ON THE FINITE DIFFERENCE OPERATOR $l_{N^2}(u,v)$

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ABSTRACT. In this work, we consider a finite difference operator $L_N^2$ corresponding to

$$Lu := -(u_{xx} + u_{yy}) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,$$

in $S_{h^2,1}$. We derive the relation between the absolute value of the bilinear form $l_{N^2}(u,v)$ on $S_{h^2,1} \times S_{h^2,1}$ and Sobolev $H^1$ norms.

1. Introduction and preliminaries

Let $\Omega := I \times I$, where $I = [0,1]$, and let $h = \frac{1}{N}$, where $N$ is a nonzero positive integer. The knots are given by the points $x_i = i h (i = 0, 1, \ldots, N)$ and the $i^{th}$- subinterval is denoted by $I_i := [x_{i-1}, x_i] (i = 1, 2, \ldots, N)$. Let $\{\xi_l\}_{l=1}^N$ be the set of local Legendre-Gauss (LG) points (see [1]) such that $\xi_l = x_{l-1} + \frac{h}{2}$. With $\xi_0 = 0$ and $\xi_{N+1} = 1$, define $S_{h,1}$ as the space of continuous piecewise linear functions on the unit interval whose restriction on each subinterval $[\xi_l, \xi_{l+1}] (l = 0, 1, \ldots, N)$ is linear satisfying the zero boundary conditions. The basis functions for $S_{h,1}$ are given by the usual hat functions $\{\phi_k\}_{k=1}^N$ satisfying $\phi_k(\xi_l) = \delta_{k,l}, \quad l = 0, 1, \ldots, N + 1$. The two dimensional space $S_{h^2,1}$ is defined by the tensor product of two one-dimensional spaces

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The basis functions \( \{ \Phi_{\mu}(x, y), \mu = 1, 2, \cdots, N^2 \} \) of \( S_{h,1} \) are given by \( \Phi_{\mu}(x, y) := \phi_k(x) \phi_l(y), \mu = k + N(l - 1) \). We will denote \( a_N \sim b_N \) if there are two positive constants \( \alpha, \beta \), independent of \( N \), such that for all \( N, \ 0 < \alpha a_N < b_N < \beta a_N \). Let \( \{ u_i \}_{i=1}^N \) be such that \( u_i := u(\xi_i), \ i = 1, 2, \cdots, N \), where \( \xi_i \) is the local LG points in \( I \). Then the one dimensional second order central finite difference operator corresponding to \( -u'' \) is given by

\[
[L_N u]_k := \frac{2}{h_k + h_{k-1}} \left\{ -\frac{u_{k+1}}{h_k} + \left( \frac{1}{h_k} + \frac{1}{h_{k-1}} \right) u_k - \frac{u_{k-1}}{h_{k-1}} \right\}
\]

where \( h_k := \xi_{k+1} - \xi_k, (k = 0, 1, \cdots, N) \).

2. Main results

In this section we will compare the finite difference scheme defined in the space \( S_{h,1} \) corresponding to

\[
Lu := -(u_{xx} + u_{yy}) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
\]

with the usual Sobolev \( H^1 \) norm of \( u \).

Let \( \{ u_\mu \}_{\mu=1}^N \) be such that

\[
u_{k,l} = u_\mu := u(P_\mu), \quad P_\mu = (\xi_k, \xi_l), \quad \mu = k + (l - 1)N,
\]

where \( P_\mu \) is the local LG points in \( \Omega \).

The finite difference operator \( L_{N^2} \) corresponding to \( L \) on \( S_{h,1} \) can be written as

\[
[L_{N^2} u]_{k,l} := \frac{2}{h_k + h_{k-1}} \left\{ -\frac{u_{k+1,l}}{h_k} + \left( \frac{1}{h_k} + \frac{1}{h_{k-1}} \right) u_{k,l} - \frac{u_{k-1,l}}{h_{k-1}} \right\} \\
+ \frac{2}{h_l + h_{l-1}} \left\{ -\frac{u_{k,l+1}}{h_l} + \left( \frac{1}{h_l} + \frac{1}{h_{l-1}} \right) u_{k,l} - \frac{u_{k,l-1}}{h_{l-1}} \right\}.
\]

Define the bilinear form on \( S_{h,1} \times S_{h,1} \) as

\[
l_{N^2}(u, v) := h^2 \sum_{i,j=1}^N [L_{N^2} u]_{i,j} \bar{v}_{i,j}.
\]
Let, for $u_1$ and $u_2$ in $C$, define

$$f(u_1, u_2) := \frac{1}{h_0} u_1 \bar{v}_1 + \frac{1}{h_1} (u_2 - u_1)(\bar{v}_2 - \bar{v}_1),$$

$$g(u_1, u_2) := \frac{4}{3h_0} u_1 \bar{v}_1 + \frac{1}{h_1} (u_2 - u_1)(\bar{v}_2 - \frac{4}{3} \bar{v}_1).$$

One can easily verify that

\begin{align*}
(2.2) \quad & \Re g(u_1, u_2) \sim f(u_1, u_2), \\
& |\Im g(u_1, u_2)| \leq C f(u_1, u_2),
\end{align*}

where $C$ is a positive constant.

Note that the bilinear form $l_N(u, v)$ defined in (2.1) can be written as, using the changes of indices and boundary conditions,

\begin{align*}
(2.3) \quad & \ell_N(u, v) \\
& = h \sum_{j=1}^{N} g(u_{1,j}, u_{2,j}) + \sum_{j=2}^{N-2} \frac{(u_{j+1,j} - u_{i,j})(\bar{v}_{j+1,j} - \bar{v}_{i,j})}{h_j} \\
& + g(-u_{N,j}, -u_{N-1,j}) \\
& + h \sum_{j=2}^{N-2} \frac{(u_{j+1,j} - u_{i,j})(\bar{v}_{j+1,j} - \bar{v}_{i,j})}{h_j} \\
& - g(-u_{N,j}, -u_{N-1,j})
\end{align*}

For the continuity, we need a simple lemma.

**Lemma** If $f$ is a linear function on $[a, b]$, then there are positive numbers $C_i, i = 1, 2$, such that

$$C_1 \int_a^b f(x)^2 dx \leq \frac{b-a}{2} \{f(a)^2 + f(b)^2\} \leq C_2 \int_a^b f(x)^2 dx.$$
PROOF. Without loss of generality, we may assume $a = 0$, $b = h > 0$. If $f(x) = \xi x + \eta$. It suffices to show that there are $C_i, i = 1, 2$ such that $C_1 B \leq A \leq C_2 B$ where

$$A = \frac{h}{2} \{\eta^2 + (\xi h + \eta)^2\} = \frac{\xi^2}{2} h^3 + \xi \eta h^2 + \eta^2 h$$

$$B = \frac{\xi^2}{3} h^3 + \xi \eta h^2 + \eta^2 h$$

By a simple comparison, the first inequality is obvious with $C_1 = 1$. For the second inequality, using the inequality $ab \leq \frac{1}{2\epsilon} a^2 + \frac{\epsilon}{2} b^2$, we have

$$C_2 B - A = \left(\frac{C_2}{3} - \frac{1}{2}\right) \xi^2 h^3 + (C_2 - 1) \xi \eta h^2 + (C_2 - 1) \eta^2 h$$

$$\geq \left(\frac{C_2}{3} - \frac{1}{2} - \frac{C_2 - 1}{2\epsilon}\right) \xi^2 h^3 + (C_2 - 1 - \frac{(C_2 - 1)\epsilon}{2}) \eta^2 h.$$  

We need to find $C_2 > 0$ with some positive $\epsilon > 0$ with

$$C_2 \left(\frac{1}{3} - \frac{1}{2\epsilon}\right) - \frac{1}{2} + \frac{1}{2\epsilon} \geq 0 \text{ and } C_2 (1 - \frac{\epsilon}{2}) - 1 + \frac{\epsilon}{2} \geq 0.$$  

Now it is easy to find $C_2 > 0$ with $\frac{3}{2} < \epsilon < 2$.

**Theorem.** For $u, v \in S_{h^2,1}$, there is a positive constant $C_3$, independent of $h$, such that

$$|\ell_{N^2}(u, v)| \leq C_3 ||u||_1 ||v||_1.$$  

**Proof.** We have, from (2.3).

$$|\ell_{N^2}(u, v)|$$

$$\leq h \sum_{j=1}^N |g(u_{1,j}, u_{2,j})| + \sum_{j=2}^{N-2} \frac{|u_{j+1,j} - u_{j,j}||\tilde{v}_{j+1,j} - \tilde{v}_{j,j}|}{h_j}$$

$$+ |g(-u_{N,j}, -u_{N-1,j})|$$

$$+ h \sum_{i=1}^N |g(u_{i,1}, u_{i,2})| + \sum_{j=2}^{N-2} \frac{|u_{i,j+1} - u_{i,j}||\tilde{v}_{i,j+1} - \tilde{v}_{i,j}|}{h_j}$$

$$+ |g(-u_{i,N}, -u_{i,N-1})|.$$
By (2.2), we have \( |g(u_1, u_2)| \leq C f(u_1, u_2) \) where \( C \) is an absolute positive constant. Therefore, using the fact \( u \in S_{h, i} \) and the boundary condition, we have

\[
|\ell_{N^2}(u, v)|
\]

\[
\leq C h \sum_{j=1}^{N} \left\{ \left( \sum_{i=0}^{N} \frac{1}{h_i} |u_{x+1,j} - u_{x,j}|^2 \right)^{\frac{1}{2}} \left( \sum_{i=0}^{N} \frac{1}{h_i} |ar{u}_{x+1,j} - \bar{u}_{x,j}|^2 \right)^{\frac{1}{2}} \right\}
\]

\[
+ C h \sum_{i=1}^{N} \left\{ \left( \sum_{j=0}^{N} \frac{1}{h_j} |u_{x,j+1} - u_{x,j}|^2 \right)^{\frac{1}{2}} \left( \sum_{j=0}^{N} \frac{1}{h_j} |ar{u}_{x,j+1} - \bar{u}_{x,j}|^2 \right)^{\frac{1}{2}} \right\}
\]

\[
= C h \sum_{j=1}^{N} \left( \sum_{i=0}^{N} \int_{\xi_i}^{\xi_{i+1}} |u_x(\cdot, \xi_j)|^2 \, dx \right)^{\frac{1}{2}} \left( \sum_{i=0}^{N} \int_{\xi_i}^{\xi_{i+1}} |v_x(\cdot, \xi_j)|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
+ C h \sum_{i=1}^{N} \left( \sum_{j=0}^{N} \int_{\xi_j}^{\xi_{j+1}} |u_y(\xi_i, \cdot)|^2 \, dy \right)^{\frac{1}{2}} \left( \sum_{j=0}^{N} \int_{\xi_j}^{\xi_{j+1}} |v_y(\xi_i, \cdot)|^2 \, dy \right)^{\frac{1}{2}}
\]

\[
= C h \sum_{j=1}^{N} \left( \int_{0}^{1} |u_x(\cdot, \xi_j)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{0}^{1} |v_x(\cdot, \xi_j)|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
+ C h \sum_{i=1}^{N} \left( \int_{0}^{1} |u_y(\xi_i, \cdot)|^2 \, dy \right)^{\frac{1}{2}} \left( \int_{0}^{1} |v_y(\xi_i, \cdot)|^2 \, dy \right)^{\frac{1}{2}}
\]

If we use Cauchy Schwarz inequality, the boundary condition, and Lemma, we have

\[
|\ell_{N^2}(u, v)| \leq dh \left( \sum_{j=1}^{N} \int_{0}^{1} |u_x(\cdot, \xi_j)|^2 \, dx \right)^{\frac{1}{2}} \left( \sum_{j=1}^{N} \int_{0}^{1} |v_x(\cdot, \xi_j)|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
- dh \left( \sum_{i=1}^{N} \int_{0}^{1} |u_y(\xi_i, \cdot)|^2 \, dy \right)^{\frac{1}{2}} \left( \sum_{i=1}^{N} \int_{0}^{1} |v_y(\xi_i, \cdot)|^2 \, dy \right)^{\frac{1}{2}}
\]
\[
\leq d \left( \int_0^1 \sum_{j=0}^N \frac{h}{2} \left[ |u_x(\cdot, \xi_j)|^2 + |u_x(\cdot, \xi_{j+1})|^2 \right] dx \right)^{\frac{1}{2}}
\cdot \left( \int_0^1 \sum_{j=0}^N \frac{h}{2} \left[ |v_x(\cdot, \xi_j)|^2 + |v_x(\cdot, \xi_{j+1})|^2 \right] dx \right)^{\frac{1}{2}}
+ d \left( \int_0^1 \sum_{i=0}^N \frac{h}{2} \left[ |u_y(\xi_i, \cdot)|^2 + |u_y(\xi_{i+1}, \cdot)|^2 \right] dy \right)^{\frac{1}{2}}
\cdot \left( \int_0^1 \sum_{i=0}^N \frac{h}{2} \left[ |v_y(\xi_i, \cdot)|^2 + |v_y(\xi_{i+1}, \cdot)|^2 \right] dy \right)^{\frac{1}{2}}
\leq C \left( \int_\Omega u_x^2 \right)^{\frac{1}{2}} \left( \int_\Omega v_x^2 \right)^{\frac{1}{2}} + C \left( \int_\Omega u_y^2 \right)^{\frac{1}{2}} \left( \int_\Omega v_y^2 \right)^{\frac{1}{2}}
\leq C \left\{ \|u_x\|^2 + |v_y|^2 \right\}^\frac{1}{2} \left\{ \|v_y\|^2 + |v_y|^2 \right\}^\frac{1}{2} \leq C \|u\|_1 \|v\|_1,
\]

where \(C\) is a constant.

REFERENCES

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