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CONDITIONS IMPLYING NORMALITY

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ABSTRACT In this paper we find some classes of operators implying normaity The main result is as follows If T is restriction-convexoid and is reduced by each of its eigenspaces corresponding to isolated eigenvalues, which is a class including hyponormal operators and if $\sigma(T)$ is countable then T is diagonal and normal

1.Introduction

Throughout this paper let $\mathcal{H}$ denote an infinite dimensional separable Hilbert space Let $\mathcal{L}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, write $N(T)$ and $R(T)$ for the null space and range of T; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T Recall ([6],[9]) that $T \in \mathcal{L}(\mathcal{H})$ is called *regular* if there is an operator $T' \in \mathcal{L}(\mathcal{H})$ for which $T = TT'T$. It is familiar that if T is regular then T has closed range and that its converse is also true in the Hilbert space setting An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Fredholm* if it has closed range $R(T)$ with finite dimensional null space and with its range of finite co-dimension. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Browder* if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$([6]) The *index* of a Fredholm operator $T \in \mathcal{L}(\mathcal{H})$ is given by

$$\text{ind}(T) = \dim N(T) - \dim \mathcal{H}/R(T).$$

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The *essential* spectrum $\sigma_e(T)$ and the *Browder* spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(\mathcal{H})$ are defined by ([1],[2],[3],[7])

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not Fredholm}\},$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not Browder}\}.$$

Evidently([6],[7],[10])

$$\sigma_e(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc}\sigma(T),$$

where we write acc$\mathbf{K}$ for the accumulation points of $\mathbf{K} \subset \mathbb{C}$. If we write

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$$

for the *Riesz point* of T([6],[9]]), then

$$\text{iso}\sigma(T) \setminus \sigma_e(T) = p_{00}(T).$$

If $T \in \mathcal{L}(\mathcal{H})$, write $r(T)$ for the spectral radius of T. It is familiar that $r(T) \leq ||T||$; $\lambda \in \pi_0(T)$ is called *normal* if also $\overline{\lambda} \in \pi_0(T^*)$ and the corresponding eigenspaces are equal, i.e.,$N(T - \lambda I) = N(T^* - \overline{\lambda} I)$. Such a subspaces reduces T, and the restriction of T is trivially normal([3]). An operator $T \in \mathcal{L}(\mathcal{H})$ is called *normaloid* if $r(T) = ||T||$ and *isoloid* if iso $\sigma(T) \subseteq \pi_0(T)$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to satisfy condition (G_1) if $(T - \lambda I)^{-1}$ is normaloid for all $\lambda \notin \sigma(T)$. If $T \in \mathcal{L}(\mathcal{H})$, write $W(T)$ for the numerical range of T. It is also familar that $W(T)$ is convex and conv $\sigma(T) \subseteq$ cl$W(T)$. An operator T is *convexoid* if conv$\sigma(T) =$ cl$W(T)$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *reguloid* ([7]) if $T - \lambda I$ is regular for each $\lambda \in$ iso $\sigma(T)$ and will be called *closoid* if $R(T - \lambda I)$ is closed for each $\lambda \in$ iso$\sigma(T)$. Let P be a property of operators. We say that an operator T is *restriction-P* if the restriction of T to every invariant subspaces has property P and that T is *reduction-P* if every direct summand of T has property P. Evidently, restriction-P $\Rightarrow$ reduction-P. It is well known that if $T \in \mathcal{L}(\mathcal{H})$ then we have([2],[3]):

$(G_1) \Rightarrow$ convexoid and isoloid;
seminormal $\Longrightarrow$ reduction-(G_1) $\Longrightarrow$ reduction-isoloid;
hyponormal $\Longrightarrow$ restriction-convexoid.

REMARK. The restriction of an operator with (G_1)-property fails to have the (G_1)-property on the invariant subspace: for example, let T_1 be an operator on l_2 with the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which is nilpotent. Let T_2 be the bounded operator defined on $\mathcal{H}$ with a countable basis $\{e_i\}_{i=1}^{\infty}$ by $T_2 e_i = \alpha_i e_i$ and where α_i are such that $\min_i |\lambda - \alpha_i| < |\lambda|^2$ for all $\lambda, |\lambda| < 1$. Then the operator $T = T_1 \oplus T_2$ has the (G_1)-property and clearly nonnormal. Note that seminormal operators are reduction-convexoid. but they may not be restriction-convexoid. for example consider the backward shift U^* on l_2, where U is the unilateral shift(cf [3]). Thus the replacement of "reduction-" by "restriction-" is very stringent. If $T \in \mathcal{L}(\mathcal{H})$ and we define P_F through Riesz functional calculus

$$P_F = \frac{1}{2\pi i} \int_\gamma (\lambda - T)^{-1} d\lambda,$$

where F is that isolated part of $\sigma(T)$ and γ is a Cauchy contour containing F. then P_F is a projection and we can decompose T into $T = T_1 \oplus T_2$ such that $\sigma(T_1) = F$ and $\sigma(T_2) = \sigma(T) \setminus F$. which is called the *spectral projection*([6])

2. Conditions implying normality

We begin with:

LEMMA 2.1 *If $T \in \mathcal{L}(\mathcal{H})$ is restriction-convexoid then T is isoloid.*

PROOF. Let λ_0 be an isolated point of $\sigma(T)$ and let D_ϵ be a circle with the center at λ_0 and radius ϵ so that $\{z : |\lambda - \lambda_0| \leq \epsilon\} \cap \sigma(T) = \{\lambda_0\}$. We define now the projection P_{λ_0}(the spectral projection at $\lambda_0 \in \mathbb{C}$) by the formula

$$P_{\lambda_0} = \frac{1}{2\pi} \int_{D_\epsilon} (\lambda - T)^{-1} d\lambda$$

Then $P_{\lambda_0}\mathcal{H}$ is an invariant subspace of T. Since the spectrum of the restriction of T to this subspace is $\{\lambda_0\}$ and the restriction is convexoid $\text{conv}\sigma(T) = \text{cl}W(T) = \{\lambda_0\}$; thus it is of the form $\lambda_0 I|_{P_{\lambda_0}}$. i.e . $\lambda_0 \in \pi_0(T)$. This completes the proof.

LEMMA 2.2. *If $T \in \mathcal{L}(\mathcal{H})$ then*
(2.2.1)
T *satisfies* $(G_1) \Longrightarrow T$ *is reguloid* $\Longrightarrow T$ *is closoid* $\Longrightarrow T$ *is isoloid.*

and

(2.2.2) *restriction-convexoid* $\Longrightarrow$ *restriction-reguloid.*

PROOF. The first implication of (2.2.1) is known from [8, Theorem 14]. For the second implication, suppose $T \in \mathcal{L}(\mathcal{H})$ is reguloid and $\lambda \in \operatorname{iso}\sigma(T)$. Then $T - \lambda I$ is regular; thus $R(T - \lambda I)$ is complemented: thus it is closed. For the third implication of (2.2.1), suppose T is closoid and $\lambda \in \operatorname{iso}\sigma(T)$. Then we claim that $\lambda \in \pi_0(T)$. Assume to the contrary that $T - \lambda I$ is one-one. Then since by assumption $T - \lambda I$ is left invertible (cf. [6, (3.8.3.12)]), so that λ cannot lie on the boundary of $\sigma(T)$. This contradicts to the fact that $\lambda \in \operatorname{iso}\sigma(T)$. For (2.2.2), suppose T is restriction-convexoid and $\mathfrak{M}$ is an invariant subspace of T. Write $S := T|\mathfrak{M}$. Then S is also restriction-convexoid. Suppose $\lambda \in \operatorname{iso}\sigma(S)$. Observe that if T is convexoid then so is $aT + bI$ for any $a, b \in \mathbb{C}$. Thus we may write S in place of $S - \lambda I$ and assume $\lambda = 0$. Using the spectral projection at $0 \in \mathbb{C}$ we can write $S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$, where $\sigma(S_1) = \{0\}$ and $\sigma(S_2) = \sigma(S) \setminus \{0\}$. Since by assumption, S_1 is convexoid it follows that $W(S_1) = \operatorname{conv}\sigma(S_1) = \{0\}$, and hence $S_1 = 0$ Thus we have

$$S = \begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix} = SS'S \quad \text{with} \quad S' = \begin{pmatrix} 0 & 0 \\ 0 & S_2^{-1} \end{pmatrix},$$

which says that S is regular, and therefore T is restriction-reguloid(;thus, we recapture Lemma 2.1).

In 1970, S.Berberian raised the following question: if T is restriction-convexoid and $\sigma(T)$ is countable, is T normal? We now give a partial answer.

Theorem 2.3. *If T is restriction-convexoid and is reduced by each of its eigenspaces corresponding to isolated eigenspaces and $\sigma(T)$ is countable then T is diagonal and normal*

Proof. Suppose T is restriction-convexoid and is reduced by each of its eigenspaces corresponding to isolated eigenspaces and $\sigma(T)$ is countable. Let δ be the set of all normal eigenvalues of T. i.e .

$$\delta = \{\lambda \in \pi_0(T) : N(T - \lambda I) = N(T^* - \overline{\lambda} I)\}$$

We first claim that $\delta \neq \emptyset$. Since $\sigma(T)$ is countable there exists a point $\lambda \in \mathrm{iso}\,\sigma(T)$, so that $\lambda \in \pi_0(T)$ because by Lemma2.2(in general, restriction-$P \Rightarrow P$). T is isoloid. Using the spectral projection at $\lambda \in \mathbb{C}$ we can represent T as the direct sum

$$T = R \oplus S, \quad \text{where} \quad \sigma(R) = \pi_0(R) = \{\lambda\} \quad \text{and} \quad \sigma(S) = \sigma(T) \setminus \{\lambda\}$$

Since by assumption R is convexoid we have that $W(R) = \mathrm{conv}\{\lambda\} = \{\lambda\}$ and thus $\lambda \in \pi_0(R) \cap \partial W(R)$.Then an argument of Bouldin[4,Lemma1] shows that λ is a normal eigenvalue of R. Since T is reduced by each of its eigenspaces corresponding to isolated eigenspaces, we can write $T^* = R^* \oplus S^*$. But since $S^* - \overline{\lambda} I$ is invertible, it follows

$$N(T - \lambda I) = N(R - \lambda I) = N(R^* - \overline{\lambda} I) = N(T^* - \overline{\lambda} I)$$

which implies that $\delta \neq \emptyset$. Now if $\mathfrak{M}$ is the closed linear span of the eigenspaces $N(T - \lambda I)(\lambda \in \delta)$, then $\mathfrak{M}$ reduces T Write

$$T_1 := T|\mathfrak{M} \quad \text{and} \quad T_2 := T|\mathfrak{M}^{\perp}.$$

Then we have ([2],[9])

(i) T_1 is normal and diagonal;
(ii) $\pi_0(T_1) = \delta$;
(iii) $\sigma(T_1) = \mathrm{cl}\,\delta$;
(iv) $\pi_0(T_2) = \pi_0(T) \setminus \delta$.

Thus it will suffice to show that $\mathfrak{M}^\perp = \{0\}$. Assume to the contrary that $\mathfrak{M}^\perp \neq \{0\}$. Then since $\sigma(T_2)$ is also countable there exists a point $\mu \in \text{iso}\sigma(T_2)$. Since by assumption T_2 is restriction-convexoid and hence by Lemma2.2 isoloid, it follows that $\mu \in \pi_0(T_2)$ and $\mu \notin \delta$. Again using the spectral projection at $\mu \in \mathbb{C}$ we can decompose T_2 as the direct sum

$$T_2 = T_3 \oplus T_4,$$

where $\sigma(T_3) = \pi_0(T_3) = \{\mu\}$ and $\sigma(T_4) = \sigma(T_2)\setminus\{\mu\}$. Since again T_3 is convexoid, the same argument as the above gives that μ is an isolated normal eigenvalue of T_3 and further by assumption $T_2^* = T_3^* \oplus T_4^*$. But since $T_1 - \mu I$ and $T_4 - \mu I$ are both one-one we have

$$N(T - \mu I) = N(T_3 - \mu I) = N(T_3^* - \overline{\mu} I)$$

Further since $\pi_0(T_1^*) = \overline{\delta}$ and $\overline{\mu} \notin \sigma(T_4^*)$, it follows that $N(T^* - \overline{\mu} I) = N(T_3^* - \overline{\mu} I)$, and therefore $N(T - \mu I) = N(T^* - \overline{\mu} I) = N(T_3^* - \overline{\mu} I)$, which implies that $\mu \in \delta$, giving a contradiction. This completes the proof.

We have been unable to answer if restriction-convexoid operators are reduced by each of its eigenspaces corresponding to isolated eigenvalues. If the answer were affirmative then we would answer Berberian question affirmatively.

COROLLARY 2.4. *If T is restriction-convexoid and is reduced by each of its eigenspaces corresponding to isolated eigenspaces and $\sigma_e(T) = \{0\}$ then T is compact and normal.*

PROOF. Since $\sigma_e(T) = \{0\}$ and $\sigma(T) \subset \{0\} \cup p_{00}(T)$, $\sigma(T)$ is countable. By Theorem 2.3, T is normal. By the argument of Theorem 2.3, $T = T_1 \oplus \{0\}$; thus $\sigma(T) = \sigma(T_1) \cup \sigma\{0\} = \pi_0(T)$, thus $\pi_0(T) \setminus \{0\} = \text{iso}\sigma(T) \setminus \sigma_e(T) = p_{00}(T)$. Therefore the nonzero eigenvalues are *Riesz* points, so that they are either finite or form a null sequence, which implies T is compact.

COROLLARY 2.5. *If T is restriction-convexoid and is reduced by each of its eigenspaces corresponding to isolated eigenspaces and all but a finite number of elements of $\sigma(T)$ are real, then T is normal.*

PROOF. We can decompose $T = T_1 \oplus T_2$ as in Theorem2.3, where T_1 is normal. Since $\sigma(T_2)$ is real and T_2 is convexoid by hypotheses, T_2 is hermitian; thus T is normal.

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