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ON THE INTERSECTION OF TWO TORUS KNOTS

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ABSTRACT We show that the intersection of two standard torus knots of type (λ_1, λ_2) and (β_1, β_2) induces an automorphism of the cyclic group $\mathbb{Z}_d$, where d is the intersection number of the two torus knots and give an elementary proof of the fact that all non-trivial torus knots are strongly invertiable knots We also show that the intersection of two standard knots on the 3-torus $S^1 \times S^1 \times S^1$ induces an isomorphism of cyclic groups.

1. Introduction

Throughout this paper, we shall denote the set of all integers by $\mathbb{Z}$ and the cyclic group of order d by $\mathbb{Z}_d = \{0, 1, \cdots, d-1\}$. For any two integers p and q, by $(p,q) = 1$ we shall mean that p and q are relatively prime integers.

Let $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 2\}$ be the 3-sphere in the complex 2-space $\mathbb{C}^2$. A simple closed curve embedded into S^3 is called a knot and the torus knots are simple closed curves embedded on the standard torus $T^2 = S^1 \times S^1 = \{(z, w) \in S^3 \mid |z| = |w| = 1\}$. For $A = (\lambda_1, \lambda_2) \in \mathbb{Z} \times \mathbb{Z}$ with $(\lambda_1, \lambda_2) = 1$, let $\alpha_A = \{\alpha_A(t) = (e^{i2\lambda_1 t}, e^{i2\lambda_2 t}) \in T^2 | t \in [0, \pi]\}$ be a standard torus knot A torus knot is said to be of *type* (λ_1, λ_2), denoted by $T(\lambda_1, \lambda_2)$ or simply T_A, if it is

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homologous to the knot α_A. The torus knot $T(\lambda_1, \lambda_2)$ is said to be trivial(or unknotted) if either $\lambda_1 = \pm 1$ or $\lambda_2 = \pm 1$. Two non-trivial torus knots $T(\lambda_1, \lambda_2)$ and $T(\beta_1, \beta_2)$ are of the *same type* if and only if (β_1, β_2) is equal to one of $(\lambda_1, \lambda_2), (\lambda_2, \lambda_1), (-\lambda_1, \lambda_2)$, and $(-\lambda_1, -\lambda_2)$[1].

In this paper, we show that the intersection of two standard torus knots of type (λ_1, λ_2) and (β_1, β_2) induces an automorphism of the cyclic group $\mathbb{Z}_d$ of order $d = |\lambda_1\beta_2 - \lambda_2\beta_1|$, the intersection number of $T(\lambda_1, \lambda_2)$ with $T(\beta_1, \beta_2)$, and give an elementary proof of the fact that all non-trivial torus knots are strongly invertiable knots(cf. [2]). We also show that the intersection of two standard knots on the 3-torus $S^1 \times S^1 \times S^1$ induces an isomorphism of cyclic groups.

2. Intersection of two torus knots

Let $A = (\lambda\lambda_1, \lambda\lambda_2)$ and $B = (\beta\beta_1, \beta\beta_2)$ with $\lambda, \beta > 0$ and $(\lambda_1, \lambda_2) = (\beta_1, \beta_2) = 1$. Suppose that $A \neq B$. Then for $t \in [0, \frac{\pi}{\lambda})$ and $s \in [0, \frac{\pi}{\beta})$, $\alpha_A(t) = \alpha_B(s)$ if and only if $\lambda\lambda_k t - \beta\beta_k s \in \pi\mathbb{Z} (k = 1, 2)$ if and only if there exist $m, n \in \mathbb{Z}$ such that

$$(*) \qquad t = \frac{|\beta_2 m - \beta_1 n|}{\lambda d}\pi \in [0, \frac{\pi}{\lambda}), s = \frac{|\lambda_2 m - \lambda_1 n|}{\beta d}\pi \in [0, \frac{\pi}{\beta}),$$

where $d = |\lambda_1\beta_2 - \lambda_2\beta_1|$.

LEMMA 2.1. *Let (λ_1, λ_2) and (β_1, β_2) be two pairs of relatively prime integers. If $\lambda_1\beta_2 - \lambda_2\beta_1 > 0$ (respectively, $\lambda_1\beta_2 - \lambda_2\beta_1 < 0$), then there exists a unique $(m_0, n_0) \in \mathbb{Z} \times \mathbb{Z}$ such that*

$$m_0\beta_2 - n_0\beta_1 = 1, \ (\lambda_2 m_0 - \lambda_1 n_0, \lambda_1\beta_2 - \lambda_2\beta_1) = 1,$$

and $0 < \lambda_2 m_0 - \lambda_1 n_0 < \lambda_1\beta_2 - \lambda_2\beta_1$ (respectively, $\lambda_1\beta_2 - \lambda_2\beta_1 < \lambda_2 m_0 - \lambda_1 n_0 < 0$).

PROOF. Replacing (β_1, β_2) by $(-\beta_1, -\beta_2)$ it suffices to show for the case $\lambda_1\beta_2 - \lambda_2\beta_1 > 0$. Since $(\beta_1, \beta_2) = 1$, we can choose $m, n \in \mathbb{Z}$ such that $m\beta_2 - n\beta_1 = 1$. Pick an integer k (unique) such that $\frac{\lambda_1 n - \lambda_2 m}{\lambda_1\beta_2 - \lambda_2\beta_1} \leq k < 1 + \frac{\lambda_1 n - \lambda_2 m}{\lambda_1\beta_2 - \lambda_2\beta_1}$. Then $\lambda_1 n - \lambda_2 m \leq k(\lambda_1\beta_2 - \lambda_2\beta_1) < \lambda_1\beta_2 -$

$\lambda_2\beta_1 + \lambda_1 n - \lambda_2 m$. This implies that $0 \le \lambda_2(m - k\beta_1) - \lambda_1(n - k\beta_2) < \lambda_1\beta_2 - \lambda_2\beta_1$ By setting $m_0 = m - k\beta_1$ and $n_0 = n - k\beta_2$, we have that $m_0\beta_2 - n_0\beta_1 = 1$.

Suppose that c is a common divisor of $\lambda_2 m_0 - \lambda_1 n_0$ and $\lambda_1\beta_2 - \lambda_2\beta_1$. Then $\lambda_2 m_0 - \lambda_1 n_0 = cx, \lambda_1\beta_2 - \lambda_2\beta_1 = cy$ for some $x, y \in \mathbb{Z}$. It then follows that $\lambda_2 m_0\beta_2 - \lambda_1 n_0\beta_2 = cx\beta_2, \lambda_1\beta_2 n_0 - \lambda_2\beta_1 n_0 = cyn_0$. By adding these two equations, we get $\lambda_2(\beta_2 m_0 - \beta_1 n_0) = c(x\beta_2 + yn_0)$. Since $\beta_2 m_0 - \beta_1 n_0 = 1, c$ is a divisor of λ_2. Similarly by adding the two equations $\lambda_2 m_0\beta_1 - \lambda_1 n_0\beta_1 = cx\beta_1, \lambda_1\beta_2 m_0 - \lambda_2\beta_1 m_0 = cym_0$, we have that c is a divisor of λ_1. Since λ_i's are relatively prime, c must be ± 1. Therefore $(\lambda_2 m_0 - \lambda_1 n_0, \lambda_1\beta_2 - \lambda_2\beta_1) = 1$

Now suppose that $m_1\beta_2 - n_1\beta_1 = 1$ and $0 < \lambda_2 m_1 - \lambda_1 n_1 < \lambda_1\beta_2 - \lambda_2\beta_1$ Then $(m_1 - m_0)\beta_2 = (n_1 - n_0)\beta_1$. Since β_1 and β_2 are relatively prime, $n_1 - n_0 = \beta_2\mu$ for some $\mu \in \mathbb{Z}$. Now since $0 < \lambda_2 m_i - \lambda_1 n_i < \lambda_1\beta_2 - \lambda_2\beta_1$ for $i = 0, 1$, it then follows that

$$-(\lambda_1\beta_2 - \lambda_2\beta_1) < \lambda_2(m_0 - m_1) + \lambda_1(n_1 - n_0) < \lambda_1\beta_2 - \lambda_2\beta_1.$$

Note that $\lambda_2(m_0 - m_1) + \lambda_1(n_1 - n_0) = k(\lambda_1\beta_2 - \lambda_2\beta_1)$ This implies that $-1 < \mu < 1$ and hence $\mu = 0$. Therefore, $n_1 = n_0$ and $m_1 = m_0$

Theorem 2.2. *Let $A = (\lambda_1, \lambda_2)$ and $B = (\beta_1, \beta_2)$ be two pairs of relatively prime integers and let $d = |\lambda_1\beta_2 - \lambda_2\beta_1| > 0$. Then the standard torus knots T_A and T_B intersect at d- points, and there is an automorphism $\sigma : \mathbb{Z}_d \to \mathbb{Z}_d$ such that $\alpha_A(\frac{k\pi}{d}) = \alpha_B(\frac{\sigma(k)\pi}{d})$ for each $k \in \mathbb{Z}_d$.*

Proof. Suppose that $\lambda_1\beta_2 - \lambda_2\beta_1 = d > 0$ By Lemma 2.1, there exist $m_0, n_0 \in \mathbb{Z}$ such that $\beta_2 m_0 - \beta_1 n_0 = 1, (\lambda_2 m_0 - \lambda_1 n_0, d) = 1$, and $0 < \lambda_2 m_0 - \lambda_1 n_0 < d$. Let σ be the automorphism on $\mathbb{Z}_d$ defined by $\sigma(1) = \lambda_2 m_0 - \lambda_1 n_0$. For $k \in \mathbb{Z}_d$, let $\sigma(k) = p$. Then $(\lambda_2 m_0 - \lambda_1 n_0)k = p + qd$ for some $q \in \mathbb{Z}$. Set $t_k = \frac{k}{d}\pi, s_k = \frac{p}{d}\pi = \frac{\sigma(k)}{d}\pi$. Since $\beta_2 m_0 - \beta_1 n_0 = 1$,

$$\begin{aligned}
\lambda_1 t_k - \beta_1 s_k &= \frac{\pi}{d}[k\lambda_1(\beta_2 m_0 - \beta_1 n_0) - \beta_1(\lambda_2 m_0 k - \lambda_1 n_0 k - qd)] \\
&= \frac{\pi}{d}[km_0(\lambda_1\beta_2 - \lambda_2\beta_1) + \beta_1 qd] \\
&= \frac{\pi}{d}(km_0 + \beta_1 q)d \in \pi\mathbb{Z}.
\end{aligned}$$

Similarly, we have $\lambda_2 t_k - \beta_2 s_k = \frac{\pi}{d}(n_0 k + q\beta_2)d \in \pi\mathbb{Z}$. Therefore (t_k, s_k) satisfies the equation $(*)$ for the case that $\lambda = \beta \doteq 1$. By observing that α_A and α_B intersect at most d- points, we conclude that these are all solutions of $\alpha_A(t) = \alpha_B(s)$ for $(t, s) \in [0, \pi) \times [0, \pi)$.

If $\lambda_1\beta_2 - \lambda_2\beta_1 < 0$, then the automorphism σ is defined by $\sigma(1) = d + (\lambda_2 m_0 - \lambda_1 n_0)$, where (m_0, n_0) is the unique pair of the integers satisfying $\beta_2 m_0 - \beta_1 n_0 = 1, \lambda_1\beta_2 - \lambda_2\beta_1 < \lambda_2 m_0 - \lambda_1 n_0 < 0$ and $(\lambda_2 m_0 - \lambda_1 n_0, d) = 1$.

EXAMPLE 2.3 (1) Let $A = (3,5), B = (2,5)$. Then $d = 5$ and so the torus knots T_A and T_B intersect at 5 points and the corresponding automorphism $\sigma : \mathbb{Z}_5 \to \mathbb{Z}_5$ is given by $\sigma(1) = 4$ $(m_0 = -1, n_0 = -3)$.

(2) Let $A = (3,4), B = (3,5)$. Then $d = 3, \sigma(1) = 2$ $(m_0 = -1, n_0 = -2)$.

(3) Let $A = (7,9), B = (3,5)$. Then $d = 8, \sigma(1) = 5 (m_0 = -1, n_0 = -2)$.

A knot K in S^3 is said to be *strongly invertible* if there exists an orientation preserving involution of S^3 such that the fixed points of the involution are exactly two points lie in the knot K.

Let $J : S^3 \to S^3$ be the orientation preserving involution of S^3 defined by $J(z, w) = (-\overline{z}, -\overline{w})$, where $\overline{z}$ denotes the complex conjugate of z. Then $Fix(J) = \{(z, w) \in S^3 | J((z, w)) = (z, w)\} = \{(ix, iy) \in \mathbb{C}^2 \mid x, y \in \mathbb{R}, x^2 + y^2 = 2\} \cong S^1$. It is easy to see that the torus knot T_A of type $A = (\lambda_1, \lambda_2)$ is invariant under J if and only if both λ_1 and λ_2 are relatively prime odd integers. In this case, we have that $Fix(J) \cap T_A = \{(i, i), (-i, -i)\}$ and T_A is a strongly invertible knot.

Now let $A = (\lambda_1, \lambda_2)$ and $B = (\beta_1, \beta_2)$ be two pairs of relatively prime odd integers such that $|\lambda_1\beta_2 - \lambda_2\beta_1| = 2$. Then it is clear that the intersection points of T_A and T_B are the points $\alpha_A(0) = \alpha_B(0) = (1, 1)$ and $\alpha_A(\frac{\pi}{2}) = \alpha_B(\frac{\pi}{2}) = (-1, -1)$. Define two simple closed curves $T_k(A, B) : [0, \pi] \to T^2 (k = 1, 2)$ by

$$T_1(A, B) = \begin{cases} \alpha_A(t) & 0 \le t \le \frac{\pi}{2} \\ \alpha_B(\pi - t) & \frac{\pi}{2} \le t \le \pi, \end{cases}$$

$$T_2(A, B) = \begin{cases} \alpha_A(t) & 0 \le t \le \frac{\pi}{2} \\ \alpha_B(t) & \frac{\pi}{2} \le t \le \pi \end{cases}$$

Then we have the following :

THEOREM 2.4.
(1) $T_1(A, B)$ *is the strongly invertible torus knot of type* $(\frac{|\lambda_1-\beta_1|}{2}, \frac{|\lambda_2-\beta_2|}{2})$.
(2) $T_2(A, B)$ *is the strongly invertible torus knot of type* $(\frac{|\lambda_1+\beta_1|}{2}, \frac{|\lambda_2+\beta_2|}{2})$

PROOF. Since T_A and T_B are invariant under the involution J, one may easily see that $T_1(A, B)$ and $T_2(A, B)$ are invariant under the involution J. Note that $Fix(J) \cap T^2 = \{(i, i), (i, -i), (-i, i), (-i, -i)\}$ and $\alpha_X(\frac{\pi}{4}) = (\epsilon_1 i, \epsilon_2 i), \alpha_X(\frac{3\pi}{4}) = (\epsilon_1' i, \epsilon_2' i)$, where $X = A$ or B and $\epsilon_k, \epsilon_k' \in \{1, -1\}(k = 1, 2)$. Thus $Fix(J) \cap T_k(A, B)$ are two points lie in $T_k(A, B)$ for each $k = 1, 2$. Hence $T_k(A, B)(k = 1, 2)$ is a strongly invertible knot

Now let $p : \mathbb{C} \to T^2$ be the universal covering projection of T^2 defined by $p(x + iy) = (e^{i2x}, e^{i2y})$ for $x, y \in \mathbb{R}$ The group of covering transformations of p is isomorphic to the group $\mathbb{Z} \oplus \mathbb{Z}$. For each pair $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$, the map $t_a : \mathbb{C} \to \mathbb{C}$ defined by

$$t_a(z) = z + \pi a, \text{ where } a = m + in \in \mathbb{C},$$

is a covering transformation and so $pt_a = p$. It is well known that a torus knot represented by a loop $K : [0, \pi] \to T^2$ is of type (u, v) if and only if K lifts to a path $\hat{K} : [0, \pi] \to \mathbb{C}$ such that $\hat{K}(\pi) - \hat{K}(0) = \pi(u + iv)$

By considering the lifts of $T_k(A, B)(k = 1, 2)$ to the universal cover $\mathbb{C}$ of the torus T^2 and using the covering transformations, it is not difficult to see that $T_1(A, B)$ is the torus knot of type $(\frac{|\lambda_1-\beta_1|}{2}, \frac{|\lambda_2-\beta_2|}{2})$ and $T_2(A, B)$ is the torus knot of type $(\frac{|\lambda_1+\beta_1|}{2}, \frac{|\lambda_2+\beta_2|}{2})$. This completes the proof.

COROLLARY 2.5. *Every torus knots is strongly invertible.*

PROOF Let $A = (p, q)$ be an arbitrary given pair of relatively prime integers. If both p and q are odd integers, then we know already that the torus knot T_A is a strongly invertible knot. Thus we may assume that $p(\neq 0)$ is even and q is odd By Theorem 2.4, it is sufficient to show that there exist two pairs of relatively prime odd integers $B = (\lambda_1, \lambda_2)$

and $C = (\beta_1, \beta_2)$ such that either $p = \frac{|\beta_1+\lambda_1|}{2}$ and $q = \frac{|\beta_2-\lambda_2|}{2}$ or $p = \frac{|\beta_1-\lambda_1|}{2}$ and $q = \frac{|\beta_2-\lambda_2|}{2}$. To do this we present a method for finding the integers satisfying the required conditions.

Step 1. By Euclidean algorithm, find m and n such that $pm - qn = 1$.

Step 2. Replace m and n by $m' := m + q$ and $n' := n + p$ if m is odd.

Step 3. Find an odd integer k such that $m' - qk > 0, n' - pk > 0$.

Step 4. Set $\lambda_1 := n' - pk$ and $\lambda_2 := m' - qk$.

Step 5. Set $\beta_1 := 2p + \lambda_1$, $\beta_2 := 2q + \lambda_2$.

One may easily check that λ_i and β_i are odd integers for $i = 1, 2$. This implies that the torus knot of type (p, q) can be represented by $T_k(A, B)$ for some k which is a strongly invertible knot.

EXAMPLE 2.6. (1) $p = 2, q = 3$:

$$\begin{aligned}(m, n) = (-1, -1) &\to (m', n') = (m + q, n + p) = (2, 1)\\ &\to (m' - qk, n' - pk) = (2 - 3k, 1 - 2k)\\ &\to k = -1\\ &\to (\lambda_1, \lambda_2) = (n' - pk, m' - qk) = (3, 5)\\ &\to (\beta_1, \beta_2) = (2p + \lambda_1, 2q + \lambda_2) = (7, 11).\end{aligned}$$

(2) $p = 8, q = 3$:

$$\begin{aligned}(m, n) = (2, 5) &\to (m', n') = (m, n) = (2, 5)\\ &\to (m' - qk, n' - pk) = (2 - 3k, 5 - 8k)\\ &\to k = -1\\ &\to (\lambda_1, \lambda_2) = (13, 5)\\ &\to (\beta_1, \beta_2) = (29, 11).\end{aligned}$$

3. Intersection of two standard knots in $S^1 \times S^1 \times S^1$

Let $A = (\lambda_1, \lambda_2, \lambda_3), B = (\beta_1, \beta_2, \beta_3) \in (\mathbb{Z}^*)^3 = \mathbb{Z}^* \times \mathbb{Z}^* \times \mathbb{Z}^*$, where $\mathbb{Z}^* = \mathbb{Z} - \{0\}$. Suppose that g.c.d.$\{\lambda_1, \lambda_2, \lambda_3\}$ = g.c.d.$\{\beta_1, \beta_2, \beta_3\} = 1$

Then we have the following simple closed curves $\alpha_A, \alpha_B : [0, \pi) \to T^3 = S^1 \times S^1 \times S^1$, the 3-torus, defined by

$$\alpha_A(t) = (e^{i2\lambda_1 t}, e^{i2\lambda_2 t}, e^{i2\lambda_3 t}),$$
$$\alpha_B(t) = (e^{i2\beta_1 t}, e^{i2\beta_2 t}, e^{i2\beta_3 t}).$$

Suppose that $A \neq \pm B$ in $(\mathbb{Z}^*)^3$ For $1 \leq i < j \leq 3$, let $D_{ij} = \lambda_i\beta_j - \lambda_j\beta_i$. Then by hypothesis $D_{ij} \neq 0$ for some $i \neq j$. Without loss of the generality, we may assume that $i = 1, j = 2$. Let $\lambda =$ g.c d.$\{\lambda_1, \lambda_2\}, \beta =$ g.c.d.$\{\beta_1, \beta_2\}$ and let $A' = (\lambda_1, \lambda_2) = (\lambda\lambda'_1, \lambda\lambda'_2), B' = (\beta_1, \beta_2) = (\beta\beta'_1, \beta\beta'_2)$, and $d = |\lambda'_1\beta'_2 - \lambda'_2\beta'_1|$. Since $D_{12} \neq 0, d \neq 0$

THEOREM 3.1. *There exist two subgroups H_1 and H_2 of $\mathbb{Z}_{\lambda d}$ and $\mathbb{Z}_{\beta d}$, respectively, and an isomorphism $\sigma : H_1 \to H_2$ such that the two simple closed curves α_A and α_B has $|H_1|$-intersection points and $\alpha_A(\frac{m}{\lambda d}\pi) = \alpha_B(\frac{\sigma(m)}{\beta d}\pi)$ for $m \in H_1$*

PROOF. Let σ' be the automorphism of $\mathbb{Z}_d$ defined in the Theorem 2.2 viewed as $A = (\lambda'_1, \lambda'_2), B = (\beta'_1, \beta'_2)$. Then for $t \in [0, \frac{\pi}{\lambda}), s \in [0, \frac{\pi}{\beta})$, $\alpha_{A'}(t) = \alpha_{B'}(s)$ if and only if $t = \frac{m}{\lambda d}\pi, s = \frac{\sigma'(m)}{\beta d}\pi$ for some $m \in \mathbb{Z}_d$ In particular, for $t, s \in [0, \pi)$, we have that $\alpha_{A'}(t) = \alpha_{B'}(s)$ if and only if $t = \frac{dk+m}{\lambda d}\pi, s = \frac{dk'+\sigma'(m)}{\beta d}\pi$ for some $m \in \mathbb{Z}_d, k \in \mathbb{Z}_\lambda, k' \in \mathbb{Z}_\beta$ Since $\alpha_A(t) = \alpha_B(s)$ for $t, s \in [0, \pi)$ if and only if $\alpha_{A'}(t) = \alpha_{B'}(s)$, and $\lambda_3 t - \beta_3 s \in \pi\mathbb{Z}$. Thus there is a bijection from $\{(t, s) \in [0, \pi) \times [0, \pi) \mid \alpha_A(t) = \alpha_B(s)\}$ to

$$F := \{(dk - m, dk' + \sigma'(m)) \in \mathbb{Z}_{\lambda d} \times \mathbb{Z}_{\beta d} \mid m \in \mathbb{Z}_d, \\ \lambda_3\frac{dk + m}{\lambda d} - \beta_3\frac{dk' + \sigma'(m)}{\beta d} \in \mathbb{Z}\}$$

Let H_1 be the image of the first projection of F, that is,

$$H_1 = \{l \in \mathbb{Z}_{\lambda d} \mid \exists\, m \in \mathbb{Z}_d, k \in \mathbb{Z}_\lambda, k' \in \mathbb{Z}_\beta \\ \text{such that } (l = dk + m, dk' + \sigma'(m)) \in F\},$$

and let H_2 be the image of the second projection of F. Since σ is an automorphism of $\mathbb{Z}_d$, H_1 and H_2 are subgroups of $\mathbb{Z}_{\lambda d}$ and $\mathbb{Z}_{\beta d}$.

respectively. The map $\sigma : H_1 \to H_2$ defined by $\sigma(dk+m) = dk'+\sigma'(m)$ is an isomorphism satisfying $\alpha_A(\frac{k}{\lambda d}\pi) = \alpha_B(\frac{\sigma(k)}{\beta d}\pi)$ for $k \in H_1$.

COROLLARY 3.2. *If the components A and B are all odd integers, then the number of the intersection points of α_A with α_B are even.*

PROOF. Since $\alpha_A(\frac{\pi}{2}) = \alpha_B(\frac{\pi}{2})$, the group H_1 contains $\frac{\lambda d}{2}$ which is an element of order 2. Hence $|H_1|$ is divisible by 2.

EXAMPLE 3.3.

(1) Let $A = (6,10,15), B = (6,15,10)$. Then in our notation, $A' = 2(3,5), B' = 3(2,5), \lambda = 2, \beta = 3, d = 5$ and by Example 2.3, $\sigma'(1) = 4$. One may check that $(1,k) \notin F$ for any $k \in \mathbb{Z}_{15}$. If $k = 0, m = 2$, and $k' = 0$ then $(2,3) \in F$. Hence $H_1 = \{0,2,4,6,8\}, H_2 = \{0,3,6,9,12\}$ and $\sigma(2) = 3$.
(2) Let $A = (7,9,15), B = (3,5,3)$ Then $A' = (7,9), B' = (3,5), d = 8$ and $\sigma'(1) = 5$ ($\lambda = \beta = 1$). It satisfies that $15\frac{m}{8} - 3\frac{\sigma'(m)}{8} \in \mathbb{Z}$ for each $m \in \mathbb{Z}_8$. Hence the number of intersection points of α_A with α_B is 8.
(3) Let $A = (6,8,7), B = (6,10,5)$. Then $A' = 2(3,4), B' = 2(3,5), d = 3$ and $\sigma'(1) = 2$ ($\lambda = \beta = 2$). One may determine that $(1,5) \in F$, and thus $H_1 = H_2 \cong \mathbb{Z}_6, \sigma(1) = 5$.

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