PRIMITIVE POLYNOMIAL RINGS

Mi Hyang Kwon, Chol On Kim and Chan Huh

Abstract. We show that the intersection of two standard torus knots of type $(\lambda_1, \lambda_2)$ and $(\beta_1, \beta_2)$ induces an automorphism of the cyclic group $\mathbb{Z}_d$, where $d$ is the intersection number of the two torus knots and give an elementary proof of the fact that all non-trivial torus knots are strongly invertible knots. We also show that the intersection of two standard knots on the 3-torus $S^1 \times S^1 \times S^1$ induces an isomorphism of cyclic groups.

Throughout this paper all rings are associative with identity. Given a ring $R$, $R[x]$ denotes the polynomial ring over $R$ with $x$ its indeterminate. In this note we study the primitivity of polynomial rings, concerning the contraposition of the condition in [8] that is both a Morita invariant property and a generalization of the following two conditions:

1. the quasi-duo condition, which was initiated by Yu in [9] and is related to the Bass’ conjecture in [2],
2. the pm condition that was studied by Birkenmeier-Kim-Park in [3].

A ring $R$ is called maximally right bounded if every maximal right ideal of $R$ contains a maximal ideal of $R$. Consider a condition: (*)

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there exists a maximal right ideal that does not contain a maximal ideal. Clearly a ring $R$ satisfies (*) if and only if $R$ is not maximally right bounded. A ring is called right (left) duo if every right ideal is two-sided, and a ring is called right (left) quasi-duo if every maximal right (left) ideal of is two-sided. Commutative rings and abelian regular rings are right duo, right duo rings are right quasi-duo, and right quasi-duo rings are maximally right bounded. The $n$ by $n$ full matrix ring over a division ring, with $n$ any positive integer $\geq 2$, is not right quasi-duo and does not satisfy (*); but it is maximally right bounded. However the ring of row finite infinite matrices over a division ring, say $R$, satisfies (*) but is not maximally right bounded because there exist maximal right ideals of $R$ that do not contain the nonzero proper ideal $\{ f \in R \mid \text{rank}(f) \text{ is finite} \}$ of $R$. A ring $R$ is said to satisfy pm if every prime ideal of $R$ is maximal. Such rings are maximally right bounded, but there are rings which are maximally right bounded but do not satisfy pm as in [3, Example 3.3]. In this note we also obtain direct proofs for the contrapositions of main results in [8].

We first take the contraposition of [8, Proposition 1] as follows.

**Proposition 1.** Given a ring $R$ the following statements are equivalent:

1. $R$ satisfies (*).
2. There exists a right primitive ideal of $R$ that is not maximal.

**Proof.** (1)$\Rightarrow$(2). Since $R$ satisfies (*), there exists a maximal right ideal $M$ of $R$ that does not contain a maximal ideal of $R$. But $M$ contains a right primitive ideal of $R$ which is the bound of $M$, say $P$. Thus $P$ is not a maximal ideal of $R$.

(2)$\Rightarrow$(1). Let $P$ be a right primitive ideal of $R$ that is not maximal. There is a maximal right ideal of $R$ whose bound is $P$, so $R$ satisfies (*).

**Corollary 2.** [8, Proposition 1] Given a ring $R$ the following statements are equivalent:

1. $R$ is a maximally right bounded ring.
2. Every right primitive ideal of $R$ is maximal.
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PROOF. By Proposition 1

We next recall some properties of maximally right bounded rings in [8]. A ring \( R \) is called a PI-ring if \( R \) satisfies a polynomial identity with coefficients in the ring of integers.

LEMMA 3. [8, Corollary 2, Corollary 3 and Lemma 4] Given a ring \( R \) we have the following statements:

1. If every right primitive factor ring of \( R \) is artinian then \( R \) is maximally right bounded.
2. If \( R \) is a PI-ring then \( R \) is maximally right bounded.
3. If \( R \) is a division ring that is finite dimensional over its center then \( R[x] \) is maximally right bounded.
4. A semiprimitive maximally right bounded ring is a subdirect product of simple rings.
5. If a ring \( R \) is maximally right bounded, then so is every homomorphic image of \( R \).

Note that if given a ring \( R \) is a right primitive, then \( eRe \) is also a right primitive ring for every nonzero idempotent \( e \in R \). The following is one of our main results in this note.

THEOREM 4. Let \( R \) be a ring and \( 0 \neq e^2 = e \in R \). Suppose that \( eRe \trianglelefteq eRe \) for each proper left ideal \( I \) of \( R \). Then the following statements are equivalent:

1. \( R \) satisfies (\( \ast \)).
2. \( eRe \) satisfies (\( \ast \)).

PROOF. (2)\( \Rightarrow \) (1) By [8, Lemma 7]

(1)\( \Rightarrow \) (2). We use the proof of [8, Theorem 8]. Let \( I \) be a maximal right ideal of \( R \) whose bound is \( P \), such that \( P \) is not maximal. Then \( P \) is a right primitive ideal of \( R \). We will show that \( ePe \) is not a maximal ideal in \( eRe \). For convenience, let \( \overline{R} = R/P \), and \( \overline{r} = r + P \) for all \( r \in R \). Then \( \overline{R} \) is a right primitive ring. Since \( ePe = eRe \cap P \) and \( ePe \neq eRe \) by hypothesis, we have \( e \notin P \) and hence \( \overline{e} \) is a nonzero idempotent in \( \overline{R} \). Thus \( \overline{eRe} \) is also a right primitive ring. Since \( eRe/ePe \trianglelefteq \overline{eRe} \), \( ePe \) is a right primitive ideal of \( eRe \). Now let \( Q \) be a maximal ideal of \( R \) that contains \( P \) (of course \( P \trianglelefteq Q \)). Then \( ePe \subseteq eQe \trianglelefteq eRe \) by hypothesis
and $eQe$ is maximal in $eRe$ by Lemma 2.6. Assume $ePe = eQe$. Then $eQe = ePe \subseteq P$, and hence $(Re)Q(Re) = R(eQe) \subseteq RP = P$. Since $P$ is right primitive and $e \notin P$, we get $Q \subseteq P$, a contradiction to the fact that $P \subseteq Q$. Therefore $ePe \subseteq eQe$ and this completes the proof.

**COROLLARY 5.** [8, Theorem 8] Let $R$ be a ring and $0 \neq e^2 = e \in R$. Suppose that $e1e \subseteq eRe$ for each proper ideal $I$ of $R$. Then the following statements are equivalent:

1. $R$ is maximally right bounded.
2. $eRe$ is maximally right bounded.

We may compare the following result with [9, Proposition 2.1].

**PROPOSITION 6.** For a ring $R$ the following statements are equivalent:

1. $R$ satisfies (*).
2. Every $n$ by $n$ upper triangular matrix ring over $R$ satisfies (*).
3. Every $n$ by $n$ lower triangular matrix ring over $R$ satisfies (*), where $n$ is any finite (in this case assume $n \geq 2$) or an infinite cardinal number.

**PROOF.** We use the proofs of [8, Corollary 9]. (1)⇒(2). Let $S$ be the $n$ by $n$ upper triangular matrix ring over $R$. Note that every right primitive ideal $J$ of $S$ is of the form the $(i,i)$-entry of $J$ is a right primitive ideal of $R$ for some $i \in \{1, 2, \ldots \}$, say $P$, and every other entry of $J$ is $R$. By Proposition 1 and the condition (1), we may take a right primitive ideal $P$ in $R$ that is not a maximal ideal of $R$. So $J$ is not maximal in $S$ and this gives (2).

(2)⇒(1). Let $e$ be the matrix such that $(1,1)$-entry of $e$ is $1_R$ and other entries of $e$ are $0_R$. Then $0 \neq e^2 = e \in S$ and $eSe \subseteq R$. So $R$ satisfies (*) by the condition (2) and [8, Lemma 7].

We next obtain the equivalence (1)⇔(3) by the symmetry.

**COROLLARY 7.** [8, Corollary 9] For a ring $R$ the following statements are equivalent:

1. $R$ is maximally right bounded.
(2) Every $n \times n$ upper triangular matrix ring over $R$ is maximally right bounded.

(3) Every $n \times n$ lower triangular matrix ring over $R$ is maximally right bounded, where $n$ is any finite or an infinite cardinal number.

We denote the $n \times n$ full matrix ring over a ring $R$ by $\text{Mat}_n(R)$ for any positive integer $n$.

**Lemma 8.** [8, Corollary 24] For a ring $R$ and any positive integer $n$, the following statements are equivalent:

1. $R$ is maximally right bounded.
2. $\text{Mat}_n(R)$ is maximally right bounded.

By Lemma 8, we have the following equivalence for rings that satisfy (*).

**Corollary 9.** For a ring $R$ and any positive integer $n$, the following statements are equivalent:

1. $R$ satisfies (*).
2. $\text{Mat}_n(R)$ satisfies (*).

Therefore we have the following by Theorem 4, Corollary 9 and [1, Corollary 22.7].

**Corollary 10.** Suppose that a ring $R$ satisfies (*). Then for every finitely generated projective right $R$-module $P$, $\text{End}_R(P)$ also satisfies (*); especially the condition (*) is a Morita invariant property, where $\text{End}_R(P)$ is the endomorphism ring of $P$ over $R$.

Next we study the primitivity of polynomial rings over division rings. First we observe the polynomial rings over rings satisfying (*).

**Proposition 11.** If a ring $R$ satisfies (*), then $R[x]$ satisfies (*).

**Proof.** Notice first that $I + R[x]x$, with $I$ a right primitive ideal of $R$, is also a right primitive ideal of $R[x]$. Since $R$ satisfies (*), we may take $I$ such that $I$ is not a maximal ideal. So $I + R[x]x$ is also not a maximal ideal of $R[x]$, but a right primitive ideal of $R[x]$, hence $R[x]$ satisfies (*) by Proposition 1.
As the converse of Proposition 11, we may raise the following question.

Question. Does a ring \( R \) satisfy (*) if \( R[x] \) satisfies (*)?

However the answer is negative by the following example.

**Example 12.** Let \( W = W_1[Q] \) be the first Weyl algebra over the field \( Q \) of rational numbers, subject to \( yx = xy + 1 \), and let \( R \) be the right quotient division ring of \( W \). Then the center of \( R \) is \( Q \), and since \( R \) is purely transcendental over \( Q \), it follows that \( A = R \otimes Q Q(t) \) is not a division ring by [5, Theorem 3.21], where \( Q(t) \) is the quotient field of the polynomial ring \( Q[t] \) in an indeterminate \( t \). Hence \( A \neq R(t) \); so \( R[t] \) is right primitive by [5, Theorem 3.21], where \( R[t] \) is the polynomial ring over \( R \) in \( t \) and \( R(t) \) is the right quotient division ring of \( R[t] \). Clearly \( R \) does not satisfy (*). But the zero ideal of \( R[t] \) is right primitive which is not maximal. Therefore \( R[t] \) satisfy (*) by Proposition 1.

The following is also one of our main results in this paper.

**Theorem 13.** For a simple ring \( R \) the following statements are equivalent:

1. \( R[x] \) satisfies (*).
2. \( R[x] \) is right primitive.

**Proof.** (2)\( \Rightarrow \) (1). Note that the zero ideal of \( R[x] \) is always not maximal. Since \( R[x] \) is right primitive by the condition, \( R[x] \) satisfies (*) by Proposition 1.

(1)\( \Rightarrow \) (2). Suppose that the condition (1) holds. Then there is a right primitive ideal \( P \) of \( R[x] \) that is not maximal by Proposition 1. Let \( M \) be a maximal ideal of \( R[x] \) such that \( P \subseteq M \). Here assume \( P \neq 0 \). Then [5, Lemma 15] implies that \( P \) is generated by a nonzero central monic polynomial in \( R[x] \) because \( R \) is simple by hypothesis, say \( P = f(x)R[x] \). Also by [5, Lemma 15], \( M = h(x)R[x] \) for some nonzero central monic polynomial \( h(x) \in R[x] \). Since \( M \) contains \( P \), \( f(x) = h(x)g(x) \) for some \( g(x) \in R[x] \) and so \( P = f(x)R[x] = h(x)R[x]g(x)R[x] \). But \( P \) is right primitive (hence prime), so \( M = h(x)R[x] \subseteq P \) (a contradiction to the fact that \( P \subseteq M \))
or \( g(x)R[x] \subseteq P \) if \( g(x)R[x] \subseteq P \), then \( g(x) = f(x)m(x) \) for some \( m(x) \in R[x] \) and so \( f(x) = h(x)f(x)m(x) = f(x)h(x)m(x) \). It then follows that \( h(x)m(x) = m(x)h(x) = 1_{R[x]} \) since \( f(x) \) is monic; hence \( M = R[x] \), a contradiction to the fact that \( M \) is a maximal ideal of \( R[x] \). Consequently \( P \) must be the zero ideal and therefore \( R[x] \) is right primitive.

By Theorem 13, we obtain the following result.

**COROLLARY 14.** [8, Theorem 16] For a simple ring \( R \) the following statements are equivalent:

1. \( R[x] \) is maximally right bounded.
2. \( R[x] \) is not right primitive.

We do not know whether the condition (*) is left-right symmetric. But if \( R \) is a division ring, then \( R[x] \) satisfies (*) if and only if \( R[x] \) satisfies the "left-handed" version of (*) as in the following.

**COROLLARY 15.** Let \( R \) be a division ring. Then the following statements are equivalent:

1. \( R[x] \) satisfies (*).
2. \( R[x] \) is right primitive.
3. \( R[x] \) is left primitive.
4. \( R[x] \) satisfies the left version of (*)

**PROOF.** By [8, Lemma 18] and Theorem 13.

Due to Jacobson [7], a ring is called strongly right (left) bounded if every nonzero right (left) ideal contains a nonzero ideal; and a ring is called right (left) bounded if every essential right (left) ideal contains a nonzero ideal. Strongly right bounded rings are clearly right bounded. In [4], we have that a ring \( R \) is right duo if and only if every factor ring of \( R \) is strongly right bounded. In the following arguments we obtain the connections among the preceding conditions, right duoness, maximally right boundedness and the condition (*).
Lemma 16. [6, Theorem 15.2] Let $R$ be a simple Artinian ring. Then the following statements are equivalent:

1. $R[x]$ is right bounded.
2. $R[x]$ is not right primitive.

A ring $R$ is called right Ore if given $a, b \in R$ with $b$ regular there exist $a_1, b_1 \in R$ with $b_1$ regular such that $ab_1 = ba_1$. It is a well-known fact that $R$ is a right Ore ring if and only if there exists the classical right quotient ring of $R$. Left case may be defined similarly. Given a division ring $D$, $D[x]$ is an Ore (i.e., both right and left Ore) domain, so every nonzero right (left) ideal is essential; hence $D[x]$ is strongly right (left) bounded if and only if it is right (left) bounded. Consequently we have the following results.

Proposition 17. Let $R$ be a simple Artinian ring. Then the following statements are equivalent:

1. $R[x]$ is right bounded.
2. $R[x]$ is not right primitive.
3. $R[x]$ is maximally right bounded.

Proof. By Corollary 14 and Lemma 16.

Corollary 18. Let $D$ be a division ring. Then the following statements are equivalent:

1. $D[x]$ is strongly right bounded.
2. $D[x]$ is right bounded.
3. $D[x]$ is not right primitive.
4. $D[x]$ is maximally right bounded.
5. The left versions of the statements (1)–(4).

Proof. By Corollary 15, Proposition 17 and the argument prior to Proposition 17.

There exists a division ring that does not satisfy the statements in Corollary 18. Let $R$ be the Weyl division algebra over a field of characteristic zero. Then $R[x]$ is right primitive by [6, Theorem 15.16].
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Mi Hyang Kwon
Department of Mathematics Education
Pusan National University
Pusan 609-735, Korea

Chol On Kim and Chan Huh
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail: chuh@hyowon.pusan.ac.kr