ON THE INTERSECTION OF TWO TORUS KNOTS

Sang Youl Lee and Yongdo Lim

Abstract We show that the intersection of two standard torus knots of type $(\lambda_1, \lambda_2)$ and $(\beta_1, \beta_2)$ induces an automorphism of the cyclic group $\mathbb{Z}_d$, where $d$ is the intersection number of the two torus knots and give an elementary proof of the fact that all non-trivial torus knots are strongly invertible knots. We also show that the intersection of two standard knots on the 3-torus $S^1 \times S^1 \times S^1$ induces an isomorphism of cyclic groups.

1. Introduction

Throughout this paper, we shall denote the set of all integers by $\mathbb{Z}$ and the cyclic group of order $d$ by $\mathbb{Z}_d = \{0, 1, \ldots, d - 1\}$. For any two integers $p$ and $q$, by $(p, q) = 1$ we shall mean that $p$ and $q$ are relatively prime integers.

Let $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 2\}$ be the 3-sphere in the complex 2-space $\mathbb{C}^2$. A simple closed curve embedded into $S^3$ is called a knot and the torus knots are simple closed curves embedded on the standard torus $T^2 = S^1 \times S^1 = \{(z, w) \in S^3 \mid |z| = |w| = 1\}$. For $A = (\lambda_1, \lambda_2) \in \mathbb{Z} \times \mathbb{Z}$ with $(\lambda_1, \lambda_2) = 1$, let $\alpha_A = \{\alpha_A(t) = (e^{2\pi i \lambda_1 t}, e^{2\pi i \lambda_2 t}) \in T^2 \mid t \in [0, \pi]\}$ be a standard torus knot. A torus knot is said to be of type $(\lambda_1, \lambda_2)$, denoted by $T(\lambda_1, \lambda_2)$ or simply $T_A$, if it is

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homologous to the knot $\alpha_A$. The torus knot $T(\lambda_1, \lambda_2)$ is said to be trivial (or unknotted) if either $\lambda_1 = \pm 1$ or $\lambda_2 = \pm 1$. Two non-trivial torus knots $T(\lambda_1, \lambda_2)$ and $T(\beta_1, \beta_2)$ are of the same type if and only if $(\beta_1, \beta_2)$ is equal to one of $(\lambda_1, \lambda_2), (\lambda_2, \lambda_1), (-\lambda_1, \lambda_2),$ and $(-\lambda_1, -\lambda_2)$ [1].

In this paper, we show that the intersection of two standard torus knots of type $(\lambda_1, \lambda_2)$ and $(\beta_1, \beta_2)$ induces an automorphism of the cyclic group $\mathbb{Z}_d$ of order $d = |\lambda_1 \beta_2 - \lambda_2 \beta_1|$, the intersection number of $T(\lambda_1, \lambda_2)$ with $T(\beta_1, \beta_2)$, and give an elementary proof of the fact that all non-trivial torus knots are strongly invertable knots (cf. [2]). We also show that the intersection of two standard knots on the 3-torus $S^1 \times S^1 \times S^1$ induces an isomorphism of cyclic groups.

2. Intersection of two torus knots

Let $A = (\lambda_1, \lambda_2)$ and $B = (\beta_1, \beta_2)$ with $\lambda, \beta > 0$ and $(\lambda_1, \lambda_2) = (\beta_1, \beta_2) = 1$. Suppose that $A \neq B$. Then for $t \in [0, \frac{\pi}{\lambda})$ and $s \in [0, \frac{\pi}{\beta})$, $\alpha_A(t) = \alpha_B(s)$ if and only if $\lambda \lambda_k t - \beta \beta_k s \in \pi \mathbb{Z}(k = 1, 2)$ if and only if there exist $m, n \in \mathbb{Z}$ such that

\[(*) \quad t = \frac{|\beta_2 m - \beta_1 n|}{\lambda d} \pi \in [0, \frac{\pi}{\lambda}), s = \frac{|\lambda_2 m - \lambda_1 n|}{\beta d} \pi \in [0, \frac{\pi}{\beta}) ,
\]

where $d = |\lambda_1 \beta_2 - \lambda_2 \beta_1|$. 

**Lemma 2.1.** Let $(\lambda_1, \lambda_2)$ and $(\beta_1, \beta_2)$ be two pairs of relatively prime integers. If $\lambda_1 \beta_2 - \lambda_2 \beta_1 > 0$ (respectively, $\lambda_1 \beta_2 - \lambda_2 \beta_1 < 0$), then there exists a unique $(m_0, n_0) \in \mathbb{Z} \times \mathbb{Z}$ such that

\[m_0 \beta_2 - n_0 \beta_1 = 1, \quad (\lambda_2 m_0 - \lambda_1 n_0, \lambda_1 \beta_2 - \lambda_2 \beta_1) = 1,
\]

and $0 < \lambda_2 m_0 - \lambda_1 n_0 < \lambda_1 \beta_2 - \lambda_2 \beta_1$ (respectively, $\lambda_1 \beta_2 - \lambda_2 \beta_1 < \lambda_2 m_0 - \lambda_1 n_0 < 0$).

**Proof.** Replacing $(\beta_1, \beta_2)$ by $(-\beta_1, -\beta_2)$ it suffices to show for the case $\lambda_1 \beta_2 - \lambda_2 \beta_1 > 0$. Since $(\beta_1, \beta_2) = 1$, we can choose $m, n \in \mathbb{Z}$ such that $m \beta_2 - n \beta_1 = 1$. Pick an integer $k$ (unique) such that $\frac{\lambda_1 n - \lambda_2 m}{\lambda_1 \beta_2 - \lambda_2 \beta_1} \leq k < 1 + \frac{\lambda_1 n - \lambda_2 m}{\lambda_1 \beta_2 - \lambda_2 \beta_1}$. Then $\lambda_1 m - \lambda_2 n \leq k(\lambda_1 \beta_2 - \lambda_2 \beta_1)$ < $\lambda_1 \beta_2 -$
\[ \lambda_2 \beta_1 + \lambda_1 n - \lambda_2 m. \] This implies that \( 0 \leq \lambda_2 (m - k \beta_1) - \lambda_1 (n - k \beta_2) < \lambda_1 \beta_2 - \lambda_2 \beta_1 \). By setting \( m_0 = m - k \beta_1 \) and \( n_0 = n - k \beta_2 \), we have that \( m_0 \beta_2 - n_0 \beta_1 = 1 \).

Suppose that \( c \) is a common divisor of \( \lambda_2 m_0 - \lambda_1 n_0 \) and \( \lambda_1 \beta_2 - \lambda_2 \beta_1 \). Then \( \lambda_2 m_0 - \lambda_1 n_0 = cx \lambda_1 \beta_2 - \lambda_2 \beta_1 = cy \) for some \( x, y \in \mathbb{Z} \). It then follows that \( \lambda_2 m_0 \beta_2 - \lambda_1 n_0 \beta_2 = cx \beta_2, \lambda_1 \beta_2 n_0 - \lambda_2 \beta_1 n_0 = cy n_0 \). By adding these two equations, we get \( \lambda_2 (\beta_2 m_0 - \beta_1 n_0) = c(x \beta_2 + y n_0) \). Since \( \beta_2 m_0 - \beta_1 n_0 = 1 \), \( c \) is a divisor of \( \lambda_2 \). Similarly by adding the two equations \( \lambda_2 m_0 \beta_1 - \lambda_1 n_0 \beta_1 = cx \beta_1, \lambda_1 \beta_2 m_0 - \lambda_2 \beta_1 m_0 = cy m_0 \), we have that \( c \) is a divisor of \( \lambda_1 \). Since \( \lambda_2 \)'s are relatively prime, \( c \) must be \( \pm 1 \). Therefore \( (\lambda_2 m_0 - \lambda_1 n_0, \lambda_1 \beta_2 - \lambda_2 \beta_1) = 1 \).

Now suppose that \( m_1 \beta_2 - n_1 \beta_1 = 1 \) and \( 0 < \lambda_2 m_1 - \lambda_1 n_1 < \lambda_1 \beta_2 - \lambda_2 \beta_1 \). Then \( (m_1 - m_0)\beta_2 = (n_1 - n_0)\beta_1 \). Since \( \beta_1 \) and \( \beta_2 \) are relatively prime, \( n_1 - n_0 = \beta_2 \mu \) for some \( \mu \in \mathbb{Z} \). Now since \( 0 < \lambda_2 m_1 - \lambda_1 n_1 < \lambda_1 \beta_2 - \lambda_2 \beta_1 \) for \( \mu = 0, 1 \), it then follows that

\[-(\lambda_1 \beta_2 - \lambda_2 \beta_1) < \lambda_2 (m_0 - m_1) + \lambda_1 (n_1 - n_0) < \lambda_1 \beta_2 - \lambda_2 \beta_1.\]

Note that \( \lambda_2 (m_0 - m_1) + \lambda_1 (n_1 - n_0) = k (\lambda_1 \beta_2 - \lambda_2 \beta_1) \). This implies that \( -1 < \mu < 1 \) and hence \( \mu = 0 \). Therefore, \( n_1 = n_0 \) and \( m_1 = m_0 \).

**Theorem 2.2.** Let \( A = (\lambda_1, \lambda_2) \) and \( B = (\beta_1, \beta_2) \) be two pairs of relatively prime integers and let \( d = |\lambda_1 \beta_2 - \lambda_2 \beta_1| > 0 \). Then the standard torus knots \( T_A \) and \( T_B \) intersect at \( d \) points, and there is an automorphism \( \sigma : \mathbb{Z}_d \rightarrow \mathbb{Z}_d \) such that \( \alpha_A (\frac{k \pi}{d}) = \alpha_B (\frac{\sigma(k) \pi}{d}) \) for each \( k \in \mathbb{Z}_d \).

**Proof.** Suppose that \( \lambda_1 \beta_2 - \lambda_2 \beta_1 = d > 0 \). By Lemma 2.1, there exist \( m_0, n_0 \in \mathbb{Z} \) such that \( \beta_2 m_0 - \beta_1 n_0 = 1, (\lambda_2 m_0 - \lambda_1 n_0, d) = 1 \), and \( 0 < \lambda_2 m_0 - \lambda_1 n_0 < d \). Let \( \sigma \) be the automorphism on \( \mathbb{Z}_d \) defined by \( \sigma (1) = \lambda_2 m_0 - \lambda_1 n_0 \). For \( k \in \mathbb{Z}_d \), let \( \sigma (k) = p \). Then \( (\lambda_2 m_0 - \lambda_1 n_0)k = p + q d \) for some \( q \in \mathbb{Z} \). Set \( t_k = \frac{k}{d} \pi, s_k = \frac{p}{d} \pi = \frac{\sigma(k) \pi}{d} \).

Since \( \beta_2 m_0 - \beta_1 n_0 = 1 \),

\[
\lambda_1 t_k - \beta_1 s_k = \frac{\pi}{d} [k \lambda_1 (\beta_2 m_0 - \beta_1 n_0) - \beta_1 (\lambda_2 m_0 k - \lambda_1 n_0 k - q d)]
= \frac{\pi}{d} [k m_0 (\lambda_1 \beta_2 - \lambda_2 \beta_1) + \beta_1 q d]
= \frac{\pi}{d} (k m_0 + \beta_1 q) d \in \pi \mathbb{Z}.
\]
Similarly, we have $\lambda_2 t_k - \beta_2 s_k = \frac{\pi}{2} (n_0 k + q \beta_2) d \in \pi \mathbb{Z}$. Therefore $(t_k, s_k)$ satisfies the equation (*) for the case that $\lambda = \beta = 1$. By observing that $\alpha_A$ and $\alpha_B$ intersect at most $d$ points, we conclude that these are all solutions of $\alpha_A(t) = \alpha_B(s)$ for $(t, s) \in [0, \pi) \times [0, \pi)$.

If $\lambda_1 \beta_2 - \lambda_2 \beta_1 < 0$, then the automorphism $\sigma$ is defined by $\sigma(1) = d + (\lambda_2 m_0 - \lambda_1 n_0)$, where $(m_0, n_0)$ is the unique pair of the integers satisfying $\beta_2 m_0 - \beta_1 n_0 = 1, \lambda_1 \beta_2 - \lambda_2 \beta_1 < \lambda_2 m_0 - \lambda_1 n_0 < 0$ and $(\lambda_2 m_0 - \lambda_1 n_0, d) = 1$.

**Example 2.3** (1) Let $A = (3, 5), B = (2, 5)$. Then $d = 5$ and so the torus knots $T_A$ and $T_B$ intersect at 5 points and the corresponding automorphism $\sigma : \mathbb{Z}_5 \to \mathbb{Z}_5$ is given by $\sigma(1) = 4 (m_0 = -1, n_0 = -3)$.

(2) Let $A = (3, 4), B = (3, 5)$. Then $d = 3, \sigma(1) = 2 (m_0 = -1, n_0 = -2)$.

(3) Let $A = (7, 9), B = (3, 5)$. Then $d = 8, \sigma(1) = 5 (m_0 = -1, n_0 = -2)$.

A knot $K$ in $S^3$ is said to be strongly invertible if there exists an orientation preserving involution of $S^3$ such that the fixed points of the involution are exactly two points lie in the knot $K$.

Let $J : S^3 \to S^3$ be the orientation preserving involution of $S^3$ defined by $J(z, w) = (-\bar{z}, -\bar{w})$, where $\bar{z}$ denotes the complex conjugate of $z$. Then $Fix(J) = \{ (z, w) \in S^3 | J((z, w)) = (z, w) \} = \{ (ix, iy) \in \mathbb{C}^2 | x, y \in \mathbb{R}, x^2 + y^2 = 2 \} \cong S^1$. It is easy to see that the torus knot $T_A$ of type $A = (\lambda_1, \lambda_2)$ is invariant under $J$ if and only if both $\lambda_1$ and $\lambda_2$ are relatively prime odd integers. In this case, we have that $Fix(J) \cap T_A = \{(i, i), (-i, -i)\}$ and $T_A$ is a strongly invertible knot.

Now let $A = (\lambda_1, \lambda_2)$ and $B = (\beta_1, \beta_2)$ be two pairs of relatively prime odd integers such that $|\lambda_1 \beta_2 - \lambda_2 \beta_1| = 2$. Then it is clear that the intersection points of $T_A$ and $T_B$ are the points $\alpha_A(0) = \alpha_B(0) = (1, 1)$ and $\alpha_A(\frac{\pi}{2}) = \alpha_B(\frac{\pi}{2}) = (-1, -1)$. Define two simple closed curves $T_k(A, B) : [0, \pi] \to T^2(k = 1, 2)$ by

$$ T_1(A, B) = \begin{cases} \alpha_A(t) & 0 \leq t \leq \frac{\pi}{2} \\ \alpha_B(\pi - t) & \frac{\pi}{2} \leq t \leq \pi, \end{cases} $$

$$ T_2(A, B) = \begin{cases} \alpha_A(t) & 0 \leq t \leq \frac{\pi}{2} \\ \alpha_B(t) & \frac{\pi}{2} \leq t \leq \pi. \end{cases} $$
Then we have the following:

**THEOREM 2.4.**
(1) \( T_1(A, B) \) is the strongly invertible torus knot of type \( \left( \frac{|\lambda_1 - \beta_1|}{2}, \frac{|\lambda_2 - \beta_2|}{2} \right) \).
(2) \( T_2(A, B) \) is the strongly invertible torus knot of type \( \left( \frac{|\lambda_1 + \beta_1|}{2}, \frac{|\lambda_2 + \beta_2|}{2} \right) \).

**PROOF.** Since \( T_A \) and \( T_B \) are invariant under the involution \( J \), one may easily see that \( T_1(A, B) \) and \( T_2(A, B) \) are invariant under the involution \( J \). Note that \( \text{Fix}(J) \cap T^2 = \{ (z, i), (i, -z), (-i, -i), (-i, i) \} \) and \( \alpha_X(z) = (\epsilon_1 z, \epsilon_2 i), \alpha_X(z) = (\epsilon_1' i, \epsilon_2' z) \), where \( X = A \) or \( B \) and \( \epsilon_k, \epsilon_k' \in \{ 1, -1 \} \) \( (k = 1, 2) \). Thus \( \text{Fix}(J) \cap T_k(A, B) \) are two points lie in \( T_k(A, B) \) for each \( k = 1, 2 \). Hence \( T_k(A, B) \) is a strongly invertible knot.

Now let \( p : \mathbb{C} \rightarrow T^2 \) be the universal covering projection of \( T^2 \) defined by \( p(x + iy) = (e^{2ix}, e^{2iy}) \) for \( x, y \in \mathbb{R} \). The group of covering transformations of \( p \) is isomorphic to the group \( \mathbb{Z} \oplus \mathbb{Z} \). For each pair \( (m, n) \in \mathbb{Z} \oplus \mathbb{Z} \), the map \( t_m : \mathbb{C} \rightarrow \mathbb{C} \) defined by

\[
t_m(z) = z + \pi a, \text{ where } a = m + in \in \mathbb{C},
\]

is a covering transformation and so \( p t_m = p \). It is well known that a torus knot represented by a loop \( K : [0, \pi] \rightarrow T^2 \) is of type \( (u, v) \) if and only if \( K \) lifts to a path \( \tilde{K} : [0, \pi] \rightarrow \mathbb{C} \) such that \( \tilde{K}(\pi) = \tilde{K}(0) = \pi(u + iv) \).

By considering the lifts of \( T_k(A, B)(k = 1, 2) \) to the universal cover \( \mathbb{C} \) of the torus \( T^2 \) and using the covering transformations, it is not difficult to see that \( T_1(A, B) \) is the torus knot of type \( \left( \frac{|\lambda_1 - \beta_1|}{2}, \frac{|\lambda_2 - \beta_2|}{2} \right) \) and \( T_2(A, B) \) is the torus knot of type \( \left( \frac{|\lambda_1 + \beta_1|}{2}, \frac{|\lambda_2 + \beta_2|}{2} \right) \). This completes the proof.

**COROLLARY 2.5.** Every torus knots is strongly invertible.

**PROOF.** Let \( A = (p, q) \) be an arbitrary given pair of relatively prime integers. If both \( p \) and \( q \) are odd integers, then we know already that the torus knot \( T_A \) is a strongly invertible knot. Thus we may assume that \( p(\neq 0) \) is even and \( q \) is odd. By Theorem 2.4, it is sufficient to show that there exist two pairs of relatively prime odd integers \( B = (\lambda_1, \lambda_2) \)
and \( C = (\beta_1, \beta_2) \) such that either \( p = \frac{|\beta_1 + \lambda_1|}{2} \) and \( q = \frac{|\beta_2 - \lambda_2|}{2} \) or \( p = \frac{|\beta_1 - \lambda_1|}{2} \) and \( q = \frac{|\beta_2 - \lambda_2|}{2} \). To do this we present a method for finding the integers satisfying the required conditions.

Step 1. By Euclidean algorithm, find \( m \) and \( n \) such that \( pm - qn = 1 \).

Step 2. Replace \( m \) and \( n \) by \( m' := m + q \) and \( n' := n + p \) if \( m \) is odd.

Step 3. Find an odd integer \( k \) such that \( m' - qk > 0, n' - pk > 0 \).

Step 4. Set \( \lambda_1 := n' - pk \) and \( \lambda_2 := m' - qk \).

Step 5. Set \( \beta_1 := 2p + \lambda_1 \), \( \beta_2 := 2q + \lambda_2 \).

One may easily check that \( \lambda_i \) and \( \beta_i \) are odd integers for \( i = 1, 2 \). This implies that the torus knot of type \((p, q)\) can be represented by \( T_k(A, B) \) for some \( k \) which is a strongly invertible knot.

Example 2.6. (1) \( p = 2, q = 3 \):

\[
(m, n) = (-1, -1) \rightarrow (m', n') = (m + q, n + p) = (2, 1) \\
\rightarrow (m' - qk, n' - pk) = (2 - 3k, 1 - 2k) \\
\rightarrow k = -1 \\
\rightarrow (\lambda_1, \lambda_2) = (n' - pk, m' - qk) = (3, 5) \\
\rightarrow (\beta_1, \beta_2) = (2p + \lambda_1, 2q + \lambda_2) = (7, 11).
\]

(2) \( p = 8, q = 3 \):

\[
(m, n) = (2, 5) \rightarrow (m', n') = (m, n) = (2, 5) \\
\rightarrow (m' - qk, n' - pk) = (2 - 3k, 5 - 8k) \\
\rightarrow k = -1 \\
\rightarrow (\lambda_1, \lambda_2) = (13, 5) \\
\rightarrow (\beta_1, \beta_2) = (29, 11).
\]

3. Intersection of two standard knots in \( S^1 \times S^1 \times S^1 \)

Let \( A = (\lambda_1, \lambda_2, \lambda_3) \), \( B = (\beta_1, \beta_2, \beta_3) \in (\mathbb{Z}^*)^3 = \mathbb{Z}^* \times \mathbb{Z}^* \times \mathbb{Z}^* \), where \( \mathbb{Z}^* = \mathbb{Z} - \{0\} \). Suppose that \( \text{g.c.d.} \{\lambda_1, \lambda_2, \lambda_3\} = \text{g.c.d.} \{\beta_1, \beta_2, \beta_3\} = 1 \)
Then we have the following simple closed curves $\alpha_A, \alpha_B : [0, \pi) \to T^3 = S^1 \times S^1 \times S^1$, the 3-torus, defined by
\[
\alpha_A(t) = (e^{i2\lambda_1 t}, e^{i2\lambda_2 t}, e^{i2\lambda_3 t}),
\alpha_B(t) = (e^{i2\beta_1 t}, e^{i2\beta_2 t}, e^{i2\beta_3 t}).
\]

Suppose that $A \neq \pm B$ in $(\mathbb{Z}^*)^3$. For $1 \leq i < j \leq 3$, let $D_{ij} = \lambda_i \beta_j - \lambda_j \beta_i$. Then by hypothesis $D_{ij} \neq 0$ for some $i \neq j$. Without loss of the generality, we may assume that $i = 1, j = 2$. Let $\lambda = \text{g.c.d.}\{\lambda_1, \lambda_2\}, \beta = \text{g.c.d.}\{\beta_1, \beta_2\}$ and let $A' = (\lambda_1, \lambda_2) = (\lambda \lambda'_1, \lambda \lambda'_2), B' = (\beta_1, \beta_2) = (\beta \beta'_1, \beta \beta'_2)$, and $d = |\lambda_1 \beta'_2 - \lambda_2 \beta'_1|$. Since $D_{12} \neq 0, d \neq 0$

**Theorem 3.1.** There exist two subgroups $H_1$ and $H_2$ of $\mathbb{Z}_{\lambda d}$ and $\mathbb{Z}_{\beta d}$, respectively, and an isomorphism $\sigma : H_1 \to H_2$ such that the two simple closed curves $\alpha_A$ and $\alpha_B$ has $|H_1|$-intersection points and $\alpha_A(m_{\lambda d}^2 \pi) = \alpha_B(m_{\beta d}^2 \pi)$ for $m \in H_1$.

**Proof.** Let $\sigma'$ be the automorphism of $\mathbb{Z}_d$ defined in the Theorem 2.2 viewed as $A = (\lambda'_1, \lambda'_2), B = (\beta'_1, \beta'_2)$. Then for $t \in [0, \pi), s \in [0, \pi)$, $\alpha_A(t) = \alpha_B(s)$ if and only if $t = \frac{m}{\lambda d} \pi, s = \frac{\sigma'(m)}{\beta d} \pi$ for some $m \in \mathbb{Z}_d$.

In particular, for $t, s \in [0, \pi)$, we have that $\alpha_A(t) = \alpha_B(s)$ if and only if $t = \frac{mk - m}{\lambda d} \pi, s = \frac{\lambda k' - \sigma'(m) \lambda}{\beta d} \pi$ for some $m \in \mathbb{Z}_d, k \in \mathbb{Z}_{\lambda}, k' \in \mathbb{Z}_{\beta}$.

Since $\alpha_A(t) = \alpha_B(s)$ for $t, s \in [0, \pi)$ if and only if $\alpha_A(t) = \alpha_B(s)$, and $\lambda_3 t - \beta_3 s \in \pi \mathbb{Z}$. Thus there is a bijection from $\{(t, s) \in [0, \pi) \times [0, \pi) \mid \alpha_A(t) = \alpha_B(s)\}$ to
\[
F := \{(mk - m, dk' + \sigma'(m)) \in \mathbb{Z}_{\lambda d} \times \mathbb{Z}_{\beta d} \mid m \in \mathbb{Z}_d, \lambda \frac{dk + m}{d} - \beta \frac{d \lambda k' - \sigma'(m)}{\beta d} \in \mathbb{Z}\}
\]

Let $H_1$ be the image of the first projection of $F$. That is,
\[
H_1 = \{l \in \mathbb{Z}_{\lambda d} \mid \exists m \in \mathbb{Z}_d, k \in \mathbb{Z}_{\lambda}, k' \in \mathbb{Z}_{\beta} \text{ such that } (l = dk + m, dk' + \sigma'(m)) \in F\},
\]
and let $H_2$ be the image of the second projection of $F$. Since $\sigma$ is an automorphism of $\mathbb{Z}_d$, $H_1$ and $H_2$ are subgroups of $\mathbb{Z}_{\lambda d}$ and $\mathbb{Z}_{\beta d}$. 

respectively. The map \( \sigma : H_1 \rightarrow H_2 \) defined by \( \sigma(dk + m) = dk' + \sigma'(m) \) is an isomorphism satisfying \( \alpha_A\left(\frac{k}{\lambda d}\right) \pi = \alpha_B\left(\frac{\sigma(k)}{\beta d}\right) \pi \) for \( k \in H_1 \).

**Corollary 3.2.** If the components \( A \) and \( B \) are all odd integers, then the number of the intersection points of \( \alpha_A \) with \( \alpha_B \) are even.

**Proof.** Since \( \alpha_A\left(\frac{n}{2}\right) = \alpha_B\left(\frac{n}{2}\right) \), the group \( H_1 \) contains \( \frac{\lambda d}{2} \) which is an element of order 2. Hence \( |H_1| \) is divisible by 2.

**Example 3.3.**

(1) Let \( A = (6, 10, 15), B = (6, 15, 10) \). Then in our notation, \( A' = 2(3, 5), B' = 3(2, 5), \lambda = 2, \beta = 3, d = 5 \) and by Example 2.3, \( \sigma'(1) = 4 \). One may check that \( (1, k) \notin F \) for any \( k \in \mathbb{Z}_{15} \). If \( k = 0, m = 2, \) and \( k' = 0 \) then \( (2, 3) \in F \). Hence \( H_1 = \{0, 2, 4, 6, 8\}, H_2 = \{0, 3, 6, 9, 12\} \) and \( \sigma(2) = 3 \).

(2) Let \( A = (7, 9, 15), B = (3, 5, 3) \). Then \( A' = (7, 9), B' = (3, 5), d = 8 \) and \( \sigma'(1) = 5 \) (\( \lambda = \beta = 1 \)). It satisfies that \( 15 \frac{m}{8} - 3 \frac{\sigma'(m)}{8} \in \mathbb{Z} \) for each \( m \in \mathbb{Z}_8 \). Hence the number of intersection points of \( \alpha_A \) with \( \alpha_B \) is 8.

(3) Let \( A = (6, 8, 7), B = (6, 10, 5) \). Then \( A' = 2(3, 4), B' = 2(3, 5), d = 3 \) and \( \sigma'(1) = 2 \) (\( \lambda = \beta = 2 \)). One may determine that \( (1, 5) \in F \), and thus \( H_1 = H_2 \cong \mathbb{Z}_6, \sigma(1) = 5 \).

**References**


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