East Asian Math J 16(2000), No 1, pp 81-87

ANOTHER TRANSFORMATION OF THE GENERALIZED HYPERGEOMETRIC SERIES

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ABSTRACT Bose and Mitra obtained certain interesting transformations of the generalized hypergeometric series by using some known summation formulas and employing suitable contour integrations in complex function theory. The authors aim at providing another transformation of the generalized hypergeometric series by making use of the technique as those of Bose and Mitra and a known summation formula, which Bose and Mitra did not use, for the Gaussian hypergeometric series

1. Introduction and Preliminaries

The Pochhammer symbol $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ (Γ denotes the familiar Gamma function) is used in this study as is the generalized hypergeometric function

$$(1.1)\qquad {}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p\,;\\ \beta_1,\cdots,\beta_q\,;\end{matrix}\; z\right] = {}_pF_q[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;\ z] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!},$$

Received January 15, 2000.

1991 Mathematics Subject Classification: Primary 33C05, Secondary 33C20

Key words and phrases Generalized hypergeometric series, Kummer theorem Watson theorem

in which any values of parameters (throughout this paper) leading to results which do not make sense are tacitly excluded.

For a detailed exposition of the properties of the generalized hypergeometric function (1.1), see (for example) Rainville [4] and Slater [5].

The following summation formulas are well known:

$$
\begin{aligned}
(1.2)\quad & {}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e & ; \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e & ; \end{matrix}\; 1\right] \\
& = \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(1+a-c-e)\Gamma(1+a-d-e)};
\end{aligned}
$$

$$
\begin{aligned}
(1.3)\quad & {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & ; \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & ; \end{matrix}\; -1\right] \\
& = \frac{\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+a-c-d)};
\end{aligned}
$$

$$
\begin{aligned}
(1.4)\quad & {}_3F_2\left[\begin{matrix} a, & b, & c & ; \\ & 1+a-b, & 1+a-c & ; \end{matrix}\; 1\right] \\
& = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)};
\end{aligned}
$$

$$
\begin{aligned}
(1.5)\quad & {}_3F_2\left[\begin{matrix} a, & b, & c; \\ & \frac{1}{2}(a+b+1), & 2c; \end{matrix}\; 1\right] \\
& = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+c)\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a+c)\Gamma(\frac{1}{2}-\frac{1}{2}b+c)};
\end{aligned}
$$

$$
(1.6)\qquad {}_2F_1[\,a,\ b;\ c;\,1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.
$$

Mitra [3] and Bose [2] used the formulas (1.2) $\sim$ (1.6) to get some transformations of the generalized hypergeometric series by employing suitable contour integrations in complex function theory. Here the authors aim at giving another additional transformation of ${}_pF_q$ by using the same technique of Bose [2] and Mitra [3] and another summation formula which was not used by the afore-cited works of Bose and Mitra.

2. A Transformation formula

Besides the summation formulas listed in Section 1, there is another well-known summation identity:

$$(2.1)\qquad {}_2F_1\left[\begin{matrix} a, & b & ; \\ & 1+a-b & : \end{matrix} -1\right] = \frac{\Gamma(1+a-b)\Gamma\left(\frac{1}{2}\right)}{2^a\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(\frac{a}{2}-b+1\right)},$$

which is due to Kummer [4, p.68].

Multiplying both sides of (2.1) by $\frac{\Gamma(a)\Gamma(b)}{\Gamma(1+a-b)}$, we, after slight rearrangement, find that

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)}{2^a\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(\frac{a}{2}-b+1\right)} = \sum_{r=0}^{\infty}\frac{\Gamma(a+r)\Gamma(b-r)(-1)^r}{\Gamma(r+1)\Gamma(1+a-b+r)},$$

which, upon setting $b=-s$ and $\frac{1}{2}+\frac{a}{2}=\rho_1$, yields

(2.2)
$$\frac{\Gamma(-s)}{\Gamma(\rho_1+\frac{1}{2}+s)} = \frac{2^{2\rho_1-1}\Gamma(\rho_1)}{\Gamma(\frac{1}{2})\Gamma(2\rho_1-1)}\sum_{r=0}^{\infty}\frac{\Gamma(2\rho_1-1+r)\Gamma(-s+r)(-1)^r}{\Gamma(r+1)\Gamma(2\rho_1+r+s)}$$

If we multiply both side of (2.2) by

$$\frac{1}{2\pi i}\frac{\Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\ \cdots\ \Gamma(\alpha_p+s)(-x)^s}{\Gamma(\rho_2+s)\ \cdots\ \Gamma(\rho_q+s)},$$

we get

$$(2.3)\qquad \begin{aligned} &\frac{1}{2\pi i}\cdot\frac{\Gamma(-s)\Gamma(\alpha_1+s)\ \cdots\ \Gamma(\alpha_p+s)(-x)^s}{\Gamma(\rho_1+\frac{1}{2}+s)\Gamma(\rho_2+s)\ \cdots\ \Gamma(\rho_q+s)} \\ &= \frac{1}{2\pi i}\cdot\frac{2^{2\rho_1-1}\Gamma(\rho_1)}{\Gamma(\frac{1}{2})\Gamma(2\rho_1-1)}\sum_{r=0}^{\infty}\frac{\Gamma(2\rho_1-1+r)}{\Gamma(r+1)} \\ &\quad\cdot\frac{\Gamma(-s+r)(-1)^r\ \Gamma(\alpha_1+s)\ \cdots\ \Gamma(\alpha_p+s)(-x)^s}{\Gamma(2\rho_1+r+s)\cdot\Gamma(\rho_2+s)\ \cdots\ \Gamma(\rho_q+s)}. \end{aligned}$$

Let us integrate both sides of (2.3) with respect to s along a contour C_N which encloses only $(N+1)$ poles of $\Gamma(-s)$ but none of the other poles of the integrands, so that $\Re(\alpha_i) > 0$. Indeed, the contour C_N is taken to be a triangle (obtained by joining the point $N+\epsilon$ on the real axis where $0 < \epsilon < 1$, to two points on the imaginary axis equidistant from the origin) except for a semicircular loop round the origin so as to include the origin.

In the limit when $N \to \infty$ (through positive integral values), C_N becomes C, a contour which consists of two infinite lines starting parallel to the positive real axis from two points on the imaginary axis (equidistant from the origin) and the finite portion of the imaginary axis between them.

Now using the well-known asymptotic formula for Γ:

$$\Gamma(\alpha+s) \sim \exp\left\{\left(s+\alpha-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log(2\pi)\right\} + O\left(\frac{1}{s}\right), \tag{2.4}$$

where $|s| \to \infty$; $|\arg(s)| \leq \pi - \epsilon$; $|\arg(s+\alpha)| \leq \pi - \epsilon$ for $0 < \epsilon < \pi$.

It is not difficult to find that (*cf.* Bose [2]; see also Rainville [4, pp. 94-102]) the general term of the summation part on the left-hand side of (2.3) tends to zero exponentially as $\Re(s) \to \infty$ along C provided that

$$x \in \mathbb{C} \ \text{ when } \ p-q-1<0; \quad |x|<0 \ \text{ when } \ p-q-1=0. \tag{2.5}$$

This argument shows that the order of integration and summation may be interchanged even when the range of integration is infinite.

We thus obtain

$$\begin{aligned} &{}_pf_q\left(\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p; \\ \rho_1+\frac{1}{2}, \rho_2, \ldots, \rho_q;\end{matrix}\ x\right) \\ &= \frac{2^{2\rho_1-1}\Gamma(\rho_1)}{\Gamma(\frac{1}{2})\Gamma(2\rho_1-1)}\sum_{r=0}^{\infty}\frac{\Gamma(2\rho_1-1+r)(-1)^r}{\Gamma(r+1)} \\ &\quad\cdot\frac{1}{2\pi i}\int_C \frac{\Gamma(r-s)\Gamma(\alpha_1+s)\ \cdots\ \Gamma(\alpha_p+s)(-x)^s}{\Gamma(2\rho_1+r+s)\Gamma(\rho_2+s)\ \cdots\ \Gamma(\rho_q+s)}ds, \end{aligned} \tag{2.6}$$

where

$$ {}_pf_q\left(\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p; \\ \rho_1, \rho_2, \ldots, \rho_q;\end{matrix}\ x\right) := \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}{\Gamma(\rho_1)\cdots\Gamma(\rho_q)}\,{}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \rho_1, \ldots, \rho_q;\end{matrix}\ x\right] $$

under the condition (2.5).
Regarding C to be the limit of C_N as mentioned above. with replaced s by $r+t$ and integrated with respect to t, we find that
(2.7)

$$ {}_pf_q\left(\begin{matrix}\alpha_1, & \alpha_2, \ldots, \alpha_p & : \\ \rho_1+\frac{1}{2}, & \rho_2, \ldots, \rho_q & ;\end{matrix}\ x\right) = \frac{2^{2\rho_1-1}\Gamma(\rho_1)}{\Gamma(\frac{1}{2})\Gamma(2\rho_1-1)} $$
$$ \cdot\sum_{r=0}^{\infty}\frac{\Gamma(2\rho_1-1+r)x^r}{\Gamma(r-1)}\,{}_pf_q\left(\begin{matrix}\alpha_1+r, \alpha_2+r, \ldots, \alpha_p+r; \\ 2\rho_1+2r, \rho_2+r, \ldots, \rho_q+r;\end{matrix}\ x\right) $$

under the condition (2.5).
The convergence of the right-hand side of (2 7) can readily be justified as in [2]. In fact, the absolute value of the general term in (2.7) is less than the absolute value of the general term of a hypergeometric series of the form ${}_{p+1}f_{q+1}$ with argument $x/4$, which evidently converges absolutely for all x when $p-q-1<0$ or for $|x|<1$ when $p-q-1=0$.

The restriction on the $\alpha's$ may be removed by analytic continuation.

Now (2.7) also holds for $|x|=1$ when $p-q-1=0$, which can be proved by appealing to Abel's theorem on continuity of power series by first proving that both sides of (2.7) converge when $|x|=1$

It is also not difficult to show that both sides of (2.7) converge for $x=1$ provided

$$ \Re\left(\sum_{i=2}^{q}\rho_i+\rho_1+\frac{1}{2}-\sum_{i=1}^{q+1}\alpha_i\right)>0, $$

and similarly for $x=-1$ provided

$$ \Re\left(\sum_{i=2}^{q}\rho_i+\rho_1+\frac{3}{2}-\sum_{i=1}^{q+1}\alpha_i\right)>0 $$

When $p > q+1$, the left-hand side of (2.7) is divergent unless either $x = 0$ or one of the $\alpha's$ is a negative integer.

We have considered the convergence of (2.7) for the case when $p = q+1$.

The special case of (2.7) when $x = 1$, $p-1 = 2 = q$, $\rho_1 = \alpha_3$, and $\rho_2 = \frac{1}{2}(\alpha_1+\alpha_2+1)$, with the aid of Watson's Theorem (1.5), becomes

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}\alpha_1, & \alpha_2, & \rho_1; \\ \rho_1+\frac{1}{2}, & \frac{1}{2}(\alpha_1+\alpha_2+1); & \end{matrix}\;1\right]\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\rho_1)}{\Gamma(\rho_1+\frac{1}{2})\Gamma\left(\frac{1}{2}+\frac{\alpha_1}{2}+\frac{\alpha_2}{2}\right)} \\
&= \frac{2^{\rho_1-1}\Gamma(\rho_1)}{\Gamma(2\rho_1-1)}\sum_{r=0}^{\infty}\frac{\Gamma(2\rho_1-1+r)}{\Gamma(r+1)}\cdot\frac{\Gamma\left(\frac{1}{2}+\rho_1+r\right)\Gamma\left(\frac{1}{2}+\frac{\alpha_1}{2}+\frac{\alpha_2}{2}+r\right)}{\Gamma\left(\frac{1}{2}+\frac{\alpha_1}{2}+\frac{r}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\alpha_2}{2}+\frac{r}{2}\right)} \\
&\cdot\frac{\Gamma\left(\frac{1}{2}-\frac{\alpha_1}{2}+\frac{\alpha_2}{2}+\rho_1\right)\Gamma(\alpha_1+r)\Gamma(\alpha_2-r)\Gamma(\rho_1+r)}{\Gamma\left(\frac{1}{2}-\frac{\alpha_1}{2}+\rho_1+\frac{r}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\alpha_2}{2}+\rho_1+\frac{r}{2}\right)\Gamma(2\rho_1+2r)\Gamma\left(\frac{1}{2}+\frac{\alpha_1}{2}+\frac{\alpha_2}{2}+r\right)}
\end{aligned}$$

which, upon dividing r into even and odd parts, yields

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}\alpha_1, & \alpha_2, & \rho_1; \\ \rho_1+\frac{1}{2}, & \frac{1}{2}(\alpha_1+\alpha_2+1); & \end{matrix}\;1\right] = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\rho_1+\frac{1}{2}\right) \\
&\cdot\Gamma\left(\frac{1}{2}+\frac{\alpha_1}{2}+\frac{\alpha_2}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\alpha_1}{2}-\frac{\alpha_2}{2}+\rho_1\right) f\left(\alpha_1,\alpha_2;\rho_1,\rho_2\right),
\end{aligned} \tag{2.8}$$

where, for convenience,

$$\begin{aligned}
&f\left(\alpha_1,\alpha_2;\rho_1,\rho_2\right) \\
&:= \frac{1}{\Gamma\left(\frac{1}{2}+\frac{\alpha_1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\alpha_2}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\alpha_1}{2}+\rho_1\right)\Gamma\left(\frac{1}{2}-\frac{\alpha_2}{2}+\rho_1\right)} \\
&\cdot{}_4F_3\left[\begin{matrix} & \rho_1-\frac{1}{2}, & \rho_1, & \frac{\alpha_1}{2}, & \frac{\alpha_2}{2}; \\ \frac{1}{2}, & \frac{1}{2}-\frac{\alpha_1}{2}+\rho_1, & \frac{1}{2}-\frac{\alpha_2}{2}+\rho_1; & & \end{matrix}\;1\right]
\end{aligned}$$

$$+\frac{(2\rho_1-1)\alpha_1\alpha_2}{4\Gamma\left(1+\frac{\alpha_1}{2}\right)\Gamma\left(1+\frac{\alpha_2}{2}\right)\Gamma\left(1-\frac{\alpha_1}{2}+\rho_1\right)\Gamma\left(1-\frac{\alpha_2}{2}+\rho_1\right)}$$
$$\cdot {}_4F_3\left[\begin{matrix}\rho_1+\frac{1}{2}, & \rho_1, \frac{\alpha_1}{2}+\frac{1}{2}, & \frac{\alpha_2}{2}+\frac{1}{2}; \\ \frac{3}{2}, & 1-\frac{\alpha_1}{2}+\rho_1, & 1-\frac{\alpha_2}{2}+\rho_1;\end{matrix}\; 1\right],$$

the two ${}_4F_3$ being well-poised, and terminating when ρ_1 is a negative integer

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