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THE RIGIDITY FOR REAL HYPERSURFACES IN $P_3(\mathbb{C})$

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Abstract. We prove that a certain class of real hypersurfaces in $P_3(\mathbb{C})$ has the rigidity. Making use of this we classify all homogeneous real hypersurfaces in $P_3(\mathbb{C})$.

0. Introduction

Let $P_n(\mathbb{C})$ denote a complex n-dimensional complex projective space. The second named author , I-B.Kim and B.H.Kim [4] proved the following rigidity theorem.

THEOREM A. *Let M be a $(2n-1)$-dimensional connected Riemannian manifold, and ι and $\hat{\iota}$ be two isometric immersions of M into $P_n(\mathbb{C})(n \geq 4)$. If the type number of (M, ι) or $(M, \hat{\iota})$ is not equal to 2 at every point of M, then ι and $\hat{\iota}$ are rigid, that is, there exists a unique isometry φ of $P_n(\mathbb{C})$ such that $\varphi \circ \iota = \hat{\iota}$.*

The purpose of this paper is to prove that Theorem A holds also for $n = 3$, which is stated as Theorem 1 in § 2. Making use of this, we obtain the following classification theorem.

THEOREM 2. *Let M be a 5 - dimensuional connected homogeneous Riemannian manifold. If M admits an isometric immersion into $P_3(\mathbb{C})$, then $\iota(M)$ is congruent to one of the so-called model spaces of type A_1 or A_2 or B ([3]).*

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1. Preliminaries

We fix the Fubini - Study metric on $P_n(\mathbb{C})$ such that its constant holomorphic sectional curvature is a positive constant 4. Let M be a 5 - dimensional Riemannian manifold, and ι an isometric immersion of M into $P_3(\mathbb{C})$.

In the sequal, the indices i, j, k, ℓ run over the range $\{1, 2, 3, 4, 5\}$ unless otherwise stated. Take an orthonormal frame field $\{e_i\}$ locally defined on M, and denote its dual forms by $\{\theta_i\}$. Then the connection forms θ_{ij} and the curvature forms Θ_{ij} of M with respect to $\{e_i\}$ are defined by

$$d\theta_i + \sum \theta_{ij} \wedge \theta_j = 0, \quad \theta_{ij} + \theta_{ji} = 0,$$

$$\Theta_{ij} = d\theta_{ij} + \sum \theta_{ik} \wedge \theta_{kj},$$

respectively.

Let I be the complex structure of $P_3(\mathbb{C})$, and ν be a unit normal vector field locally defined on $\iota(M)$. Then the almost contact structure (ϕ, ξ) of (M, ι) is defined by

$$I(\iota_* e_i) = \sum \phi_{ji} \iota_* e_j + \xi_i \nu,$$

where it is the differential of i.

Then (ϕ, ξ) satisfies

$$\sum \phi_{ik}\phi_{kj} = \xi_i \xi_j - \delta_{ij}, \tag{1.1}$$

$$\sum \phi_{ij}\xi_j = 0, \tag{1.2}$$

$$\sum {\xi_i}^2 = 1. \tag{1.3}$$

Later, we shall quote (1.1) (*resp.*(1.2)) in the form $(1.1)_{ij}$(resp. $(1.2)_i$) when we need a precise explanation, and do not quote (1.3) to avoid too often repeats. The vector field $\xi = \sum \xi_i e_i$ is uniquely determined on M up to sign, which is called the structure vector field of (M, ι).

Denoting the shape operator or the second fundamental tensor by $A = (A_{ij})$, we have the equation of Gauss ;

$$\begin{aligned}\Theta_{ij} &= \sum A_{ik}A_{j\ell}\theta_k \wedge \theta_\ell + \theta_i \wedge \theta_j \\ &\quad + \sum(\phi_{ik}\phi_{j\ell} + \phi_{ij}\phi_{k\ell})\theta_k \wedge \theta_\ell.\end{aligned} \tag{1.4}$$

The rank of A is called the type number of (M, ι).

For another isometric immersion $\hat{\iota}$ of M into $P_5(\mathbb{C})$, we shall denote the differential forms and tensor fields of $(M, \hat{\iota})$ by the same symbols as ones of (M, ι) but with a hat. Then, since $\theta_i = \hat{\theta}_i$ and $\Theta_{ij} = \hat{\Theta}_{ij}$, we have from (1.4)

$$\begin{aligned}&A_{ik}A_{j\ell} - A_{i\ell}A_{jk} + (\phi_{ik}\phi_{j\ell} - \phi_{i\ell}\phi_{jk} + 2\phi_{ij}\phi_{k\ell}) \\ &= \hat{A}_{ik}\hat{A}_{j\ell} - \hat{A}_{i\ell}\hat{A}_{jk} + (\hat{\phi}_{ik}\hat{\phi}_{j\ell} - \hat{\phi}_{i\ell}\hat{\phi}_{jk} + 2\hat{\phi}_{ij}\hat{\phi}_{k\ell}).\end{aligned} \tag{1.5}$$

Hereafter we use the following index convention :

$$i, j, k = 1, 2, 3, 4\ ;\ p, q, r = 1, 2, 3$$

unless otherwise stated and adopt a special orthonormal frame field $\{e_1, ..., e_5\}$ such that

$$\hat{A}_{i5} = 0, \tag{1.6}$$

$$A_{p5} = 0, \tag{1.7}$$

$$\hat{\phi}_{15} = \hat{\phi}_{25} = 0, \tag{1.8}$$

$$\phi_{14} = 0. \tag{1.9}$$

This is possible. In fact, we define a subset V of M by

$$V = \{x \in M; \sum \hat{A}_{i5}e_5 + \hat{A}_{55}e_5 \neq 0 \ \ at \ \ x\}$$

and choose a new orthonormal frame field $\{\widetilde{e}_1, ..., \widetilde{e}_5\}$ on V so that $\widetilde{e}_5$ is in the direction of $\sum \hat{A}_{i5}e_i + \hat{A}_{55}e_5$. Denote this new orthonormal frame field by the same letter $\{e_1, ..., e_5\}$. Then (1.6) holds on V with respect to the new one. Of course, (1.6) holds also on $M - \bar{V}$ if it is not empty. In the future, using (1.6), we shall prove that e_5 is a common principal direction of ι and $\hat{\iota}$ everywhere on $V \cup (M - \bar{V})$, and so everywhere on M. Thus we may assume that (1.6) holds on whole M. In the sequel we shall omit this argument. Next, we choose a further new orthonormal frame field $\{\widetilde{e}_1, ..., \widetilde{e}_5\}$ so that $\widetilde{e}_5 = e_5$ and $\widetilde{e}_4$ is in the direction of $\sum A_{i5}e_i$, .. etc. In such a way, we have a desired frame field.

Now from (1.5) $\sim$ (1.7) we have

$$(1.10) \qquad \begin{aligned} &\phi_{p4}\phi_{q5} - \phi_{p5}\phi_{q4} + 2\phi_{pq}\phi_{45} \\ &= \hat{\phi}_{p4}\hat{\phi}_{q5} - \hat{\phi}_{p5}\hat{\phi}_{q4} + 2\hat{\phi}_{pq}\hat{\phi}_{45}, \end{aligned}$$

$$(1.11) \qquad \begin{aligned} &\phi_{pr}\phi_{q5} - \phi_{p5}\phi_{qr} + 2\phi_{pq}\phi_{r5} \\ &= \hat{\phi}_{pr}\hat{\phi}_{q5} - \hat{\phi}_{p5}\hat{\phi}_{qr} + 2\hat{\phi}_{pq}\hat{\phi}_{r5}. \end{aligned}$$

Putting $(i, j, k) = (p, 4, q), (p, 4, 4), (p, 5, q), (p, 5, 4)$ and $(4, 5, 4)$ in (1.5), we have

$$(1.12) \qquad \begin{aligned} A_{pq}A_{45} = &-\phi_{pq}\phi_{45} + \phi_{p5}\phi_{4q} - 2\phi_{p4}\phi_{q5} \\ &+ \hat{\phi}_{pq}\hat{\phi}_{45} - \hat{\phi}_{p5}\hat{\phi}_{4q} + 2\hat{\phi}_{p4}\hat{\phi}_{q5}, \end{aligned}$$

$$(1.13) \qquad A_{p4}A_{45} = -3\phi_{p4}\phi_{45} + 3\hat{\phi}_{p4}\hat{\phi}_{45},$$

$$(1.14) \qquad A_{pq}A_{55} = -3\phi_{p5}\phi_{q5} + 3\hat{\phi}_{p5}\hat{\phi}_{q5},$$

$$(1.15) \qquad A_{p4}A_{55} = -3\phi_{p5}\phi_{45} + 3\hat{\phi}_{p5}\hat{\phi}_{45},$$

$$(1.16) \qquad A_{44}A_{55} - {A_{45}}^2 = -3{\phi_{45}}^2 + 3{\hat{\phi}_{45}}^2,$$

respectively.

Taking the symmetric part of (1.12) we have

$$(1.17) \qquad \begin{aligned} 2A_{pq}A_{45} = &-3(\phi_{p4}\phi_{q5} + \phi_{q4}\phi_{p5}) \\ &+ 3(\hat{\phi}_{p4}\hat{\phi}_{q5} + \hat{\phi}_{q4}\hat{\phi}_{p5}). \end{aligned}$$

By the same method as the one in the proof of Lemma 3.1 in [2] we obtain from (1.11)

$$(1.18) \qquad \phi_{pq}\phi_{r5} = \hat{\phi}_{pq}\hat{\phi}_{r5}.$$

2. Theorems and their proofs

First we prove

THEOREM 1. *Let M be a 5− dimensional connected Riemannian manifold, and ι and $\hat{\iota}$ be two isometric immersions of M into $P_3(\mathbb{C})$. If the type number of (M,ι) or $(M,\hat{\iota})$ is not equal to 2 at every point of M, then ι and $\hat{\iota}$ are congruent, that is, there exists an isometry φ of $P_3(\mathbb{C})$ such that $\varphi\circ\iota=\hat{\iota}$.*

Proof. We may assume that rank $A \geq 3$. In fact, it is known ([3]) that there is not a non-empty open set U on M such that rank $A \leq 1$ on U or rank $\hat{A} \leq 1$ on U. Thus, by exchanging the roles of ι and $\hat{\iota}$ if necessary, we can set in such a way that for any point p of M there is an open neighborhood V of p such that rank $A \geq 3$ on V, which proves our assersion.

If $A_{45}=0$, then from (1.6) and (1.7) we see that the vector e_5 is a common principal direction of ι and $\hat{\iota}$. Then, the main theorem in [2] implies that ι and $\hat{\iota}$ are congruent.

In the remainder we assume

$$A_{45}\neq 0, \tag{2.1}$$

and derive a contradiction.

First we have from (1.14) and $(1.17)_{11}$

$$A_{11}=0,\ \phi_{15}=0. \tag{2.2}$$

This and $(1.14)_{12}$ give

$$A_{12}=0. \tag{2.3}$$

Hereafter, to avoid the complexity of indices, we put $a=\phi_{12}, b=\phi_{13}, \alpha=\phi_{14}, x=\phi_{15}, c=\phi_{23}, \beta=\phi_{24}, y=\phi_{25}, \gamma=\phi_{34}, z=\phi_{35}$, and $w=\phi_{45}$. Remark that we already set $\alpha=\hat{x}=\hat{y}=0$ and showed $x=0$. Now (1.10) and (1.18) amount to

$$aw=\hat{a}\hat{w}, \tag{2.4}$$

$$2bw=\hat{\alpha}\hat{z}+2\hat{b}\hat{w}, \tag{2.5}$$

$$\beta z-\gamma y+2cw=\hat{\beta}\hat{z}+2\hat{c}\hat{w}, \tag{2.6}$$

$$ay=by=cy=0, \tag{2.7}$$

$$az=\hat{a}\hat{z},\ bz=\hat{b}\hat{z},\ cz=\hat{c}\hat{z}. \tag{2.8}$$

Assertion 1. $A_{55} \neq 0$.

If $A_{55} = 0$, then (1.8), $(1.14)_{22}$ and $(1.14)_{33}$ imply $y = 0$ and $z^2 = \hat{z}^2$. Moreover, $(1.15)_3$ and (1.16) imply $zw = \hat{z}\hat{w}$ and $w^2 \neq \hat{w}^2$. These yield $z = \hat{z} = 0$. If follows from (1.17) that $A_{pq} = 0$, which gives rise to a contradiction rank $A \leq 2$. □

Now from $(1.14)_{13}$ and $(1.15)_1$ we have

$$A_{13} = A_{14} = 0. \tag{2.9}$$

Assertion 2. $\hat{\alpha} = 0$.

In fact, from $(1.13)_1$ and $(1.17)_{13}$ we have $\hat{\alpha}\hat{w} = \hat{\alpha}\hat{z} = 0$. If $\hat{\alpha} \neq 0$, then we have $\hat{z} = \hat{w} = 0$. If $a = b = c = 0$, then from (2.6) we see $\beta z - \gamma y = 0$, which gives rise to a contradiction rank $\phi = 3$.

Thus $a^2 + b^2 + c^2 > 0$. Then from (2.4) $\sim$ (2.8) we have $y = z = w = 0$. This, (1.4) and (1.15) give $A_{pq} = 0$ and $A_{p4} = 0$, which contradicts rank$A \leq 2$. □

Assertion 3. $y = 0$.

Assume $y \neq 0$. Then from (2.7) we have $a = b = c = 0$ and so $e_1 = \xi$. If $\hat{z} \neq 0$ or $\hat{w} \neq 0$, then from (2.4) and (2.5) we have $\hat{a} = \hat{b} = 0$ and so $e_1 = \hat{\xi} = \xi$. Then by a theorem in [1], ι and $\hat{\iota}$ are congruent. In particular, $A_{45} = \hat{A}_{45} = 0$, which is a contrdiction. Therefore we have $\hat{z} = \hat{w} = 0$. This and (2.6) give rise to a contradiction rank $\hat{\phi} \leq 3$. □

Now from $(1.14)_{22}, (1.14)_{23}$ and $(1.15)_2$ we have

$$A_{22} = A_{23} = A_{24} = 0. \tag{2.10}$$

This, together with $(1.13)_2$ and $(1.17)_{23}$, gives

$$\beta z = \hat{\beta}\hat{z}, \ \beta w = \hat{\beta}\hat{w}. \tag{2.11}$$

Assertion 4. *Neither of the following cases can occur.*

$$z^2 = \hat{z}^2, \ zw = \hat{z}\hat{w}. \tag{1}$$

$$\gamma = \hat{\gamma} = 0. \tag{2}$$

$$z = \hat{z} = 0. \tag{3}$$

In fact, since rank $A \geq 3$, from (2.2), (2.3), (2.9) and (2.10) we find ${A_{33}}^2 + {A_{34}}^2 > 0$. Then our assertion follows from (1.7), $(1.13)_3$, $(1.14)_{33}$, $(1.15)_3$ and $(1.17)_{33}$. □

Assertion 5. *For a non - zero real number ε with $\varepsilon^2 \neq 1$, we have $\hat{a} = \varepsilon a$, $\hat{b} = \varepsilon b$, $\hat{c} = \varepsilon c$, $\hat{\beta} = \varepsilon\beta$, $z = \varepsilon\hat{z}$, and $w = \varepsilon\hat{w}$.*

In fact, first we shall show $\hat{z} \neq 0$. If $\hat{z} = 0$, then $z \neq 0$ by (3) of Assertion 4. Then from (2.8) and (2.11) we find $a = b = c = \beta = 0$, which contradicts rank $\phi = 4$. Similarly we have $z \neq 0$. By a similar argument, we see $w = 0$ if and only if $\hat{w} = 0$. Then, putting $\varepsilon = z/\hat{z}$, we have assertion 5 from (2.4), (2.5), (2.6), (2.8) and (2.11). From (1) of Assertion 4 we have $\varepsilon^2 \neq 1$. □

Assertion 6. $b\hat{b} \neq 0$.

Assume $b = 0$. Then $\hat{b} = 0$ and $a\hat{a} \neq 0$ since $\xi \neq \hat{\xi}$ as in a previous argument. From $(1.1)_{15}$ and $\widehat{(1.1)}_{15}$ we have $\xi_1\xi_5 = \hat{\xi}_1\hat{\xi}_5 = 0$. The case $\xi_1 = \hat{\xi}_1 = 0$ can not occur since if not so, then we have $\varepsilon^2 = 1$ from $a^2 = \hat{a}^2 = 1$. Then case $\xi_5 = \hat{\xi}_5 = 0$ can not also occur since if not so, then we have $\gamma = \hat{\gamma} = 0$ by $(1.1)_{45}$ and $\widehat{(1.1)}_{45}$, which contradicts (2) of Assertion 4 . Hence $\xi_1\hat{\xi}_5 \neq 0$, and so $\xi_5 = \hat{\xi}_1 = 0$. It follows from $\widehat{(1.1)}_{14}$ and $(1.1)_{25}$ that $\hat{\beta} = c = 0$ and so $\beta = \hat{c} = 0$, which contrdicts $(1.2)_2$. □

Assertion 7. $a\hat{a} \neq 0$.

Assume $a = 0$. Then $\hat{a} = 0$. Multiply $(1.1)_{45}$ by ${\xi_1}^2$ and use $(1.1)_{14}$ and $(1.1)_{15}$. Then we have $\gamma = 0$. Similarly we have $\hat{\gamma} = 0$, which contradicts (2) of Assertion 4. □

Assertion 8. $\hat{\gamma} = \varepsilon\gamma$.

In fact, multiply $(1.2)_2$ by ξ_5 and use $(1.1)_{15}$, $(1.1)_{35}$ and $(1.1)_{45}$. Then we have

$$abz + (\beta z - cw)\gamma = 0. \tag{2.12}$$

Similarly we have

$$\hat{a}\hat{b}\hat{z} + (\hat{\beta}\hat{z} - \hat{c}\hat{w})\hat{\gamma} = 0.$$

This and Assertion 5 imply

$$\hat{a}bz + (\beta z - cw)\hat{\gamma} = 0. \tag{2.13}$$

It follows from (2.12) and (2.13) that $\hat{a}\gamma = a\hat{\gamma}$. □

Now, applying Assertions 5 and 8 to $\widehat{(1.2)}$, we see that the vector $(\hat{\xi}_1, ..., \hat{\xi}_4, \hat{\xi}_5/\varepsilon^2)$ belongs to the kernel of ϕ. Thus there is a scalar λ such that $\hat{\xi}_i = \lambda\xi_i (i = 1,2,3,4)$ and $\hat{\xi}_5 = \lambda\varepsilon^2\xi_5$. Then from (2.20) and $\widehat{(2.20)}$ we obtain $\lambda^2 = 1$ and so a contradiction $\varepsilon^2 = 1$, which proves Theorem 1.

The proof of Theorem 2 can be done in the same way as the one in [4].

Remark 2.1. Theorem A holds also for a real hypersurface in the hyperbolic complex space form $H_n(\mathbb{C})$.

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