

## SOME GEOMETRIC APPLICATIONS OF EXTREMAL LENGTH(II)

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**Abstract.** In this paper, we introduce the extremal length and examine its properties and consider the applications of extremal length to conformal mappings. We obtain the theorems in the connection with “the extremal length zero” and “the fundamental sequences”.

### 1. Extremal length

Our tool is “extremal length”, a conformally invariance associated with a family of curves.

Let  $D$  be a domain in the complex plane,  $\{\gamma\}$  a family of curves of  $D$ , and let  $\rho(z)$  be a non-negative real-valued function defined on  $D$ . We set

$$A_D(\rho) = A(\rho) = \int \int_D \rho^2(z) dx dy.$$

Let  $\gamma$  be a curve of the family  $\{\gamma\}$ . If there is countable sequence  $\{\gamma_i\}$  of disjoint rectifiable arcs, which are parameterized by their arc lengths (i.e., the arc  $\gamma_i$  is given by  $z = z_i(s_i)$ , where  $0 \leq s_i \leq l_i$ , and  $l_i$  is the length of  $\gamma_i$ ), such that  $(\gamma) = \cup_i^\infty (\gamma_i)$ , we set

$$l_\rho(\gamma) = \sum_{i=1}^{\infty} \int_{\gamma_i} \rho(z_i(s_i)) ds_i.$$

We introduce

$$L(\rho) = L_{\{\gamma\}}(\rho) = \inf_{\gamma \in \{\gamma\}} l_\rho(\gamma),$$

where  $L_{\{\gamma\}}(\rho) = \infty$ , the family  $\{\gamma\}$  is empty.

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DEFINITION 1.1 ([1]). The function  $\rho(z)$  shall be called *admissible* with respect to  $D$  and  $\{\gamma\}$ , if  $A(\rho)$  and  $L(\rho)$  are not both zero or both infinite.

DEFINITION 1.2 ([1]). The quantity

$$(1) \quad \lambda_D(\gamma) = \lambda(\gamma) = \sup \frac{L^2(\rho)}{A(\rho)},$$

where the supremum is taken over all admissible functions  $\rho(z)$ , is called the *extremal length* of the family of curves  $\{\gamma\}$  with respect to the domain  $D$ .

PROPOSITION 1.3 ([9]). Let  $B$  be the interior of the annulus formed by two concentric circles of radii  $a$  and  $b$  ( $a < b$ ). The family  $\{\gamma\}$  of curves of  $B$  which connect the two circles has extremal length

$$(2) \quad \lambda(\gamma) = \frac{1}{2\pi} \log \frac{b}{a},$$

while the family  $\{\gamma^*\}$  of simple closed curves of  $B$  which separate the two boundary components has extremal length

$$(3) \quad \lambda(\gamma^*) = \frac{2\pi}{\log \frac{b}{a}}.$$

The following theorem is an immediate consequence of the definition.

THEOREM 1.4 ([1]). (Conformal invariance of extremal length) Let  $z^* = f(z)$  be a 1-1 conformal mapping on  $D$  upon a domain  $D^*$  and  $\{\gamma\}$  be a family of curves in  $D$ , then

$$\lambda(\gamma) = \lambda(f(\gamma)).$$

THEOREM 1.5. Let  $\{\gamma\}$  and  $\{\gamma'\}$  be two families of curves of a domain  $D$  such that each  $\gamma \in \{\gamma\}$  contains a  $\gamma' \in \{\gamma'\}$ . Then

$$\lambda(\gamma) \geq \lambda(\gamma').$$

*Proof.* Let  $\varepsilon > 0$  be given and let  $\rho'(z)$  be admissible for the family  $\{\gamma'\}$ , subject to the condition that

$$(4) \quad \frac{L^2_{\{\gamma'\}}(\rho')}{A(\rho')} > \lambda(\gamma') - \varepsilon.$$

Define a function  $\rho_0(z)$  in  $D$  by

$$\rho_0(z) = \begin{cases} \rho'(z) & z \text{ belongs to some } \gamma' \\ 0 & \text{otherwise} \end{cases}.$$

For  $\gamma \in \{\gamma\}$ , let  $\gamma'$  be one of the elements of  $\{\gamma'\}$  contained in  $\gamma$ . Then

$$(5) \quad l_{\rho_0}(\gamma) \geq L_{\{\gamma'\}}(\rho'),$$

since either  $l_{\rho_0}(\gamma) = \infty$ , or

$$l_{\rho_0}(\gamma) = \int_{\gamma} \rho_0 ds \geq \int_{\gamma'} \rho' ds = l_{\rho'}(\gamma') \geq L_{\{\gamma'\}}(\rho').$$

For each  $\gamma \in \{\gamma\}$ , (5) holds, then

$$(6) \quad L_{\{\gamma\}}(\rho_0) \geq L_{\{\gamma'\}}(\rho').$$

Also, we have

$$(7) \quad A(\rho_0) \leq A(\rho').$$

Show that  $\rho_0$  is admissible for the family  $\{\gamma\}$ . If  $\lambda(\gamma') = 0$ , the theorem holds trivially. Assume that  $A(\rho') < \infty$ , and  $A(\rho_0) < \infty$ .

If  $A(\rho_0) = 0$ , then either (i)  $A(\rho') = 0$ , or (ii)  $A(\rho') > 0$ . In case (i),  $L_{\{\gamma'\}}(\rho') > 0$ , since  $\rho'$  is admissible for  $\{\gamma'\}$ , while in case (ii), we choose  $\varepsilon$  so that  $0 < \varepsilon < \lambda(\gamma')$  in (4), and we conclude from that formula that  $L_{\{\gamma'\}}(\rho') > 0$ . In either case, therefore,  $A(\rho_0) = 0$  implies, by (6), that  $L_{\{\gamma\}}(\rho_0) > 0$ ; hence,  $\rho_0$  is admissible for  $\{\gamma\}$ . By (6) and (7), we obtain

$$\frac{L_{\{\gamma\}}^2(\rho_0)}{A(\rho_0)} \geq \frac{L_{\{\gamma'\}}^2(\rho')}{A(\rho')},$$

and therefore, by (4),  $\lambda(\gamma) > \lambda(\gamma') - \varepsilon$ , then

$$\lambda(\gamma) \geq \lambda(\gamma').$$

This completes the proof of the theorem.

COROLLARY 1.6. *In particular, if  $\{\gamma\}$  is contained in  $\{\gamma'\}$ , then*

$$\lambda(\gamma) \geq \lambda(\gamma').$$

## 2. Geometric applications of extremal length

An important application of extremal length is to the boundary correspondence between two simply connected domains which are mapped conformally on each other.

Throughout this section,  $D$  is a domain in the complex plane.

DEFINITION 2.1 ([1]). A *crosscut* of  $D$  is a Jordan curve  $\gamma$  in  $D$  which in both directions tends to a boundary point.

It is well known that  $D - \gamma$  consists of two simply connected components.

DEFINITION 2.2 ([1]). Choose a fixed  $z_0 \in D$  and consider sequences  $P = \{p_n\}$  of points in  $D$ . With the sequence  $P$ , we associate the family  $\Gamma_P$  of all clusters of crosscuts of  $D$  which separate  $z_0$  from almost all  $p_n$ . The sequence  $P$  is said to be *fundamental* if  $\lambda(\Gamma_P) = 0$ .

The definition is independent of the choice of  $z_0$ .

Recall that  $\lambda(\Gamma_P) = 0$  if and only if  $L(\rho) = 0$  for all  $\rho$  with  $A(\rho) < \infty$ .

LEMMA 2.3 ([1]). *Let  $D$  and  $D^*$  be simply connected domains in the complex plane, and consider sequence  $P = \{p_n\}$  of points in  $D$ . Let  $f$  be a conformal mapping from  $D$  to  $D^*$ . Then the sequences  $\{p_n\}$  and  $\{f(p_n)\}$  are simultaneously fundamental.*

LEMMA 2.4. *Let  $P = \{p_n\}$  be a fundamental sequence in  $D$ . Then all accumulation points of the  $P = \{p_n\}$  lie on the boundary of  $D$ .*

*Proof.* Let  $\alpha \in D$  be an accumulation point of  $P$ . Then  $\alpha \notin E$ , since otherwise  $\{\gamma\}$  would be empty, and  $\lambda(\gamma) = \infty$ , where  $E$  is the set of  $z_0$  in definition 2.2. Therefore, the open set  $D - E$  contains a closed disk  $F$  with center  $\alpha$  such that  $p_n \in F$  for an infinite number of indices. If  $\xi$  is a curve contained in  $D$  which connects  $\alpha$  to a point  $\beta$  of  $E$ , then every crosscut  $\gamma$  of the family  $\{\gamma\}$  intersects the continuum  $T = \xi \cup F$ .  $T$  has positive spherical distance  $\delta$  from the boundary. Let  $\rho^*(z) = \psi(z, \infty)$ , where  $\psi(z, w)$  is the spherical distance. Then, for each  $\gamma \in \{\gamma\}$ ,  $l_{\rho^*}(\gamma) \geq 2\delta$ . Hence  $L_{\{\gamma\}}(\rho^*) \geq 2\delta$ . Since  $0 < A(\rho^*) < \pi$ ,  $\rho^*$  is admissible. We have

$$\lambda(\gamma) \geq \frac{L^2(\rho^*)}{A(\rho^*)} \geq \frac{4}{\pi} \cdot \delta^2 > 0.$$

This contradiction shows that  $\alpha$  cannot be an interior point of  $D$ . This completes the proof.

LEMMA 2.5. *A sequence  $P = \{p_n\}$  of points of  $\Delta = \{z \mid |z| < 1\}$  is a fundamental if and only if it converges to a point  $\zeta = e^{i\theta}$  of the boundary of  $\Delta$ .*

*Proof.* Let  $\lim_{n \rightarrow \infty} p_n = \zeta$ , and set  $r_j = |p_j - \zeta|$ ,  $C_j = \{z \mid |z - \zeta| = r_j\}$ . Then the sequence  $\{r_j\}$  converges to 0. By restriction to a suitable subsequence of  $\{p_n\}$ , we assume that  $C_1$  separates  $C_n$ , ( $n > 1$ ) from  $z_0$ . Let  $\{\gamma\}$  be the family of crosscuts of  $\Delta$  which separate  $z_0$  from almost all  $p_n$ . For  $j$ , every simple closed curve, which separates the  $C_0$  and  $C_j$ , contains a crosscut which separates  $z_0$  from almost all  $p_n$  and which belongs to the family  $\{\gamma\}$ . Let  $\{\delta_k\}$  be the family of all such simple closed curves by Proposition 1.3 and Theorem 1.5, we have

$$\lambda(\gamma) \leq \lambda(\delta_k) = \frac{2\pi}{\log \frac{r_0}{r_k}}.$$

The right side converges to zero as  $k \rightarrow \infty$ , and the result,  $\lambda(\gamma) = 0$ . Thus,  $P = \{p_n\}$  is a fundamental sequence in  $D$ .

On the other hand, if  $\zeta_1 = e^{i\theta_1}$  and  $\zeta_2 = e^{i\theta_2}$  are distinct points of the boundary. Let  $\{p_m\}$  and  $\{q_n\}$  be sequences converging to  $\zeta_1$  and  $\zeta_2$ , respectively. Let  $z_0 = 0$ . Each member of the family  $\{\gamma\}$  of crosscuts of  $\Delta$  which separate  $z_0$  from almost all points of the join of our two sequences have Euclidean length at least  $|\zeta_1 - \zeta_2|$ . Denoting by  $\rho_0$  the function  $\rho_0(z) = 1$  for all  $z$  of  $\Delta$ , we have  $L_{\{\gamma\}}(\rho_0) \geq |\zeta_1 - \zeta_2|$ , while  $A(\rho_0) = \pi$ ; hence,

$$\lambda(\gamma) \geq \frac{1}{\pi} |\zeta_1 - \zeta_2|^2 > 0.$$

A sequence of points of  $\Delta$  with more than one accumulation point on the boundary cannot be a fundamental sequence. This completes the proof.

THEOREM 2.6. *Let  $D$  be a Jordan domain in the complex plane, and let  $w = f(z)$  be a conformal mapping on  $\Delta = \{z \mid |z| < 1\}$  upon  $D$ . Let  $w_0$  be a point on the boundary of  $D$ , and consider a sequence  $P = \{w_n\}$  of points of  $D$  converging to  $w_0$ . Let  $f^{-1}(P) = \{z_n\}$  be a inverse image of  $P$ . Then the sequence of points  $\{z_n\}$  has no point of accumulation in the interior of  $\Delta$ , and  $\{z_n\}$  has one and only one point of accumulation  $z_0$  on the boundary of  $\Delta$ .*

*Proof.* Since  $w_0$  is a point on the boundary of  $D$ ,  $P = \{w_n\}$  is a fundamental by Lemma 2.4. Thus,  $\{z_n\}$  is a fundamental by Lemma 2.3.

Therefore, point of accumulation  $z_0$  of  $\{z_n\}$  exist on the boundary of  $\Delta$ , by Lemma 2.4.

On the other hand,  $z_0$  is one and only one point of accumulation by Lemma 2.5. This completes the proof of the theorem.

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