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HYPERBOLIC HOMOTHETIC MOTIONS OF CONICS

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1. Introduction

Let G be an affine group and V be an affine algebric set. We recall that an action of G on V is given by a mapping

$$\cdot : K \times V \to V \text{ denoted by } (k, v) \to k \cdot v$$

such that (i) $(k \cdot t) \cdot v = k \cdot (t \cdot v)$ and (ii) $1 \cdot v = v$ for all $k, t \in K$, $v \in V$, 1 being the identity of G. The orbit of an element v of V is

$$K \cdot v = \{k \cdot v \mid k \in K\}.$$

An element $v \in V$ is G invariant or, simply, invariant if $k \cdot v = v$ for all $g \in G$[1,2,3].

The study of the Euclidean motions of conics, by defining the action of the Euclidean group on conics and using some matrix techniques, was given in [3].

By analogy with the Euclidean case, Mendes and Ruas have studied the behavior of conics in the plane under the Lorentzian group action[4].

The present paper deal with the hyperbolic homothetic motion of conics. By defining the hyperbolic homothetic group we analyse its action on a conic. We present h-invariant polynomials and show that certain polynomials in the coefficients of the conics are h-invariant under the group mentioned above. Some geometric interpretation related to standard form of conics are also given. Finally the h-invariant subalgebra I, generated by some polynomials $\tilde{\tau}, \delta, \Delta$ are investigated.

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2. A hyperbolic homothetic group action

A 2×2 matrix N is called h-special hyperbolic homothetic if there exists h and θ such that

$$N = N(\theta, h) = h \begin{pmatrix} ch\theta & sh\theta \\ sh\theta & ch\theta \end{pmatrix}, \quad h, \theta \in IR, h \neq 0 \tag{2.1}$$

where $ch\theta, sh\theta$ are usual hyperbolic functions.

The set of all such matrices, with usual multiplication constitutes a group, called the group of hyperbolic homothetic rotations, or proper homothetic Lorentz group.

Each $N \in G$ defines a linear transformation

$$N : IL^2 \to IL^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix} = N \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then the Lorentz inner product is defined by $< \vec{v}, \vec{w} >= x_1x_2 - y_1y_2$, where $\vec{v} = (x_1, y_1)$, $\vec{w} = (x_2, y_2)$.

It follows easily that the eigenvectors of N are $\vec{e} = (1, -1)$ and $\vec{e} = (1, 1)$ and corresponding eigenvalues $\lambda_1 = h(ch\theta - sh\theta)$ and $\lambda_2 = h(ch\theta + sh\theta)$ are positive real numbers.

Let I, II, III, IV, V and VI be the sets in figure 1. The I ∪ III, II ∪ IV and V ∪ VI remain h-invariant, that is, if $(x, y) \in A$, then $N(x, y) \in A$, $\forall N \in G$, where A denotes any one of these sets.

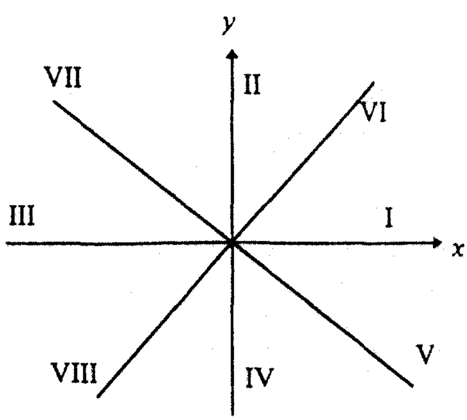

Figure1

To see, for instance, that I $\cup$ III G h-invariant, let $\vec{v} = (x, y)$ be any element in this set(In I or in III). Then $\vec{v} = \alpha\vec{e}_1 + \beta\vec{e}_2$, where α and β are positive numbers. Hence, $N\vec{v} = (\alpha\lambda_1)\vec{e}_1 + (\beta\lambda_2)\vec{e}_2$ with $\alpha\lambda_1$, $\beta\lambda_2$ are positive and so $N\vec{v} \in$ I or $N\vec{v} \in$ III.

Vectors in I or III are called spacelike, in II or IV are timelike, and in V or VI are null.

Furthermore, the points of the G h-invariant sets,, I $\cup$ III, II $\cup$ IV, slide along the hyperbolas $x^2 - y^2 =$ constant. Now, $\lambda_1\lambda_2 = h^2$ for all θ. If $\theta > 0$, the plane shrinks λ_1 times to the straight line $x = -y$ and stretches in the orthogonal direction away from $x = y$. When $\theta < 0$, we just reverse the directions of stretching and compression, and the directin of motion of the points along the hyperbolas.

In analogy with the Euclidean case [3] and with the hyperbolic case [4], we shall denote by K, consists of all 3×3 real matrices of the form

$$k(\theta, h) = \begin{pmatrix} N(\theta) & B \\ 0 & 1 \end{pmatrix}, \tag{2.2}$$

where $N(\theta)$ is a special hyperbolic homothetic matrix and B is a 2×1 column matrix.

We can identify the plane IR^2 with the plane $z = 1$ in IR^3, that is, we can write

$$p = \begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}. \tag{2.3}$$

Then each matrix $k \in K$ gives rise to a motion $M_k(p) = kP = Np + B$.

Let V be the vector space of all 3×3 real symmetric matrices. For $k \in K$ and $Q \in V$, we define

$$k \cdot Q = (k^{-1})^t Q k^{-1}. \tag{2.4}$$

It is easy to verify that (2.2) gives an action of G on V.

We want to study the orbit under G of an element Q in V. To do this let us denote a typical element Q in V by

$$Q = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} = \begin{pmatrix} A(Q) & D(Q) \\ D^t(Q) & f(Q) \end{pmatrix}, \tag{2.5}$$

where $A(Q)=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, $D(Q)=\begin{pmatrix} d \\ e \end{pmatrix}$ and $f(Q)=f$.

We can identify a matrix $Q \in V$ with the polynomial $Q(P)$ $=Q(x,y)$ $=ax^2$ $+2bxy+cy^2+2dx+2ey+f$. Note that $Q(p)=P^tQP$.

Let $Q'=k\cdot Q=(k^{-1})^tQk^{-1}$ and C, C' be the conics associated to Q, Q', respectively; that is $C=\{p\,|\,Q(p)=0\}$ and $C'=\{p\,|\,Q'(p)=0\}$. The equation

$$Q(p)=P^tQP=P^tk^tQ^tkP=(kP)^tQ'(kP)=Q'(kp)$$

shows that p lies on C if and only if $M_k(p)=kp$ is on C'. The following explicit expression can be given for the matrix $A(Q')$:

$$(2.6)\quad A(Q')=h^2\begin{pmatrix} ach^2\theta+bsh2\theta+csh^2\theta & \left(\frac{(a+c)}{2}\right)sh2\theta+bch2\theta \\ \left(\frac{(a+c)}{2}\right)sh2\theta+bch2\theta & ash^2\theta+bsh2\theta+csh^2\theta \end{pmatrix}.$$

3. h-invariant polynomials on K

Let $R=IR[z_1,z_2,z_3,z_4,z_5,z_6]$ be the polynomial ring in six indeterminates over the reals. We can identify R with the ring of polynomial functions on V by defining $P(Q)=P(a,b,c,d,e,f)$, where $P=P(z_1,z_2,z_3,z_4,z_5,z_6)$ is in R and Q is in V.

A polynomial P in R is called K h-invariant with respect to the action of the hyperbolic homothetic group if $P(k\cdot Q)=\sum_{i=1}^{n}p_i$ for all $k\in K$ and $Q\in V$, where

$$p_i=p_i(k\cdot Q)=h^{t_i}p_i(Q),\quad t_i\in Z.$$

Thus p_i's, $(1\le i\le n)$, are also h-invariant polynomials. The set of such h-invariants polynomials, denoted by I, is an algebra over reals.

Now, we define three polynomials in R and show that they are K h-invariant. Let

$$\delta(Q)=\det A(Q),\quad \Delta(Q)=\det(Q)\text{ and }\tilde{\tau}(Q)=a-c.$$

LEMMA 3.1. *δ,Δ and $\tilde{\tau}$ are K h-invariant.*

Proof. We see that $\tilde{\tau}$ (respectively, δ) K h-invariant by taking the trace(respectively, determinant) of the equation $A(Q')=NA(Q)N$. The h-invariance of Δ follows by taking the determinant of both sides of $Q'=k\cdot Q=(k^{-1})^tQk^{-1}$.

4. Canonical forms

As in (2.5), let

$$Q = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \text{ and } A(Q) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Under the hyperbolic homothetic motions we want to find conditions to reduces $A(Q)$ to a diagonal form. As we shall see the possibility of this reduction will depend upon the sign of

$$\Lambda(Q) = [\tilde{\tau}(Q)]^2 + 4\delta(Q) = (a-c)^2 + 4(ac-b^2) = (a+c)^2 - 4b^2.$$

So a Lemma follows.

LEMMA 4.1. *If $b \neq 0$, $\Lambda(Q) > 0$ if and only if $|\frac{a+c}{2b}| > 1$ and $\Lambda(Q)$ is h-invariant.*

PROPOSITION 4.1. *If $\Lambda(Q) > 0$, then in the orbit of Q there exists a Q' which is of the form*

$$Q' = \begin{pmatrix} a' & 0 & d' \\ 0 & c' & e' \\ d' & e' & f' \end{pmatrix}.$$

Proof. We can assume $b \neq 0$. From (2.6) we need θ such that

$$b' = h^2 \left\{ \left(\frac{a+c}{2} \right) sh2\theta + bch2\theta \right\} = 0,$$

or equivalently

$$coth2\theta = -\frac{a+c}{2b}.$$

This is possible if and only if $|\frac{a+c}{2b}| > 1$, or by the Lemma 4.1, $\Lambda(Q) > 0$.

Now, to obtain Q', we take $k^{-1} = \begin{pmatrix} N(\theta, h) & 0 \\ 0 & 1 \end{pmatrix}$, as in (2.2).

The diagonal element a' and c' are solutions of the system

$$\begin{cases} a' - c' & = h^2 \tilde{\tau}(Q) = h^2(a-c) \\ a'c' & = h^4 \delta(Q) = h^4(ac - b^2) \end{cases}.$$

Thus from (2.6) we get ${a'}^2 - {c'}^2 = (a'-c')(a'+c') = h^4 \frac{(a-c)}{(a+c)} \Lambda(Q) ch2\theta$. Since $\Lambda(Q) > 0$, it follows that the sign of $(a')^2 - (c')^2$ is eqaul to the sign $a^2 - c^2$. These conditions determine a' and c' uniquely.

PROPOSITION 4.2. *If $\Lambda(Q) <)(b \neq 0)$, then the orbit of Q contains the matrix*

$$Q' = \begin{pmatrix} h^2\tilde{\tau}(Q) & \pm h^2\sqrt{-\delta(Q)} & d' \\ \pm h^2\sqrt{-\delta(Q)} & 0 & e' \\ d' & e' & f \end{pmatrix}$$

(where the sign of the square root is equal to the sign of b).

Proof. We want to make $c' = 0$.

It follows from (2.6) that we need θ such that

$$\text{(4.1)} \qquad \begin{cases} a' & = h^2\{ach^2\theta + bsh2\theta + csh^2\theta\} \\ b' & = h^2\{\frac{a+c}{2}sh2\theta + bch2\theta\} \\ 0 & = h^2\{ash^2\theta + bsh2\theta + cch^2\theta \end{cases} .$$

The h-invariance of $\tilde{\tau}(Q)$ and $\delta(Q)$ imply $a' = h^2(a-c) = h^2\tilde{\tau}(Q)$ and $b' = \mp h^2\sqrt{-\delta(Q)}$. We shall be assuming that b' has the same sign as b, that is, $bb' > 0$. Because of the first and last equation of (4.1) are linearly dependent we obtain

$$\text{(4.2)} \qquad \begin{cases} b' & = h^2\{\frac{a+c}{2}sh2\theta + bch2\theta\} \\ 0 & = h^2\{ash^2\theta + bsh2\theta + cch^2\theta \end{cases} ,$$

or

$$\begin{cases} 2b' & = h^2\{(a+c)sh2\theta + 2bch2\theta\} \\ a-c & = 2bsh2\theta + (a+c)ch2\theta \end{cases} .$$

For a moment, let us consider the system:

$$\text{(4.3)} \qquad \begin{cases} 2b' & = h^2\{(a+c)y + 2bx\} \\ a-c & = 2by + (a+c)x \end{cases} .$$

Since the determinant $\Lambda(Q) = (a+c)^2 - 4b^2 < 0$, (4.3) has unique solutions

$$x = \frac{h^2(a^2-c^2) - 4bb'}{h^2\Lambda(Q)} \text{ and } y = \frac{2b'(a+c) - 2bh^2(a-c)}{h^2\Lambda(Q)}.$$

It is easy to show that $x^2 - y^2 = 1$. To finish the proof, we must show that $x \geq 0$. In fact, since $\Lambda(Q) < 0$, it is only necessary to show that $h^2(a^2-c^2) - 4bb' < 0$. Now,

$$\text{(4.4)} \qquad \Lambda(Q) = [\tilde{\tau}(Q)]^2 + 4\delta(Q) = [\tilde{\tau}(Q)]^2 + \frac{4\delta(Q')}{h^4}.$$

From the h-invariance of $\delta(Q)$, we also have

$$\Lambda(Q) = [\tilde{\tau}(Q)]^2 - \frac{4b'^2}{h^4}. \tag{4.5}$$

From (4.4) and (4.5), the condition $\Lambda(Q) < 0$ is equivalent to the inequalities

$$(i) \quad |a+c| < 2|b| \text{ or } (ii) \quad |a-c| < \frac{2|b'|}{h^2}.$$

By multiplying (i) and (ii) we obtain: $a^2 - c^2 \leq |a^2 - c^2| < \frac{4}{h^2}|b||b'|$ or, since $bb' > 0$, $h^2(a^2 - c^2) - 4bb' < 0$, as desired.

PROPOSITION 4.3. *If $\Lambda(Q) = 0$, then :*

(i) *If $a^2 - c^2 > 0$, the orbit of Q contains the matrix*

$$Q' = \begin{pmatrix} h^2\tilde{\tau}(Q) & \mp\frac{h^2|\tilde{\tau}(Q)|}{2} & d' \\ \mp\frac{h^2|\tilde{\tau}(Q)|}{2} & 0 & e' \\ d' & e' & f \end{pmatrix}$$

plas sign if $b > 0$ minus sign if $b < 0$.

(ii) *If $a^2 - c^2 < 0$, the matrix*

$$Q'' = \begin{pmatrix} 0 & \mp\frac{h^2|\tilde{\tau}(Q)|}{2} & d' \\ \mp\frac{h^2|\tilde{\tau}(Q)|}{2} & -h^2\tilde{\tau}(Q) & e' \\ d' & e' & f \end{pmatrix}$$

plas sign if $b > 0$ minus sign if $b < 0$, belong to the orbit of Q.

(iii) *If $a = c$, then $b = \pm a$ and*

$$Q = \begin{pmatrix} a & \pm a & d \\ \pm a & a & e \\ d & e & f \end{pmatrix}.$$

(iv) *If $a = -c$, then $b = 0$ and*

$$Q = \begin{pmatrix} a & 0 & d \\ 0 & -a & e \\ d & e & f \end{pmatrix}.$$

Proof. First, let us suppose $a^2 - c^2 > 0$. In this case, we shall assume that b' has the same sign as b. As in the proof of Proposition 4.2 we obtain the system

$$\begin{aligned} 2b' &= (a+c)y + 2bx \\ a - c &= 2by + (a+c)x \end{aligned} \tag{4.6}$$

The determinant $\Lambda(Q) = (a+c)^2 - 4b^2$ is equal to zero. The system (4.6) admits a line of solutions, with slope $m = \pm 1$, which intercepts the x-axis in $x_0 = \frac{a-c}{a+b} > 0$.

Thus, this line intercepts the hyperbola $x^2 - y^2 = 1$, $x > 0$ and we can take θ such that $x = ch2\theta$ and $y = sh2\theta$.

When $a^2 - c^2 < 0$, we shall make $a' = 0$. In this case, we proceed as before, just solving the system

$$\begin{cases} 2b' & = h^2\{(a+c)y + 2bx\} \\ c - a & = 2by + (a+c)x \end{cases}.$$

Of course, (iii) and (iv) follow directly from the hypothesis.

Up to now, we analysed the action of a hyperbolic homothetic rotation on the matrix Q. Now, we shall be using translation to bring Q to a normal form.

PROPOSITION 4.4. *If $\delta(Q) \neq 0$ and (i) $\Lambda(Q) > 0$ or (ii) $\Lambda(Q) = 0$, with $b = 0$, then the orbit of Q contains the matrix:*

$$\begin{pmatrix} h^2a' & 0 & 0 \\ 0 & h^2c' & 0 \\ 0 & 0 & \frac{\Delta(Q)}{\delta(Q)} \end{pmatrix}. \tag{4.7}$$

In (ii), we also have $a = -c'$.

Proof. (i) Since $\Lambda(Q) > 0$, from the Proposition 4.1, the orbit of Q contains the matrix:

$$Q' = \begin{pmatrix} a' & 0 & d' \\ 0 & c' & e' \\ d' & e' & f \end{pmatrix}. \tag{4.8}$$

Because $a' \cdot c' = \delta(Q') = h^4\delta(Q) \neq 0$, there exist real numbers s and t such that:

$$a's + d' = 0 \text{ and } c't + e' = 0.$$

If we take k^{-1} as in the form (2.2) with $N = hI_2$ and $B^t = [s\ t]$, where I_2 is 2×2 unit matrix. Then $Q'' = k \cdot Q'$ is diagonal, with entries a', c' and f'. From $h^4\Delta(Q) = \Delta(Q'') = a'c'f' = \delta(Q) \cdot f'$, it follows that Q'' is as we expected.

(*ii*) When $b = 0$ and $\Lambda(Q) = 0$, it also follows that $a' = -c'$.

PROPOSITION 4.5. *If $\delta(Q) \neq 0$ and $\Lambda(Q) < 0$, then the orbit of Q contains the matrix*

$$\begin{pmatrix} h^4\tilde{\tau}(Q) & \mp h^4\sqrt{-\delta(Q)} & 0 \\ \mp h^4\sqrt{-\delta(Q)} & 0 & 0 \\ 0 & 0 & -\frac{\Delta(Q)}{\delta(Q)} \end{pmatrix}$$

(Sgn $\sqrt{-\delta(Q)}$ equals sgn of b).

Proof. Let us suppose $b > 0$(the case $b < 0$ is analogous). From Proposition 4.2, the orbit of Q contains the matrix

$$\begin{pmatrix} h^2\tilde{\tau}(Q) & \mp h^2\sqrt{-\delta(Q)} & d' \\ \mp h^2\sqrt{-\delta(Q)} & 0 & e' \\ d' & e' & f \end{pmatrix}.$$

Since $\delta(Q) \neq 0$, there exist real numbers s and t such that

$$sh^2\tilde{\tau}(Q) + th^2\sqrt{-\delta(Q)} + d' = 0 \text{ and } sh^2\sqrt{-\delta(Q)} + e' = 0.$$

Now, let k^{-1} replace k in (2.2), with $N = hI_2$ and $B^t = [s\ t]$. Then, $Q'' = k \cdot Q'$ is as above.

PROPOSITION 4.6. *If $\delta(Q) \neq 0$ and $\Lambda(Q) = 0$, then*

(i) $a^2 - c^2 > 0$, *the orbit of Q contains the matrix*

$$\begin{pmatrix} h^4\tilde{\tau}(Q) & \mp\frac{h^4|\tilde{\tau}(Q)|}{2} & 0 \\ \mp\frac{h^4|\tilde{\tau}(Q)|}{2} & 0 & 0 \\ 0 & 0 & -\frac{\Delta(Q)}{\delta(Q)} \end{pmatrix}.$$

(ii) *If $a^2 - c^2 < 0$, the orbit of Q contains the matrix*

$$\begin{pmatrix} 0 & \mp\frac{h^4|\tilde{\tau}(Q)|}{2} & 0 \\ \mp\frac{h^4|\tilde{\tau}(Q)|}{2} & -h^4\tilde{\tau}(Q) & 0 \\ 0 & 0 & -\frac{\Delta(Q)}{\delta(Q)} \end{pmatrix}.$$

The proof is analogous to the proof of the Proposition 4.5, and we shall omit it.

PROPOSITION 4.7. *If* $\delta(Q) = 0$, $\tilde{\tau}(Q) \neq 0$*(hence,* $\Lambda(Q) > 0$*) and* $\Delta(Q) \neq 0$, *then*

(i) *If* $\tilde{\tau}(Q) \cdot \Delta(Q) < 0$, *the orbit of* Q *contains the matrix*

$$\begin{pmatrix} h^4\tilde{\tau}(Q) & 0 & 0 \\ 0 & 0 & e'' \\ 0 & e'' & 0 \end{pmatrix},$$

where $e'' = \mp|h|\sqrt{\frac{\Delta(Q)}{\tilde{\tau}(Q)}}$.

(ii) *If* $\tilde{\tau}(Q) \cdot \Delta(Q) > 0$, *the orbit of* Q *contains the matrix*

$$\begin{pmatrix} 0 & 0 & d'' \\ 0 & -h^4\tilde{\tau}(Q) & 0 \\ d'' & 0 & 0 \end{pmatrix},$$

where $d'' = \mp|h|\sqrt{\frac{\Delta(Q)}{\tilde{\tau}(Q)}}$.

Proof. (i) The hypothesis imply that $\Lambda(Q) > 0$.
Since $\delta(Q) = 0$, and $\tilde{\tau}(Q) \cdot \Delta(Q) < 0$, we may assume in (4.8) that:

$$\begin{pmatrix} h^2\tilde{\tau}(Q) & 0 & d' \\ 0 & 0 & e' \\ d' & e' & f \end{pmatrix}.$$

We note that $\Delta(Q') = h^4\Delta(Q) = -e'^2h^2\tilde{\tau}(Q) \neq 0$, so that $e' \neq 0$. Then, we can find s and t such that $h^2s\tilde{\tau}(Q) + d' = 0$ and $h^2s^2\tilde{\tau}(Q) + 2(d's + e't) + f = 0$. To obtain the normal form, let $Q'' = k \cdot Q'$, where k^{-1} has the form (2.2) with $N = hI_2$ and $B^t = [s\ t]$.

(ii) Follows analogously.

PROPOSITION 4.8. *If* $\delta(Q) = 0$, $\tilde{\tau}(Q) \neq 0$*(hence,* $\Lambda(Q) > 0$*) and* $\Delta(Q) = 0$, *then the orbit of* Q *contains either the matrix*

(i)

$$\begin{pmatrix} h^4\tilde{\tau}(Q) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f' \end{pmatrix},$$

when $|c| < |b|$ *or* ($b = 0$ *and* $a \neq 0$),

or

(ii)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -h^4\tilde{\tau}(Q) & 0 \\ 0 & 0 & f' \end{pmatrix},$$

when $|c| > |b|$ *or* ($b = 0$ *and* $c \neq 0$).

In either case, $f' = f - \frac{d^2-e^2}{\tilde{\tau}(Q)}$.

Proof. From the Proposition 4.1, we may assume

$$Q' = \begin{pmatrix} a' & 0 & d' \\ 0 & c' & e' \\ d' & e' & f \end{pmatrix},$$

where $a' \cdot c' = 0$, since $\delta = 0$

If $c' = 0$, $\Lambda(Q) = 0$ implies $e' = 0$.

Now, choosing k^{-1} as in (4.8), with $N = hI_2$ and $B^t = [s\ 0]$ we have

$$Q' = k \cdot Q = \begin{pmatrix} h^4\tilde{\tau}(Q) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f' \end{pmatrix},$$

where $h^2 s\tilde{\tau}(Q) + d' = 0$.

With the same procedure, when $a' = 0$, we obtain (ii).

Using the expression for $f' = f(Q')$ as in (2.5) and the invariance of $d^2 - e^2$ with respect to a hyperbolic homothetic rotation, we can find f'.

PROPOSITION 4.9. *If* $\delta(Q) = \tilde{\tau}(Q) = 0$*(hence,* $\Lambda(Q) = 0$*), then*

(i) *If* $b = a$*, the orbit of* Q *contains either the matrix*

$$\begin{pmatrix} h^2a & h^2a & d' \\ h^2a & h^2a & 0 \\ d' & 0 & 0 \end{pmatrix}, \text{ when } d \neq e$$

or the matrix

$$\begin{pmatrix} h^2a & h^2a & 0 \\ h^2a & h^2a & 0 \\ 0 & 0 & f' \end{pmatrix}, \text{ when } d = e.$$

Furthermore, $d' = (d - e)h^2$ *and* $f' = f - \frac{d^2}{a}$.

(ii) *If* $b = -a$, *the orbit of* Q *contains either the matrix*

$$\begin{pmatrix} h^2a & -h^2a & d' \\ -h^2a & h^2a & 0 \\ d' & 0 & 0 \end{pmatrix}, \text{ when } d \neq -e$$

or the matrix

$$\begin{pmatrix} h^2a & -h^2a & 0 \\ -h^2a & h^2a & 0 \\ 0 & 0 & f' \end{pmatrix}, \text{ when } d = -e.$$

Furthermore, $d' = (d+e)h^2$ *and* $f' = f - \frac{d^2}{a}$.

The proof in analogous to the previous Propositions, and we shall omit it.

5. Geometric Interpretation

As we saw in the section 4, according to $\Lambda(Q) > 0$ and $\Lambda(Q) < 0$, a given hyperbola may or may not reduced to the form $a'x^2 + c'y^2 + f' = 0$. Now, we want to find a geometric interpretation for this fact.

We recall here that the group of hyperbolic homothetic rotations leave h-invariant some sets of figure 1, in section 2 to simplify notation, we shall denote by $S = \mathrm{I} \cup \mathrm{III}$, the set of spacelike vectors and by $T = \mathrm{II} \cup \mathrm{IV}$, the set of timelike vectors.

PROPOSITION 5.1. *If* $\delta(Q) < 0$ *and* $\Lambda(Q) > 0$, *then both asymptotes of the hyperbola* C *associated to* Q *are timelike or both are spacelike. That is if* $a^2 - c^2 > 0$, *they are both in* T *and if* $a^2 - c^2 < 0$, *they are in* S.

Proof. Since S and T are h-invariant, we may analyse the position of the asymptotes of the reduced form.

First, let $a^2 - c^2 > 0$. The standard equation of C after a hyperbolic homothetic rotation is $a'x^2 + c'y^2 = -\frac{\Delta(Q)}{\delta(Q)}$, where ${a'}^2 - {c'}^2 > 0$(Proposition 4.1). Then, the asymptotes of this reduced conic are the lines

$$L_1 : |h|\sqrt{|a'|}x + |h|\sqrt{|c'|}y = 0,$$
$$L_2 : |h|\sqrt{|a'|}x - |h|\sqrt{|c'|}y = 0.$$

respectively.

Since ${a'}^2 - {c'}^2 > 0$, $\frac{|a'|}{|c'|} > 1$ and it follows that L_1 lies in T. The second case $a^2 - c^2 < 0$ is analogous. In this case, L_1 and L_2 are both in S.

Proposition 5.2. *If $\delta(Q) < 0$ and $\Lambda(Q) < 0$, then the hyperbola C has one asymptote which is timelike and one which is spacelike.*

Proof. From the Proposition 4.1, the standard equation of C is

$$(i) \qquad h^4\tilde{\tau}(Q)x^2 + 2h^4\sqrt{-\delta(Q)}xy - \frac{\Delta(Q)}{\delta(Q)} = 0$$

or

$$(ii) \qquad h^4\tilde{\tau}(Q)x^2 - 2h^4\sqrt{-\delta(Q)}xy - \frac{\Delta(Q)}{\delta(Q)} = 0.$$

Let's assume that (i) holds. The asymptotes are the lines:

$$L_1 : x = 0 \text{ and } L_2 : h^4\tilde{\tau}(Q)x + 2h^4\sqrt{-\delta(Q)}y = 0.$$

To finish the proof, we just observe that the slope of L_2 is in the interval $(-1, 1)$. The case (ii) is analogous.

Proposition 5.3. *If $\delta(Q) < 0$ and $\Lambda(Q) = 0$, at least one of the lines $y = x$ and $y = -x$ is an asymptotes of C(may be both).*

Proof. (i) Let $b = 0$. Then, from Proposition 4.1, the standard equation is

$$h^2a'x^2 + h^2c'y^2 + \frac{\Delta(Q)}{\delta(Q)} = 0.$$

which has $y = x$ and $y = -x$ as asymptotes.

Since these lines are h-invariant, the result is also true for the original C.

(ii) Let $b \neq 0$. Then $\Lambda(Q) = 0 \Leftrightarrow \tilde{\tau}(Q) = \mp 2\sqrt{-\delta(Q)}$. From the Proposition 4.3, the standard equation is

$$h^4\tilde{\tau}(Q)x^2 \mp h^4|\tilde{\tau}(Q)|xy = \frac{\Delta(Q)}{\delta(Q)} \tag{5.1}$$

or

$$\mp h^4|\tilde{\tau}(Q)|xy - h^4\tilde{\tau}(Q)y^2 = \frac{\Delta(Q)}{\delta(Q)}. \tag{5.2}$$

If, for instance, (5.1) holds, then the asymptotes are

$$L_1 : x = 0 \text{ and } L_2 : y = x \text{ or } y = -x.$$

The orginal C inherits the properties: L_1 lies in S and L_2 is $y = x$ or $y = -x$.

6. The Ring of h-invariants

The exposition of this paragraph is devoted to showing that any polynimial in $R = IR[z_1, z_2, z_3, z_4, z_5, z_6]$ h-invariant under the action of the hyperbolic homothetic group K is a polynomial in $\tilde{\tau}, \delta$ and Δ. Now, we prove that they are algebraically independent over IR.

THEOREM 6.1. *If $R[\tilde{\tau}, \delta, \Delta]$denote the subalgebra of R, generated by $\tilde{\tau}, \delta$ and Δ, and K is the algebra of h-invariant polynomials, then $R = IR[\tilde{\tau}, \delta, \Delta]$. Moreover, $\tilde{\tau}, \delta$ and Δ are algebraically independent over R.*

To simplify notation, we shall denote a matrix

$$Q = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \text{ by } (a, b, c, d, e, f).$$

We recall that R can be identifed with the ring of polynomial functions over V, as we saw in §3.

Now, we define the following subsets of V:

$$\begin{aligned} X_1 &= \{(h^4 a, h^4 b, 0, 0, 0, f \mid b \neq 0, a^2 - 4b^4 < 0\} \\ X_2 &= \{(h^4 a, 0, h^4 c, 0, 0, f \mid ac \neq 0, a \neq -c\} \\ U_1 &= \{Q \in V \mid \delta(Q) < 0 \text{ ve } \Lambda(Q) < 0\} \\ U_2 &= \{Q \in V \mid \delta(Q) \neq 0 \text{ ve } \Lambda(Q) > 0\} \end{aligned}$$

Any element of U_1 is transformed by H in an element in X_1. Also, any element in U_2 is K-invariant to an element in X_2.

We need the following lemma:

LEMMA 6.1. *Any h-invariant polynomial is even in the variable b.*

Proof. First, we observe that if $(a, b, c, d, e, f) \in U_2$, then $(a, -b, c, d, e, f) \in U_2$. Hence, given any h-invariant polynomial P, we define in U_2 the polynomial:

$$\tilde{P}(a, b, c, d, e, f) = P(a, b, c, d, e, f) - P(a, -b, c, d, e, f).$$

Now, we saw that any element (a, b, c, d, e, f) in U_2 is equvalent to an element $(a, 0, c, 0, 0, f)$. The h-invariance of P implies

$$P(a, b, c, d, e, f) = P(h^4 a, 0, h^4 c, 0, 0, f). \tag{6.1}$$

Hence, $\tilde{P} \equiv 0$ in U_2. Since U_2 is an open set in V, it follows that $\tilde{p} \equiv 0$ in V, that is, P is even in b.

We divide the proof of Theorem 6.1 into three main steps:

Proof of Theorem 6.1. We divide the prrof of Theorem 6.1 into three main steps:

Step 1. Let B be the ring of the polynomial functions over X_1, and, as before, P be the set of h-invariant polynomials. We define a mapping $H : I \to A$ by $P \to H(P) = P|_{X_1}$, restriction of P to X_1, that is

$$P(a,b,c,d,e,f) \to P(h^4 a, h^4 b, 0,0,0,f). \tag{6.2}$$

It follows that H is a one-to-one homomorphism. In fact, from the invariance of P, $H(P) = P|_{X_1} = 0$ implies $P|_{U_1} = 0$. Since U_1 is open in V, the polynomial P must be identically zero in V.

We conclude this step by noting that:

$$\begin{aligned} H(\tilde{\tau}(Q')) &= \tilde{\tau}(Q')|_{X_1} = h^2 \tilde{\tau}(Q)|_{X_1} = h^2 a = h^2 H(\tilde{\tau}(Q)), \\ H(\delta(Q')) &= \delta(Q')|_{X_1} = h^4 \delta(Q)|_{X_1} = -h^4 b^2 = h^4 H(\delta(Q)), \\ H(\Delta(Q')) &= \Delta(Q')|_{X_1} = h^4 \Delta(Q)|_{X_1} = -h^4 b^2 f = h^4 H(\Delta(Q)). \end{aligned}$$

Step 2. Let P be a h-invariant polynomial. There are unique polynomials $g_0(a,b), \cdots, g_m(a,b)$, such that

$$H(P) = g_0(a,b) f^m + \cdots + g_m(a,b). \tag{6.3}$$

From the previous Lemma 6.1, P is even in the variable b. Hence

$$H(P) = h_0(a,-b^2) f^m + \cdots + h_m(a,-b^2). \tag{6.4}$$

Next, let us consider the $h$invariant polynomials $\delta^m P$ and $h_0(\tilde{\tau},\delta)\Delta^m + h_1(\tilde{\tau},\delta)\delta\Delta^{m-1} + \cdots + h_m(\tilde{\tau},\delta)\delta^m$. The image by H of each of one these polynomials is equal to $(-b^2)^m H(P)$.

As we saw in step 1. H is 1-1, and we have the equality:

$$\delta^m P = h_0(\tilde{\tau},\delta)\Delta^m + h_1(\tilde{\tau},\delta)\delta\Delta^{m-1} + \cdots + h_m(\tilde{\tau},\delta)\delta^m. \tag{6.5}$$

Step 3. This part of the proof follows as in §3. We repeat it here for completeness. We rewrite the right-hand side of (6.5) as:

$$\delta^m P = \delta^n [j_0(\tilde{\tau},\Delta)\delta^k + \cdots + j_k(\tilde{\tau},\Delta)], \tag{6.6}$$

where $j_k(\tilde{\tau}, \Delta) \neq 0$.
If $n \geq m$, it follows that P is a polynomial in $\tilde{\tau}, \delta$ and Δ.

Now, assuming $n < m$, we obtain a contradiction. In fact,

$$\delta^{n-m} = j_0(\tilde{\tau}, \Delta)\delta^k + \cdots + j_k(\tilde{\tau}, \Delta), \text{ where } m - n > 0.$$

Since $j_k(\tilde{\tau}, \Delta) \neq 0$, there are real numbers α and β such that $j_k(\alpha, \beta) \neq 0$ and $\alpha\beta < 0$. Now, we take $\gamma = \sqrt{-\frac{\beta}{\alpha}}$ and $Q = (\alpha, 0, 0, 0, \gamma, 0)$.

Evaluating both sides of (6.6) in Q, gives $0 = j_k(\tilde{\tau}, \Delta) \neq 0$. Finally, $\tilde{\tau}, \delta$ and Δ are algebraically independent since their images under H are.

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