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I–TORSION–FREE MODULES OVER PULLBACK RINGS

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1. Introduction

A ring is prime if $AB = 0$ for (2-sided) ideals A, B implies that $A = 0$ or $B = 0$. A ring is semi-prime if it has no non-zero nilpotent ideals. If A is an ideal in a semi-prime ring R, then the left and right annihilators of A, in R, have zero intersection with A (the squares of these intersections are zero) and hence they coincide. Therefore we will write merely $\mathrm{Ann}_R A$. Let R be any ring. A left R-module S is **I-torsion-free** if $JT = 0$, for some left non-zero submodule T of S and some ideal J, implies that $KJ = 0$ for some non-zero ideal K. For semi-prime rings this can be restated: if $\mathrm{ann}_R J = 0$ then $\mathrm{ann}_M J = 0$. Let S be a left R-module. For each (2-sided) ideal J in R, set $\mathrm{ann}_S J = \{s \in S : Js = 0\}$. An **affiliated submodule** of S is any submodule of the form $\mathrm{ann}_S J$ where J is an ideal of R maximal among the annihilators of non-zero submodules of S.

Systems of linear equations can be regarded as conjunctions of linear equations and repeated conjunction will be denoted by use of $\bigwedge$ (in the same way that $\sum$ is used for repeated addition). Let M be an R-submodule of N. Then M is **pure** in N if any finite system $\bigwedge_{i=1}^{n} \sum_{j=1}^{m} r_{ij} x_j$ $= c_i$ of equations over M (that is, R-linear equations with constants from M) with $r_{ij} \in R, c_i \in M$ which is solvable in N is also solvable in M. A module I is pure-njective if any (infinite) system of equations in I which is finitely solvable in I, is solvable in I [7, Theorem 2.8]. The module N is a **pure essential extension** of M if M is pure in N and for all non-zero submodules L of N, if $M \cap L = 0$ then $(L \oplus M)/L$ is not pure in N/L. A **pure-injective hull** $H(M)$ of a module M is a pure essential extension of M which is pure-injective. Every module has a pure-injective hull which is unique to isomorphism [8, Proposition 6].

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Let $v_1 : R_1 \to \bar{R}$ and $v_2 : R_2 \to \bar{R}$ be homomorphisms of two prime rings R_i onto a common prime ring $\bar{R}$. Denote the **pullback**

$$R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\} \tag{1}$$

by $(R_1 \overset{v_1}{\longrightarrow} \bar{R} \overset{v_2}{\longleftarrow} R_2)$. Then R is a ring under coordinate-wise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i and let $P = P_1 \times P_2$. Then $R/P \cong \bar{R} \cong R_i/P_i$, $i = 1, 2$. So P_1, P_2, and P are prime ideals over R_1, R_2, and R respectively and $P_1 P_2 = P_2 P_1 = 0$ (so R is not a prime ring). Furthermore, for $i \neq j$, the sequence $0 \to P_i \to R \to R_j \to 0$ is an exact sequence of R-modules (see [3]).

An R-module S is called **separated** (here $\bar{R}$ is semi-simple artinian) if there exists an R_i-module S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by $(r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2)$). Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2 S \to S/PS \leftarrow S/P_1 S)$ [3, Corollary 3.3] and $S \leq (S/P_2 S) \oplus (S/P_1 S)$. Also S is separated if and only if $P_1 S \cap P_2 S = 0$ [3, lemma 2.9]. A **separated representation** of an R-module M is an R-module epimorphism $\varphi : S \to M$ such that S is separated and such that, if φ admits a factorization $\varphi : S \overset{f}{\longrightarrow} S' \to M$ with S' separated, then f is one-to-one. The module $K = Ker(\varphi)$ is then an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [3, Proposition 2.3]. For undefined termes we refer to [2] and [6]. Our aim here to prove the following results:

2. Results

The notation below will be kept in this paper. Let R be the pullback ring as described in (1), let J be an ideal in R, and set

$$J_1 = \{r \in R_1 : (r, s) \in J \text{ for some } s \in R_2\}$$
$$J_2 = \{r \in R_2 : (r, s) \in J \text{ for some } r \in R_1\}.$$

Then for each i, $i = 1, 2$, J_i is an ideal in R_i and $J \subseteq J_1 \times J_2$. Put for simplicity $J \times 0 = (J, 0)$ and $0 \times J = (0, J)$. Moreovere, if $J^n = 0$ for some n then $J_1^n = 0 = J_2^n$. This shows that R is semi-prime since R_i is prime.

THEOREM 2.1. *The uniform dimension (or Goldie dimension) bimodule ${}_R R_R$ is equal to 2. In particular, the list of mimimal prime ideals of R are $(P_1, 0)$ and $(0, P_2)$.*

Proof. Since $R/(0,P_2) \cong R_1$ and $R/(P_1,0) \cong R_2$, so $(0,P_2)$ and $(P_1,0)$ are prime ideals of R. By [6, 2.15 p.45], it is enough to show that the list of annihilator ideals in R are:

$$R,\ 0,\ (P_1,0),\ (0,P_2).$$

Clearly, $\text{ann}_R(0) = R$. Let J be a non-zero ideal in R. We divided the proof into three cases:

case 1: Suppose $J_i \neq 0$, $i = 1,2$, and $Jr = 0$ where $r = (r_1,r_2) \in R$. Therefore $J_i r_i = 0$, and so $r_i = 0$ since R_i is a prime ring. This shows that $r = 0$, so $\text{ann}_R J = 0$.

case 2: $J_1 = 0$, $J_2 \neq 0$. If $(r_1,r_2) \in J$ then $r_1 = 0$ and $v_1(r_1) = 0 = v_2(r_2)$, so $J \subseteq (0,P_2)$. Thus $J(P_1,0) \subseteq (0,P_2)(P_1,0) = 0$, and hence $(P_1,0) \subseteq \text{ann}_R J$. To see that $\text{ann}_R J \subseteq (P_1,0)$, suppose that $(r_1,r_2)J = 0$. Then $r_2 J_2 = 0$, so $r_2 = 0$ since R_2 is prime. Thus $v_1(r_1) = 0$, so $(r_1,r_2) \subseteq (P_1,0)$, as required.

case 3: $J_1 \neq 0$, $J_2 = 0$. Applying the proof case 2 to this case the ideal $(0,P_2)$ obtained is $\text{ann}_R J = (0,P_2)$.

LEMMA 2.2. *Let* $R = (R_1 \xrightarrow{v_1} \bar{R} \xrightarrow{v_2} R_2)$ *be a pullback ring with* $Kerv_i = I_i$, $i = 1,2$. *Then* R_1 *and* R_2 *are prime rings if and only if*

1. *R is semi-prime; and*
2. *Every non-zero annihilator ideal of R different from R is either equal to $(I_1,0)$ or is equal to $(0,I_2)$.*

Proof. (1) and (2) are clear from Theorem 1. Conversely, to see that $R/(0,I_2) \cong R_1$ is prime, suppose that $AB \subseteq (0,I_2)$ for ideals A and B in R. Then $(I_1,0) = \text{ann}_R(0,I_2) \subseteq \text{ann}_R(AB)$, so $A(B(I_1,0) \subseteq (AB)\text{ann}_R(AB) = 0$. If $B(I_1,0) = 0$ then $B \subseteq (0,I_2)$. If $B(I_1,0) \neq 0$ then since $J(I_1,0) \subseteq (I_1,0)$ we have $(0,I_2) \subseteq \text{ann}_R(B(I_1,0))$. So by (2), $A \subseteq \text{ann}_R(B(I_1,0)) = (0,I_2)$. Thus R_1 is a prime ring. Similarly, R_2 is prime.

Why separated R-modules. the classification problem for the class indecomposable modules over a pullback ring R is classical and consists of two parts: 1) the description of all indecomposable separated modules over R and 2) the classification of all indecomposable non-separated modules over R by using indecomposable separated modules. Let

$$R = (R_1 \to \bar{R} \leftarrow R_2) \qquad (2)$$

be the pullback of two local dedekind domaims R_1, R_2 with maximal ideals P_1, P_2 and $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ a field, and let $\varphi : S \to M$ be a separated representation of M. By [5, ch. 11], The indecomposable finitely generated non-separated modules in terms of "moduled" graphs where each vertex is replaced by a separated indecomposable and where the kernel of the map S to M is defined in terms of the vertices where two edges meet (also see [1]).

EXAMPLE. Let R be the pullback ring as described in (2), and let $P_i = Rp_i$, $i = 1, 2$. Given the simple R-graph G

$$\begin{array}{ccccccc} \circ & \longrightarrow & \bullet & \Longrightarrow & \bullet & \longrightarrow & \bullet \\ 1 & & 2 & & 1 & & 2 \\ R_1 & & R_2/P_2^3 & & R_1/P_1^7 & & R_2/P_2^2 \end{array}$$

Set $S = (R_1 \to \bar{R} \leftarrow R_2/P_2^3) = Ra$ with $P_2^3 a = 0$ and $S' = (R_1/P_1^7 \to \bar{R} \leftarrow R_2/P_2^2) = Ra'$ with $P_1^7 a' = P_2^2 a' = 0$ (which are separated R-modules). Then one can form the non-separated module

$$M(G) = (S \oplus S')/R(p_2^2 a - p_1^6 a') = Rc + Rc'$$

where $c = a + R(p_2^2 a - p_1^6 a')$, $c' = a' + R(p_2^2 a - p_1^6 a')$, $P_2^3 c = 0 = P_1^7 c' = P_2^2 c'$, and $p_2^2 c = p_1^6 c'$ which is obtained by identifying the "P_2-part" of the socle of S_1 with the "P_1-part" of the socle of S_2.

PROPOSITION 2.3. *Let R be the pullback ring as described in (1) with $\bar{R}$ a semi-simple artinian ring. Then every left R-I-torsion-free module is a separated R-module, in fact, if $0 \longrightarrow K \longrightarrow S \overset{\varphi}{\longrightarrow} M \longrightarrow 0$ is a separated representation of M with M I-torsion-free then $S \cong M$.*

Proof. Let M be an I-torsion-free R-module, and let $]T_1 = \mathrm{ann}_M(P_1, 0)$, $T_2 = \mathrm{ann}_M(0, P_2)$. Then

$$T_1 \cap T_2 = \mathrm{ann}_M((P_1, 0) + (0, P_2)) = \mathrm{ann}_M P.$$

Since M is I-torsion-free, theorem 2.1 shows that $T_1 \cap T_2 = \mathrm{ann}_M P = 0$. As $(0, P_2)(P_1, 0)M = 0 = (P_1, 0)(0, P_2)M$, it follows that $(P_1, 0)M \cap (0, P_2)M \subseteq T_1 \cap T_2 = 0$, so M is separated by [3, Lemma 2.9].

Suppose that M is an I-torsion-free R-module. Then there is a factorization $\varphi : S \overset{\varphi}{\longrightarrow} M \overset{i}{\longrightarrow} M$ (i is the inclusion mapping) with M separated because every I-torsion-free is separated. So $\varphi : S \to M$ is one-to-one, hence $M \cong S$.

THEOREM 2.4. *Let R be the pullback ring as described in (1) with $\bar{R}$ a semi-simple artinian ring, and let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be a separated R-module.*

(i) *Each S_i is I-torsion-free as an R-module if and only if it is I-torsion-free as an R_i-module.*

(ii) *S is an R-I-torsion-free if and only if each S_i is an R_i-I-torsion-free.*

Proof. (i) Let S_1 be an R_1-I-torsion-free module, and let J be an ideal in R such that $\mathrm{ann}_R J = 0$. If $JT = 0$ for some left R-submodule T of S_1 (note that S_1 is a module over R) then $J_1 T = 0$. Then since $J_1 \neq 0$ (theorem 2.1) and R_1 is prime we have $\mathrm{ann}_{R_1} J_1 = 0$, hence $T = 0$ and S_1 is R-I-torsion-free. Conversely, let K be a non-zero ideal in R_1 and U a left R_1-submodule of S_1 such that $KU = 0$. As R_1 is a prime ring, this implies that $K(P_1, 0) \neq 0$. Set $L = K(P_1, 0) + (0, P_2)$. Then L is an ideal in R such that $LT = 0$, so $T = 0$ since S_1 is R-I-torsion-free, as required.

(ii) Let S be an R-I-torsion-free. By (i), it is enough to show that each S_i is R-I-torsion-free. Suppose J is an ideal in R such that $\mathrm{ann}_R J = 0$, and let $s_1 \in S_1$ such that $s_1 J = 0$. By [3, Lemma 2.9], we can consider $S \subseteq S_1 \oplus S_2$. Call the projection maps π_i. Let $s \in S$ have its 1th projection equal to s_1. So there is an element $s_2 \in S_2$ such that $s = (s_1, s_2)$. Then $(P_1, 0)Js \subseteq Js_1 = 0$. Since S is R-I-torsion-free $(P_1, 0)s = 0$, and hence $P_1 s_1 = 0$. It follows that $P(s_1, 0) = 0$, so $s_1 = 0$ since $\mathrm{ann}_S P = 0$. Thus S_1 is R_1-I-torsion-free. Similarly, S_2 is R_2-I-torsion-free. Let each S_i be an R_i-I-torsion-free, and let J be an ideal in R such that $\mathrm{ann}_R J = 0$, so $J_1 \neq 0$, $J_2 \neq 0$ (by Theorem 2.1). Suppose that $s = (s_1, s_2) \in \mathrm{ann}_S J$, so $J_i s_i = 0$, $i = 1, 2$. As each S_i is R_i-I-torsion-free, this implies that $s_i = 0$, so $s = 0$, as required.

PROPOSITION 2.5. *Let R be the pullback ring as described in (1) with $\bar{R}$ a field, and let S be R-I-torsion-free. Then the list of non-zero affiliated submodules of S different from S are:*

$$(0, P_2 S), \ (P_1 S, 0).$$

Proof. By 2.3, we can write $S = (S_1 \xrightarrow{f_1} \bar{S} \xrightarrow{f_2} S_2)$. Let J be an ideal in R such that $\mathrm{ann}_S J \neq 0$, $\mathrm{ann}_S J \neq S$. So either $J_1 = 0$, $J_2 \neq 0$ or $J_1 \neq 0$, $J_2 = 0$. We divided the proof into two cases.

Case 1: $J_1 = 0, J_2 \neq 0$. Clearly, $(0, P_2 S) \subseteq \mathrm{ann}_S J$. If $s = (s_1, s_2) \in \mathrm{ann}_S J$ then for each i, $J_i s_i = 0$, $s_1 = 0$. So $s_2 \in \mathrm{Ker} f_2 = P_2 S_2 \cong P_2 S$. It follows that $\mathrm{ann}_S J \subseteq (0, P_2 S)$, hence $\mathrm{ann}_S J = (0, P_2 S) = \mathrm{Ann}_S(0, P_2)$.

Case 2: $J_1 \neq 0, J_2 = 0$. By a similar argument as in case (1), $\mathrm{ann}_S J = (P_1S, 0) = \mathrm{Ann}_S(P_1, 0)$.

PROPOSITION 2.6. *Let R and S be as described in 2.5. Then $Ass(S) = (P_1, 0), (0, P_2)$.*

Proof. Let T be an R-submodule of S. First, we show that $\mathrm{ann}_R T = 0$ if and only if $T_1 \neq 0, T_2 \neq 0$ where

$$T_1 = \{t_1 \in S_1 : (t_1, t_2) \in T \text{ for some } t_2 \in S_2\}$$
$$T_2 = \{t_2 \in S_2 : (t_1, t_2) \in T \text{ for some } t_1 \in S_1\}$$

Let for each i, $T_i \neq 0$, and let $r = (r_1, r_2) \in \mathrm{ann}_R T$. Then $r_1T_1 = 0 = r_2T_2$, so $r_i = 0$, $i = 1, 2$, since over prime rings, every non-zero submodule of R_i-module T_i is faithful. Thus $\mathrm{ann}_R T = 0$. Conversely, if $T_1 = 0$ and $(t_1, t_2) \in T$ then $f_2(r_2) = 0$, so $T \subseteq (0, P_2S)$. Hence $(P_1, 0)T \subseteq (P_1, 0)(0, P_2S) = 0$, a contradiction. Second, suppose that $\mathrm{Ann}_R T \neq 0$. By above consideration we carry out the proof in two cases.

Case 1: $T_1 = 0, T_2 \neq 0$. Clearly, $(P_1, 0) \subseteq \mathrm{ann}_R T$. If $(r_1, r_2)T = 0$ then $r_iT_i = 0$, so $r_2 = 0$, $r_1 \in P_1$ and $(r_1, r_2) \in (P_1, 0)$, as required.

Case 2: $T_1 \neq 0, T_2 = 0$. By a similar argument as in case (1), $\mathrm{ann}_R T = (0, P_2)$, as required.

PROPOSITION 2.7. *Let R and S be as described in 2.5. Then $H = H(S)$ (the pure-injective hull of S) and $E(S)$ (the injective hull of S) are separated.*

Proof. By 2.3, it is enough to show that H and $E(S)$ are I-torsion-free. If L is a submodule of H and J an ideal of R such that $\mathrm{ann}_R J = 0$, then if $JL = 0$, $J(L \cap S) = 0$. Since S is R-I-torsion-free, $S \cap L = 0$. Assume that $L \neq 0$. Since S is pure-essential in H and $S \cap L = 0$, it follows that the embedding $(S \oplus L)/L$ into H/L is not pure. We derive a contradiction from this. There is a system of equations:

$$\bigwedge_{i=1}^{n} \sum_{j=1}^{m} r_{ij}x_j = a_i + L \quad \text{with} \quad r_{ij} \in R,\ a_i \in S,$$

which has a solution in H/L, say, $\bigwedge_{i=1}^{n} \sum_{j=1}^{m} r_{ij}(b_j + L) = a_i + L$, but not in $(S + L)/L$. Let $r \in J$. As $JL = 0$, the following is true in H:

$$\bigwedge_{i=1}^{n} \sum_{j=1}^{m} rr_{ij}b_j = ra_i.$$

Since S is pure in H, there are elements $c_i \in S$ such that

$$r(\bigwedge_{i=1}^{n}\sum_{j=1}^{m} r_{ij}c_j - a_i) = 0.$$

From this and $\mathrm{ann}_S J = 0$ we have $\bigwedge_{i=1}^{n}\sum_{j=1}^{m} r_{ij}(c_j + L) = a_i + L$, a contradiction. Finally, since for every non-zero submodule T of $E(S)$, $S \cap T \neq 0$ we have $E(S)$ is R-I-torsion-free.

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