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BOUNDARIES AND PEAK POINTS OF LIPSCHITZ ALGEBRAS

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Abstract. We determine the Shilov and Choquet boundaries and the set of peak points of Lipschitz algebras $Lip(X, \alpha)$ for $0 < \alpha \leq 1$, and $\ell ip(X, \alpha)$ for $0 < \alpha < 1$, on a compact metric space X. Then, when X is a compact subset of $\mathbb{C}^n$, we define some subalgebras of these Lipschitz algebras and characterize their Shilov and Choquet boundaries. Moreover, for compact plane sets X, we determine the Shilove boundary of them. We also determine the set of peak points of these subalgebras on certain compact subsets X of $\mathbb{C}^n$.

1. Introduction

Let X be a compact Hausdorff space, and let $C(X)$ be the Banach algebra of all continuous complex-valued functions on X under the uniform norm, $\| f \|_X = \sup_{x \in X} |f(x)|$ for f in $C(X)$. A subalgebra A of $C(X)$ which separates the points of X, contains the constants, and which is a Banach algebra with respect to some norm $\| \cdot \|$, is a *Banach function algebra* on X. If the norm of a Banach function algebra is equivalent to the uniform norm, then it is a *uniform algebra.* If A is a function algebra on X, then $\bar{A}$, the uniform closure of A, is a uniform algebra on X. A Banach function algebra A on X is *natural* if M_A, the maximal ideal space of A, is X; equivalently, if every homomorphism on A is given by evaluation at a point of X.

Let A be a function algebra on X. A closed subset P of X is a *peak set* of A if there exists a function $f \in A$ such that $f = 1$ on P and $|f| < 1$ on $X \backslash P$. If $P = \{p\}$, then p is a *peak point* of A and the set of all peak points of A is denoted by $S_0(A)$. A subset E of X is a *boundary* for A if every $f \in A$ attains its maximum modulus on E. It is clear that every boundary contains $S_0(A)$. The *Shilov boundary* of A is the smallest closed boundary of A and it is denoted by $\Gamma(A)$. The Shilov boundary for function algebras exists and it is in fact the intersection

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of all closed boundaries. Also, for a Banach function algebra A on a compact metrizable space X, we have $\overline{S_0}(A) = \Gamma(A)$ [1], [4]. If A^* is the dual (conjugate) space of $(A, \|\cdot\|_X)$, then $K(A)$, the state space of A is defined by $K(A) = \{\varphi \in A^* : \|\varphi\| = \varphi(1) = 1\}$. $K(A)$ is a weak*-compact Hausdorff convex subset of the closed unit ball in A^*. The *Choquet boundary* of A is the set of all $x \in X$ for which φ_x is an extreme point of $K(A)$ and it is denoted by $Ch(A)$. It is known that $\overline{Ch}(A) = \Gamma(A)$.

In the sequel, we will need the following important remark due to T.G. Honary.

REMARK 1 [3]. Let A be a function algebra on X and $\bar{A}$ be the uniform closure of A. Then we have

$$Ch(A) = Ch(\bar{A}) \text{ and } \Gamma(A) = \Gamma(\bar{A}).$$

2. Boundaries and peak points of Lipschitz algebras

Let (X, d) be a compact metric space and $\alpha > 0$. The algebra of all complex-valued functions f on X for which

$$p_\alpha(f) = \sup_{x,y\in X\ x\neq y} \frac{|f(x) - f(y)|}{d^\alpha(x,y)} < \infty,$$

is denoted by $Lip(X, \alpha)$ and the subalgebra of those functions f for which

$$\lim_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d^\alpha(x,y)} = 0,$$

is denoted by $\ell ip(X, \alpha)$. These are called Lipschitz algebras of order α and were first studied by D.R. Sherbert in [7]. The algebras $Lip(X, \alpha)$ for $\alpha \leq 1$, and $\ell ip(X, \alpha)$ for $\alpha < 1$, are Banach function algebras on the compact metric space X under the norm $\| f \| = \| f \|_X + p_\alpha(f)$. Moreover, $\ell ip(X, \alpha)$ is a proper subalgebra of $Lip(X, \alpha)$, and for every $\alpha < 1$ we have $Lip(X, 1) \subseteq \ell ip(X, \alpha)$. Since these Lipschitz algebras are self-adjoint and separate the points of X, they are uniformly dense in $C(X)$, by the Stone-Weierstrass Theorem. Hence these Lipschitz algebras are natural Banach function algebras on X, and by Remark 1 we have

$$\Gamma(Lip(X, \alpha)) = \Gamma(\ell ip(X, \alpha)) = \Gamma(C(X)) = X,$$
$$Ch(Lip(X, \alpha)) = Ch(\ell ip(X, \alpha)) = Ch(C(X)) = X.$$

As we know $S_0(C(X)) = X$. Now we show that $S_0(Lip(X,\alpha))$ $= S_0(\ell ip(X,\alpha)) = X$:

Let $x_0 \in X$, define

$$f(x) = 1 - \frac{d(x,x_0)}{\mathrm{diam}(X)} \quad (x \in X).$$

It is easy to see that $f \in Lip(X,1), 0 \leq f \leq 1, f(x_0) = 1$ and $f(x) < 1$ for $x \in X \backslash \{x_0\}$; that is f is peak at $x_0 \in X$. So x_0 belongs to $S_0(Lip(X,\alpha))$ and $S_0(\ell ip(X,\alpha))$.

3. Boundaries and peak points of subalgebras of Lipschitz algebras

Let X be a compact subset of $\mathbb{C}^n$, by the Functional Calculus Theorem [2, Theorem 3.4.5], $Lip(X,1)$ contains all functions which are analytic in some neighbourhood of X. So we can define the following subalgebras of $Lip(X,\alpha)$ for $\alpha \leq 1$, and $\ell ip(X,\alpha)$ for $\alpha < 1$, when X is a compact subset of $\mathbb{C}^n$.

DEFINITION 1. The closed subalgebra of $Lip(X,\alpha)$ $(\ell ip(X,\alpha))$ which is generated by the polynomials in $z_1, z_2, \ldots, z_n$ by the rational functions with poles off X, or by the analytic functions in some neighbourhood of X, is denoted by $Lip_P(X,\alpha)$ $(\ell/ip_P(X,\alpha))$, by $Lip_R(X,\alpha)$ $(\ell ip_R(X,\alpha))$, or by $Lip_H(X,\alpha)$ $(\ell ip_H(X,\alpha))$, respectively.

DEFINITION 2. The subalgebra of $Lip(X,\alpha)$ $(\ell ip(X,\alpha))$ which is generated by those elements of $Lip(X,\alpha)$ $(\ell ip(X,\alpha))$ which are analytic in the interior of X, is denoted by $Lip_A(X,\alpha)$ $(\ell ip_A(X,\alpha))$.

So we have $Lip_A(X,\alpha) = Lip(X,\alpha) \cap A(X)$ $(\ell ip_A(X,\alpha) = \ell ip(X,\alpha) \cap A(X))$, where $A(X)$ is the algebra of contiuous functions on X which are analytic in the interior of X.

Clearly these subalgebras are all Banach function algebras on X, and the maximal ideal spaces of most of them have been determined in [5], [6]. Here we investigate their Shilov and Choquet boundaries and also the set of peak points of them. For this purpose, we consider the standard uniform algebras $P(X), R(X)$ and $H(X)$ which are the uniform closures of polynomials, rational functions with poles off X and analytic functions in a neighbourhood of X, respectively. It is clear that

$$\overline{Lip_P}(X,\alpha) = \overline{\ell ip_P}(X,\alpha) = P(X),$$
$$\overline{Lip_R}(X,\alpha) = \overline{\ell ip_R}(X,\alpha) = R(X),$$
$$\overline{Lip_H}(X,\alpha) = \overline{\ell ip_H}(X,\alpha) = H(X).$$

Hence by Remark 1 we have

$$Ch(Lip_P(X,\alpha)) = Ch(\ell ip_P(X,\alpha)) = Ch(P(X)),$$
$$Ch(Lip_R(X,\alpha)) = Ch(\ell ip_R(X,\alpha)) = Ch(R(X)),$$
$$Ch(Lip_H(X,\alpha)) = Ch(\ell ip_H(X,\alpha)) = Ch(H(X)),$$
$$\Gamma(Lip_P(X,\alpha)) = \Gamma(\ell ip_P(X,\alpha)) = \Gamma(P(X)) \subseteq bd(\hat{X}),$$
$$\Gamma(Lip_R(X,\alpha)) = \Gamma(\ell ip_R(X,\alpha)) = \Gamma(R(X)) \subseteq bd(R-hull X),$$
$$\Gamma(Lip_H(X,\alpha)) = \Gamma(\ell ip_H(X,\alpha)) = \Gamma(H(X)),$$

where $\hat{X}$ is the polynomial convex hull of X and $R-hull(X)$ is the rational convex hull of X. In particular, when X is a compact plane set we have

$$\Gamma(Lip_P(X,\alpha)) = \Gamma(\ell ip_P(X,\alpha)) = bd(\hat{X})$$
$$\Gamma(Lip_R(X,\alpha)) = \Gamma(\ell ip_R(X,\alpha)) = bd(X).$$

(As we know in this case $Lip_R(X,\alpha) = Lip_H(X,\alpha)$, $\ell ip_R(X,\alpha) = \ell ip_H(X,\alpha)$.)

In general we cannot say anything about the boundaries of $Lip_A(X,\alpha)$ and $\ell ip_A(X,\alpha)$. However in the case that $A(X) = H(X)$ (in particular, $A(X) = R(X)$ or $A(X) = P(X)$), we have

$$Ch(Lip_A(X,\alpha)) = Ch(\ell ip_A(X,\alpha)) = Ch(A(X))$$
$$\Gamma(Lip_A(X,\alpha)) = \Gamma(\ell ip_A(X,\alpha)) = \Gamma(A(X)).$$

Since in these cases we can conclude $\overline{Lip_A}(X,\alpha) = \overline{\ell ip_A}(X,\alpha) = A(X)$. For example, when X is a polynomially convex plane set,

$$\Gamma(Lip_A(X,\alpha)) = \Gamma(\ell ip_A(X,\alpha)) = bd(X).$$

Now we determine the set of peak points of these Lipschitz algebras on certain compact subsets X of $\mathbb{C}^n$.

THEOREM 1. *Let X be a compact subset of $\mathbb{C}^n$ such that $\ell ip_P(X,\alpha)$ ($\ell ip_R(X,\alpha)$) is self-adjoint. Then the set of peak points of $\ell ip_P(X,\alpha)$ ($\ell ip_R(X,\alpha)$) is X.*

Proof. The algebra $\ell ip_P(X,\alpha)$ contains the constant functions and the coordinate functions $f_k(z) = z_k$ ($z = (z_1, z_2, \ldots, z_n)$, $1 \le k \le n$). For $\zeta \in X$, define the function f on X by

$$f(z) = 1 - \frac{\| z - \zeta \|^2}{(\text{diam} X)^2} = 1 - \frac{1}{(\text{diam} X)^2} \sum_{k=1}^{n} (z_k - \zeta_k)(\bar{z}_k - \bar{\zeta}_k).$$

So by hypothesis, f is an element of $\ell ip_P(X,\alpha)$ for which $0 \le f \le 1$, $f(\zeta) = 1$ and $f(z) < 1$ if $z \ne \zeta$; i.e. f is peak at ζ. Hence X is the set of peak points of $\ell ip_P(X,\alpha)$. Similarly, one can conclude that $S_0(\ell ip_R(X,\alpha)) = X$.

Note that by the hypothesis of this theorem, X will be polynomially convex (rationally convex).

REMARK 2. Let $P_0(X)$ be the algebra of polynomials in $z_1, \ldots, z_n$ on X and $R_0(X)$ be the algebra of rational functions with poles off X. Then this theorem remains valid when the hypothesis "$\ell ip_P(X,\alpha)$ ($\ell ip_R(X,\alpha)$) is self-adjoint "is replaced by the weaker hypothesis "$P_0(X)$ ($R_0(X)$) is self-adjoint".

COROLLARY 1. *Let $X = [z_1; z_2]$ be a straight line segment in the complex plane. Then the peak points of $\ell ip_P(X,\alpha)$ is X.*

Proof. Let $X = \{z = x + iy : y = ax + b, Rez_1 \le x \le Rez_2\}$ for some $a, b \in \mathbb{R}$. Then $\bar{z}$ is a polynomial in z on X. Thus for any polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$, its complex conjugate $\bar{p}(z)$ is a polynomial in z on X, that is, $P_0(X)$ is self-adjoint. Hence, by Remark 2, $S_0(\ell ip_P(X,\alpha)) = X$.

COROLLARY 2. *If $T = \{z \in \mathbb{C} : |z - z_0| = r\}$, then $S_0(\ell ip_R(T,\alpha)) = T$.*

Proof. Let z be the coordinate function on T. Then $\bar{z} = \frac{r^2}{z - z_0} + \bar{z}_0 \in R_0(T)$, in fact, $\bar{z}$ is a rational function with the only pole z_0. It follows clearly that $R_0(T)$ is self-adjoint. Hence, by Remark 2, $S_0(\ell ip_R(T,\alpha)) = T$.

The inverse of theorem 1 is not true:

EXAMPLE. Let T be unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Clearly $\ell ip_P(T, \alpha)$ is not self-adjoint and $\Gamma(\ell ip_P(T, \alpha)) = T$. But for every $\zeta \in T$, the function

$$f(z) = \frac{1}{2}(1 + z\bar{\zeta}) \quad (z \in T)$$

is a polynomial on T and it is peak at ζ since $|f(z)| = |\cos \frac{\theta - \alpha}{2}|$ where $z = e^{i\theta} \in T$ and $\zeta = e^{i\alpha}$. So $S_0(\ell ip_P(T, \alpha)) = T$.

Now, in general, we will determine the set of peak points of $\ell ip_R(X, \alpha)$, when X is a compact subset of the complex plane with planar measure zero.

THEOREM 2. *If X is a compact subset of the complex plane with planar measure zero, then the set of peak points of $\ell ip_R(X, \alpha)$ is X.*

Proof. Since the function $f(z) = \bar{z}$ has continuous partial derivatives in every neighbourhood of X and the planar measure of X is zero, $\bar{z} \in \ell ip_R(X, \alpha)$ [5, theorem 1]. For any polynomial $p(z)$, its complex conjugate, $\bar{p}(z)$ is a polynomial in $\bar{z}$, so $\bar{p} \in \ell ip_R(X, \alpha)$. By the naturality of $\ell ip_R(X, \alpha)$, if the polynomial $p(z)$ does not vanish on X then $\bar{p}(z)$ is invertible in $\ell ip_R(X, \alpha)$. Hence complex conjugate of any rational function with poles off X is an element of $\ell ip_R(X, \alpha)$. Since $\ell ip_R(X, \alpha)$ is $\ell ip(X, \alpha)$-clouser of $R_0(X)$, so $\ell ip_R(X, \alpha)$ is self-adjoint. Therefore by Theorem 1, the set of peak points of $\ell ip_R(X, \alpha)$ is X.

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