{ as } n \to \infty.$$

Hence $\int_A f_n d\mu \to 0$ weakly in $X$.

Conversely, suppose that $\int_A f_n d\mu \to 0$ weakly in $X$ for each $A \in \Sigma$. Let $(f_n)$ be a sequence in $P_c(\mu, X)$. Then there is a sequence $(G_{f_n})$ in $K(\mu, X)$ such that $\|f_n\|_{p_1} = \|G_{f_n}\|_{S.V}$ for each $n \in \mathbf{N}$. Because $\mathcal{L}^1(\mu, X)$ is considered as a dense subspace of $K(\mu, X)$[4], for each $n \in \mathbf{N}$ there is $f_n' \in \mathcal{L}^1(\mu, X)$ with $\|G_{f_n'} - G_{f_n}\|_{S.V} < \frac{1}{n}$ where $T(f_n') = G_{f_n'}$. Hence $\|f_n' - f_n\|_{p_1} = \|G_{f_n'} - G_{f_n}\|_{S.V} < \frac{1}{n}$ for each $n \in \mathbf{N}$. We can show easily that $\int_A f_n' d\mu \to 0$ weakly in $X$ for each $A \in \Sigma$. By Theorem 1.1 , $(f_n')$ converges to 0 weakly in $\mathcal{L}^1(\mu, X)$.

Now let $g \in P_c(\mu, X)^*$. Then $g \in \mathcal{L}^1(\mu, X)^*$ . Hence $g(f_n') \to 0$ weakly as $n \to \infty$ . We have $|g(f_n) - g(f_n')| \leq \|g\|\|f_n - f_n'\|_{p_1} \leq \|g\|\frac{1}{n} \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} g(f_n) = \lim_{n\to\infty} g(f_n') = 0$. This implies that $(f_n)$ converges to 0 weakly in $P_c(\mu, X)$. □

THEOREM 2.5. *A bounded subset $K$ of $P_c(\mu, X)$ is relatively weakly compact if*

(1) *The set $\{\mu(A)^{-1}\int_A f d\mu : f \in K, A \in \Sigma, \mu(A) > 0\}$ is relatively weakly compact in $X$,*
(2) $\lim_{\pi} E_\pi f = f$ *weakly in $P_c(\mu, X)$ uniformly on $f \in K$.*

*Proof.* For every $A \in \Sigma$, we have

$$\left\{\int_A f d\mu : f \in K\right\} \subset \mu(A)\left\{\frac{1}{\mu(B)}\int_B f d\mu : f \in K, B \in \Sigma, \mu(B) > 0\right\}$$

Hence by (1), $\left\{\int_A f d\mu : f \in K\right\}$ is relatively weakly compact in $X$. By Theorem 2.2, $H$ is weakly precompact in $P_c(\mu, X)$.

Now let $(f_n)$ be a sequence in $K$. Then $(f_n)$ has a weak Cauchy subsequence, say $(f_{n_k})$. For every $A \in \Sigma$, the sequence $(\int_A f_{n_k} d\mu)$ is also a weak Cauchy sequence in $\{\int_A f d\mu : f \in K\}$ . Since $\{\int_A f d\mu : f \in K\}$ is relatively weakly compact in $X$, there is an $m(A) \in X$ such that $< m(A), x^* >= \lim_{k\to\infty} \int_A < f_{n_k}(s), x^* > d\mu,$ for all $x^* \in X^*$.

The set function $m : \Sigma \to X$ is a $\mu$ -continuous vector measure and the average range $\{m(A)/\mu(A) : A \in \Sigma, \mu(A) > 0\}$ is contained in the weak closed convex hull of $\{\mu(A)^{-1}\int_A f d\mu : f \in K, A \in \sum, \mu > 0\}$ , which is a weakly compact convex set in $X$. Hence the average range $\{m(A)/\mu(A) : A \in \Sigma, \mu(A) > 0\}$ is relatively weakly compact in $X$. Thus there is an $f \in P_c(\mu, X)$ such that $m(A) = \int_A f d\mu$ for all $A \in \Sigma$. Hence $\int_A < f(s), x^* > d\mu = \lim_{k\to\infty} \int_A < f_{n_k}(s), x^* > d\mu$ for all $A \in \Sigma$ and $x^* \in X^*$ . By Lemma 2.4, $\lim_{k\to\infty} f_{n_k} = f$ weakly in $P_c(\mu, X)$. Thus $K$ is relatively weakly compact in $P_c(\mu, X)$. □

We obtain the following corollary from Theorem 1.3 and Lemma 2.4.

COROLLARY 2.6. *Let $(f_n)$ be a sequence in $P_c(\mu, X)$. Then $(f_n)$ converges to 0 weakly in $P_c(\mu, X)$ if and only if $\int_A f_n d\mu \to 0$ in the $\sigma_e$ -topology for each $A \in \Sigma$.*

We obtain the following corollaries from Theorem 2.2, Theorem 2.5 and Corollary 2.6.

COROLLARY 2.7. *Let $K$ be a bounded subset of $P_c(\mu, X)$. Then $K$ is weakly precompact in $P_c(\mu, X)$ if and only if*

(1) $\{\int_A f d\mu : f \in K\}$ *is precompact in the $\sigma_e$ -topology for each $A \in \Sigma$,*

(2) $\lim_{\pi} \int_A E_\pi f d\mu = \int_A f d\mu$ *uniformly on $f \in K$ in the $\sigma_e$ -topology for each $A \in \Sigma$.*

COROLLARY 2.8. *Let $K$ be a bounded subset of $P_c(\mu, X)$. Then $K$ is relatively weakly compact if*

(1) $\{\mu(A)^{-1} \int_A f d\mu : f \in K, A \in \Sigma, \mu(A) > 0\}$ *is relatively compact in the $\sigma_e$-topology,*

(2) $\lim_{\pi} \int_A E_\pi f d\mu = \int_A f d\mu$ *uniformly on $f \in K$ in the $\sigma_e$ -topology for each $A \in \Sigma$.*

## References

1. J. K. Brooks and N. Dinculeanu, *Weak and strong compactness in the space of Pettis integrable functions*, Contemporary Math., **2** (1980), 116-187.
2. S. Diaz, *Weak compactness in $L^1(\mu, X)$*, Proc. Amer. Math. Soc., **124(9)** (1996), 2685-2693.
3. J. Diestel, *Sequences and Series in Banach spaces*, Springer-Verlag, New York, 1984.
4. J. Diestel and J.J.Uhl Jr., *Vector measures*, Math. Surveys Monographs, vol.15, Amer. Math. Soc., Providence, R.I., 1977.
5. J.Diestel , W.M. Ruess and W. Schachermayer, *Weak compactness in $L^1(\mu, X)$*, Proc. Amer. Math. Soc., **118 (2)** (1993), 447-453.
6. G. Emmanuele, *Precompactness in the space of Pettis integrable functions*, Acta Math. Hung., **62** (1993), 333-335.
7. D. R. Lewis, *Conditional weak compactness in certain inductive tensor products*, Math. Ann., **201** (1973), 201-209.
8. W.B. Moors, *A characterization of weak compactness in Banach spaces*, Bull. Austral. Math. Soc., **55** (1997), 497-501.

Department of Mathematics
Kangwon National University
Chuncheon 200-701, Korea