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A NOTE ON A THEOREM OF GROTHENDIECK IN C_p-THEORY

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ABSTRACT. We study some properties of og-spaces and show some sufficient conditions that a space X satisfy the conclusion of the Grothendieck theorem.

1. Introduction

We restrict our attention to the space of all real-valued continuous functions on a Tychonoff space X with the topology of pointwise convergence. This space is denoted by $C_p(X)$.

A. Grothendieck proved in [4] that if X is countably compact and A is countably compact in $C_p(X)$, then the closure of A in $C_p(X)$ is compact And this theorem was generalized by A. V. Arhangel'skii [2], M. O. Asanov and N. V. Velichko [3] and the authors [5]

In this paper, we study some properties of og-spaces and show sufficient conditions that a space X satisfies the conclusion of the Grothendieck theorem.

Throughout this paper, we assume that all spaces are Tychonoff spaces. Standard notations, not explained below, are the same as in [1], [6] and [7].

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2. Results

Let R denote the real line with the usual topology and let us recall some definitions.

A subset A of a space X is *bounded* in X if $f(A)$ is bounded in R for each $f \in C_p(X)$. It is not difficult to prove that if a subset A of a space X is bounded. then the closure of A is also bounded in X.

A space X is an *og-space* [2] if for each bounded subset A of X, the closure of A in X is compact. It is obvious that every compact space is an og-space.

We know that if $C_p(X)$ is an og-space, then X satisfies the conclusion of the Grothendieck theorem.

THEOREM 1. *Every realcompact space is an og-space.*

PROOF. Let X be a realcompact space. It is sufficient to prove that every bounded closed subset of X is compact. By compactness of X, there exists an index set Λ such that X is homeomorphic to a closed subspace of the product space $\Pi_{\alpha\in\Lambda} R_\alpha$, where $R_\alpha = R$ for each $\alpha \in \Lambda$ Let A be a bounded closed subset of X and let $p_\alpha : X \to R_\alpha$ denote the αth projection. Since each p_α is a real-valued continuous function on X and A is bounded in X, $p_\alpha(A)$ is bounded in R, and so $p_\alpha(A) \subset I_\alpha$ for some close d interval I_α of R. Hence, $A \subset \Pi_{\alpha\in\Lambda} I_\alpha$. From the facts that A is closed in X and X is homeomorphic to a closed subspace of $\Pi_{\alpha\in\Lambda} R_\alpha$, we have that A is homeomorphic to a closed subset V of $\Pi_{\alpha\in\Lambda} R_\alpha$. So, since A is homeomorphic to V and $A \subset \Pi_{\alpha\in\Lambda} I_\alpha$.

$$Cl_{\Pi I_\alpha}(V) = Cl_{\Pi R_\alpha}(V) \cap \Pi_{\alpha\in\Lambda} I_\alpha = V \cap \Pi_{\alpha\in\Lambda} I_\alpha = V,$$

i.e., V is closed in $\Pi_{\alpha\in\Lambda} I_\alpha$. By compactness of $\Pi_{\alpha\in\Lambda} I_\alpha$. V is compact and thus A is compact.

We call a function $f : X \to Y$ *strictly τ-continuous* if for each subset A of X with $|A| \leq \tau$, where $|A|$ denotes the cardinal of A, there exists a continuous functions $g : X \to Y$ such that $g|_A = f|_A$, where $f|_A$ denotes the restriction function of f on A *The weak functional tightness*(or R-tightness) of a space X [1], denoted by $t_R(X)$, is the smallest infinite cardinal τ such that every real-valued strictly τ-continuous function on X is continuous

From Theorem 1. we have the following corollaries.

COROLLARY 2. *If the weak functional tightness of $C_p(X)$ is countable, then X is an og-space.*

PROOF. Since $t_R(C_p(X))$ is countable, by [1, II.4.17 Corollary and 0.5.5.Corollary], $C_p(C_p(X))$ is realcompact and hence X is realcompact. Thus, by Theorem 1, X is an og-space.

COROLLARY 3 [1, III.4.12.THEOREM] AND [2, COROLLARY 2.8]. *If the weak functional tightness of a space X is countable, then $C_p(X)$ is an og-space.*

PROOF. By [1, II 4 17.Corollary], $C_p(X)$ is realcompact. Hence, by Theorem 1. $C_p(X)$ is an og-space

REMARK. It is clear that for a separable space $C_p(X)$, the weak functional tightness of $C_p(X)$ is countable Hence for a separable space $C_p(X)$, X is an og-space.

We say that a family Ω of subspaces of a space X *strongly functionally generates* X if for each real-valued discontinuous function f on X, there exists a member Y of Ω such that the function $f|_Y : Y \to R$ is discontinuous [1]

THEOREM 4. *If X is a normal space, then $C_p(X)$ is an og-space.*

PROOF. Let Y be a bounded closed subset of X and let f be a real-valued continuous function on Y. Since every closed subset of a normal space is C-embedded, f has a continuous extension on X. And since every bounded closed subset of a normal space is pseudocompact [1, III.4.4 Proposition] and normal, Y is countably compact. Hence, by [3] or [1, III.4.1.Theorem], we have that every bounded closed set in $C_p(Y)$ is compact Thus X is strongly functionally generated by the family of bounded closed subsets Y of X such that every bounded closed set in $C_p(Y)$ is compact Therefore, according to [1, III 4.14.Proposition], $C_p(X)$ is an og-space.

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