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THE INCOMPLETE BETA AND THEIR ASSOCIATED FUNCTIONS

INHYOK PARK. YOUNG JOON CHO.
TAE YOUNG SEO AND JUNESANG CHOI

ABSTRACT The authors aim at providing some identities involving the hypergeometric function via some known or presumably new formulas for the incomplete Beta and their associated functions Some properties of the Beta and Gamma functions are also considered

1. Introduction

The incomplete Beta function $B_x(\alpha, \beta)$ is defined by

$$(1.1) \qquad B_x(\alpha, \beta) := \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad (0 < x < 1;\ \mathrm{Re}(\alpha) > 0)$$

whose associated function (as normalized version) is given

$$(1.2) \qquad I_x(\alpha, \beta) := \frac{B_x(\alpha, \beta)}{B(\alpha, \beta)},$$

where the Beta function $B(\alpha, \beta)$ is a function of two complex variables α and β, defined by the Eulerian integral of the first kind

$$(1.3) \qquad B(\alpha, \beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad (\mathrm{Re}(\alpha) > 0;\ \mathrm{Re}(\beta) > 0),$$

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which, upon setting $t = \sin^2\theta$, is equivalently written in the form:
(1.4)
$$B(\alpha,\,\beta) := 2\int_0^{\pi/2} (\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}\,d\theta \quad (\mathrm{Re}(\alpha) > 0;\ \mathrm{Re}(\beta) > 0).$$

Note that Choi *et al.* [1] obtained and proved some interesting known or presumably new identities for the Beta function by introducing another definition via the generalized (or Hurwitz) zeta function. The incomplete Beta function is related to the beta distribution in statistics

The Beta function is closely related to the Gamma function

$$B(\alpha,\,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (\alpha,\,\beta \neq 0,\,-1,\,-2,\,\ldots), \tag{1.5}$$

where Γ denotes the well known Gamma function which has several equivalent definitions Some of which are introduced as follows:

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \quad (\mathrm{Re}(z) > 0,\ \text{L Euler}); \tag{1.6}$$

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)(z+2)\cdots(z+n)} \qquad (z \neq 0,\,-1,\,-2,\,\ldots;\ \text{K. F. Gauss}); \tag{1 7}$$

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)\exp\left(-\frac{z}{k}\right)\right] \quad (\text{K. Weierstrass}), \tag{1.8}$$

where γ is the Euler-Mescheroni constant defined by

$$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \log n\right] \cong 0.577\,215\,664\,\ldots\,. \tag{1.9}$$

We summarize some properties of the Gamma function.

$$\Gamma(z+1) = z\Gamma(z); \tag{1.10}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad (z \neq 0, \pm 1, \pm 2, \ldots); \tag{1.11}$$

$$\Gamma(1) = 1, \ \Gamma(1/2) = \sqrt{\pi}, \ \Gamma(n+1) = n! \ (n \in \mathbf{N} \cup \{0\}), \tag{1.12}$$

where $\mathbf{N}$ denotes the set of positive integers.

The hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (c \neq 0, -1, -2, \ldots), \tag{1.13}$$

where the Pochhammer symbol $(\alpha)_n$ is defined by, α any complex number,

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1) & \text{if } n \in \mathbf{N} \\ 1 & \text{if } n = 0, \end{cases} \tag{1.14}$$

which, in terms of the Gamma functions, we find

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad (\alpha \neq 0, -1, -2, \ldots) \tag{1.15}$$

The hypergeometric function ${}_2F_1$ is, more precisely, called Gauss's hypergeometric function after the famous German mathematician Carl Friedrich Gauss (1777-1855) who in the year 1812 introduced this function into analysis and gave the F-notation for it. One of Gauss's important identities for ${}_2F_1$ is the following well-known summation formula:

(1.16)

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\mathrm{Re}(c-a-b) > 0; \ c \neq 0, -1, -2, \quad)$$

A number of summation theorems for the hypergeometric function (1.13) when z takes on other special values are recorded in various literature

In this note some identities involving ${}_2F_1$ are provided by making use of some formulas for the associated $I_x(\alpha, \beta)$.

2. Some Identities for ${}_2F_1$ via I_x

The Maclaurin series of the following function is given by, for any complex number α,

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \quad (|z|<1), \tag{2.1}$$

where the generalized binomial coefficient $\binom{\alpha}{n}$ is defined by

$$\binom{\alpha}{n} := \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} & \text{if } n \in \mathbf{N} \\ 1 & \text{if } n=0, \end{cases} \tag{2.2}$$

which is equivalently written in terms of the Pochhammer symbol

$$\binom{\alpha}{n} = \frac{(-1)^n\,(-\alpha)_n}{n!} \quad (n \in \mathbf{N} \cup \{0\}) \tag{2.3}$$

We thus find that

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n \quad (|z|<1). \tag{2.4}$$

From (1.1) and (1.2), we find that, upon using (2.4) and integration termwise,

$$I_x(\alpha,\,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} x^\alpha {}_2F_1(\alpha,\,1-\beta\,;\,\alpha+1\,;\,x). \tag{2.5}$$

Indeed,

$$\begin{aligned} I_x(\alpha,\,\beta) &= \frac{1}{B(\alpha,\,\beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt \\ &= \frac{1}{B(\alpha,\,\beta)} \sum_{n=0}^{\infty} \frac{(1-\beta)_n}{n!} \frac{x^{n+\alpha}}{n+\alpha} \\ &= \frac{1}{B(\alpha,\,\beta)} x^\alpha \sum_{n=0}^{\infty} \frac{(1-\beta)_n}{n!} \frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha+1)} x^n, \end{aligned}$$

which, in view of (1.15), reaches at the desired identity (2.5) (cf. [4, p 128]).

From (1 2) we find the following identities which are readily verifiable (cf., e.g [3, pp 27-28]; see also [4, pp. 288-290])

$$(2.6)\qquad I_x(\alpha,\, \beta) = 1 - I_{1-x}(\beta,\, \alpha)$$

$$(2.7)\qquad I_x(\alpha,\, \beta) = x\, I_x(\alpha - 1,\, \beta) + (1-x)\, I_x(\alpha,\, \beta - 1);$$

$$(2.8)\quad (\alpha + \beta - \alpha x)\, I_x(\alpha,\, \beta) = \alpha(1-x)\, I_x(\alpha + 1,\, \beta) + \beta\, I_x(\alpha,\, \beta + 1);$$

$$(2.9)\quad \{\alpha + (\beta - \alpha)x\}\, I_x(\alpha,\, \beta) = \alpha(1-x)\, I_x(\alpha+1,\, \beta) + \beta x\, I_x(\alpha,\, \beta + 1),$$

$$(2.10)\qquad (\alpha + \beta)\, I_x(\alpha,\, \beta) = \alpha\, I_x(\alpha + 1,\, \beta) + \beta\, I_x(\alpha,\, \beta + 1);$$

$$(2.11)\qquad I_x(\alpha,\, \beta) = I_x(\alpha - 1,\, \beta + 1) - \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta + 1)} x^{\alpha - 1}(1-x)^{\beta};$$

$$(2.12)\qquad I_x(\alpha,\, \beta) = I_x(\alpha + 1,\, \beta - 1) + \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + 1)\Gamma(\beta)} x^{\alpha}(1-x)^{\beta - 1}.$$

$$(2.13)\qquad I_x(\alpha,\, \beta) = I_x(\alpha + 1,\, \beta) + \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + 1)\Gamma(\beta)} x^{\alpha}(1-x)^{\beta}.$$

$$(2.14)\qquad I_x(\alpha,\, \beta) = I_x(\alpha,\, \beta + 1) - \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta + 1)} x^{\alpha}(1-x)^{\beta}.$$

$$(2\,15)\qquad (x + \alpha)\, I_x(\alpha,\, \beta) = x\, I_x(\alpha - 1,\, \beta + 1) + \alpha\, I_x(\alpha + 1,\, \beta),$$

$$I_x(k,\, n-k+1) = \sum_{j=k}^{n} \binom{n}{j} x^j (1-x)^{n-j} \quad (1 \le k \le n), \tag{2.16}$$

one of whose proofs may be given by using the principle of mathematical induction.

Now application (2.5) to the identities (2.6)-(2.15) yields immediately various formulas involving the hypergeometric function as follows:
(2.17)

$$\beta x^{\alpha}{}_2F_1(\alpha,\, 1-\beta\,;\, \alpha+1\,;\, x) + \alpha(1-x)^{\beta}{}_2F_1(\beta,\, 1-\alpha\,;\, \beta+1\,;\, 1-x) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta)};$$

(2.18)

$$(\alpha+\beta-1){}_2F_1(\alpha,\, 1-\beta\,;\, \alpha+1\,;\, x) = \alpha{}_2F_1(\alpha-1,\, 1-\beta\,.\, \alpha\,;\, x) + (\beta-1)(1-x){}_2F_1(\alpha,\, 2-\beta\cdot\, \alpha+1\,;\, x);$$

$$\begin{aligned}(\alpha+1)(\alpha+\beta-\alpha x)&{}_2F_1(\alpha,\, 1-\beta\,;\, \alpha+1 : x)\\ &= \alpha(\beta-1)x(1-x){}_2F_1(\alpha+1,\, 2-\beta\,;\, \alpha+2\,;\, x)\\ &\quad + (\alpha+1)(\alpha-\beta){}_2F_1(\alpha,\, -\beta : \alpha+1\,;\, x);\end{aligned} \tag{2.19}$$

$$\begin{aligned}(\alpha+1)(\beta-1)\{\alpha+(\beta-\alpha)x\}&{}_2F_1(\alpha,\, 1-\beta\,;\, \alpha+1\,;\, x)\\ &= \alpha\beta(\beta-1)x(1-x){}_2F_1(\alpha+1,\, 2-\beta\,;\, \alpha+2\,;\, x)\\ &\quad + \alpha(\alpha+1)\beta\,{}_2F_1(\alpha-1,\, -\beta\,;\, \alpha\,;\, x);\end{aligned} \tag{2.20}$$

$$\begin{aligned}{}_2F_1(\alpha,\, 1-\beta\,;\, \alpha+1\,;\, x) &= {}_2F_1(\alpha+1,\, -\beta\,;\, \alpha+1 : x)\\ &\quad + \frac{\alpha}{\alpha+1}{}_2F_1(\alpha+1,\, 1-\beta\,;\, \alpha+2\,;\, x),\end{aligned} \tag{2.21}$$

$$\begin{aligned}\beta x{}_2F_1&(\alpha,\, 1-\beta\,;\, \alpha+1\,;\, x)\\ &= \alpha{}_2F_1(\alpha-1,\, -\beta\,;\, \alpha\,;\, x) - \alpha(1-x)^{\beta}:\end{aligned} \tag{2.22}$$

(2.23)

$$\begin{aligned}&(\alpha+1)_2F_1(\alpha,\,1-\beta\,;\,\alpha+1\,;\,x)\\&\quad=(\beta-1)x\,{}_2F_1(\alpha+1,\,2-\beta\,;\,\alpha+2\,;\,x)+(\alpha+1)(1-x)^{\beta-1};\end{aligned}$$

$$\begin{aligned}&{}_2F_1(\alpha,\,1-\beta\,;\,\alpha+1\,;\,x)\\&\quad=\frac{\alpha+\beta}{\alpha+1}x\,{}_2F_1(\alpha+1,\,1-\beta\,;\,\alpha+2\,;\,x)+(1-x)^{\beta};\end{aligned}\tag{2.24}$$

$$\begin{aligned}&\beta\,{}_2F_1(\alpha,\,1-\beta\,;\,\alpha+1:x)\\&\quad=(\alpha+\beta)\,{}_2F_1(\alpha,\,-\beta:\,\alpha+1\,;\,x)-\alpha(1-x)^{\beta},\end{aligned}\tag{2.25}$$

$$\begin{aligned}&(\alpha+1)\beta(x+\alpha)\,{}_2F_1(\alpha,\,1-\beta\,;\,\alpha+1\,;\,x)\\&\quad=\alpha(\alpha+1)\,{}_2F_1(\alpha-1,\,-\beta\,;\,\alpha\,,\,x)\\&\quad+\alpha\beta(\alpha+\beta)x\,{}_2F_1(\alpha+1,\,1-\beta\,,\,\alpha+2\,,\,x).\end{aligned}\tag{2.26}$$

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Inhyok Park, Young Joon Cho
and Tae Young Seo
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail: Tyseo@hyowon.pusan.ac.kr

Junesang Choi
Department of Mathematics
Dongguk University
Kyongju 780-714, Korea
E-mail: junesang@email.dongguk.ac.kr