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ON THE FINITE DIFFERENCE OPERATOR $l_{N^2}(u,v)$

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ABSTRACT. In this work, we consider a finite difference operator L_N^2 corresponding to

$$Lu := -(u_{xx} + u_{yy}) \text{ in } \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

in $S_{h^2,1}$. We derive the relation between the absolute value of the bilinear form $l_{N^2}(u,v)$ on $S_{h^2,1} \times S_{h^2,1}$ and Sobolev H^1 norms.

1. Introduction and preliminaries

Let $\Omega := I \times I$, where $I = [0,1]$, and let $h = \frac{1}{N}$, where N is a nonzero positive integer. The knots are given by the points $x_i = ih(i = 0,1,\ldots,N)$ and the i^{th}- subinterval is denoted by $I_i := [x_{i-1}, x_i](i = 1,2,\cdots,N)$. Let $\{\xi_i\}_{i=1}^N$ be the set of local Legendre-Gauss[LG] points (see [1]) such that $\xi_i = x_{i-1} + \frac{h}{2}$. With $\xi_0 = 0$ and $\xi_{N+1} = 1$, define $S_{h,1}$ as the space of continuous piecewise linear functions on the unit interval whose restriction on each subinterval $[\xi_i, \xi_{i+1}], (i = 0,1,\cdots,N)$ is linear satisfying the zero boundary conditions. The basis functions for $S_{h,1}$ are given by the usual hat functions $\{\phi_k\}_{k=1}^N$ satisfying $\phi_k(\xi_l) = \delta_{k,l}, \quad l = 0,1,\cdots,N+1$. The two dimensional space $S_{h^2,1}$ is defined by the tensor product of two one-dimensional spaces

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$S_{h,1}$. The basis functions $\{\Phi_\mu(x,y), \mu = 1,2,\cdots,N^2\}$ of $S_{h^2,1}$ are given by $\Phi_\mu(x,y) := \phi_k(x)\phi_l(y),\ \mu = k + N(l-1)$. We will denote $a_N \sim b_N$ if there are two positive constants α, β, independent of N, such that for all N, $0 < \alpha a_N < b_N < \beta a_N$. Let $\{u_i\}_{i=1}^N$ be such that $u_i := u(\xi_i),\ i = 1,2,\cdots,N$, where ξ_i is the local LG points in I. Then the one dimensional second order central finite difference operator corresponding to $-u''$ is given by

$$[L_N u]_k := \frac{2}{h_k + h_{k-1}}\{-\frac{u_{k+1}}{h_k} + \left(\frac{1}{h_k} + \frac{1}{h_{k-1}}\right)u_k - \frac{u_{k-1}}{h_{k-1}}\}$$

where $h_k := \xi_{k+1} - \xi_k, (k = 0, 1, \cdots, N)$.

2. Main results

In this section we will compare the finite difference scheme defined in the space $S_{h^2,1}$ corresponding to

$$Lu := -(u_{xx} + u_{yy}) \text{ in } \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

with the usual Sobolev H^1 norm of u.

Let $\{u_\mu\}_{\mu=1}^{N^2}$ be such that

$$u_{k,l} = u_\mu := u(P_\mu), \quad P_\mu = (\xi_k, \xi_l), \quad \mu = k + (l-1)N,$$

where P_μ is the local LG points in Ω.

The finite difference operator L_{N^2} corresponding to L on $S_{h^2,1}$ can be written as

$$\begin{aligned}[L_{N^2}u]_{k,l} := &\frac{2}{h_k + h_{k-1}}\{-\frac{u_{k+1,l}}{h_k} + \left(\frac{1}{h_k} + \frac{1}{h_{k-1}}\right)u_{k,l} - \frac{u_{k-1,l}}{h_{k-1}}\} \\ &+ \frac{2}{h_l + h_{l-1}}\{-\frac{u_{k,l+1}}{h_l} + \left(\frac{1}{h_l} + \frac{1}{h_{l-1}}\right)u_{k,l} - \frac{u_{k,l-1}}{h_{l-1}}\}.\end{aligned}$$

Define the bilinear form on $S_{h^2,1} \times S_{h^2,1}$ as

$$l_{N^2}(u,v) := h^2 \sum_{i,j=1}^{N} [L_{N^2}u]_{i,j}\bar{v}_{i,j}. \tag{2.1}$$

Let, for u_1 and u_2 in $\mathbf{C}$, define

$$f(u_1,u_2) := \frac{1}{h_0}u_1\bar{v_1} + \frac{1}{h_1}(u_2-u_1)(\bar{v_2}-\bar{v_1}),$$
$$g(u_1,u_2) := \frac{4}{3h_0}u_1\bar{v_1} + \frac{1}{h_1}(u_2-u_1)(\bar{v_2}-\frac{4}{3}\bar{v_1})$$

One can easily verify that

$$\begin{aligned} Re(g(u_1,u_2)) &\sim f(u_1,u_2), \\ |Im(g(u_1,u_2))| &\leq Cf(u_1,u_2), \end{aligned} \tag{2.2}$$

where C is a positive constant.

Note that the bilinear form $l_{N^2}(u,v)$ defined in (2.1) can be written as, using the changes of indices and boundary conditions,

(2.3)

$$\begin{aligned} &\ell_{N^2}(u,v) \\ &= h\sum_{j=1}^{N}[g(u_{1,j},u_{2,j}) + \sum_{i=2}^{N-2}\frac{(u_{i+1,j}-u_{i,j})(\bar{v}_{i+1,j}-\bar{v}_{i,j})}{h_i} \\ &+ g(-u_{N,j},-u_{N-1,j})] \\ &+ h\sum_{i=1}^{N}[g(u_{i,1},u_{i,2}) + \sum_{j=2}^{N-2}\frac{(u_{i,j+1}-u_{i,j})(\bar{v}_{i,j+1}-\bar{v}_{i,j})}{h_j} \\ &+ g(-u_{i,N},-u_{i,N-1})] \end{aligned}$$

For the continuity, we need a simple lemma.

LEMMA *If f is a linear function on $[a,b]$, then there are positive numbers $C_i, i=1,2,$ such that*

$$C_1\int_a^b f(x)^2dx \leq \frac{b-a}{2}\{f(a)^2+f(b)^2\} \leq C_2\int_a^b f(x)^2dx.$$

PROOF. Without of loss of generality, we may assume $a = 0, b = h > 0, f(x) = \xi x + \eta$. It suffices to show that there are $C_i, i = 1, 2$ such that $C_1 B \leq A \leq C_2 B$ where

$$A = \frac{h}{2}\{\eta^2 + (\xi h + \eta)^2\} = \frac{\xi^2}{2}h^3 + \xi\eta h^2 + \eta^2 h$$
$$B = \frac{\xi^2}{3}h^3 + \xi\eta h^2 + \eta^2 h$$

By a simple comparision, the first inequality is obvious with $C_1 = 1$ For the second inequality, using the inequality $ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$, we have

$$\begin{aligned} C_2 B - A &= (\frac{C_2}{3} - \frac{1}{2})\xi^2 h^3 + (C_2 - 1)\xi\eta h^2 + (C_2 - 1)\eta^2 h \\ &\geq (\frac{C_2}{3} - \frac{1}{2} - \frac{C_2 - 1}{2\epsilon})\xi^2 h^3 + (C_2 - 1 - \frac{(C_2 - 1)\epsilon}{2})\eta^2 h. \end{aligned}$$

We need to find $C_2 > 0$ with some positive $\epsilon > 0$ with

$$C_2(\frac{1}{3} - \frac{1}{2\epsilon}) - \frac{1}{2} + \frac{1}{2\epsilon} \geq 0 \text{ and } C_2(1 - \frac{\epsilon}{2}) - 1 + \frac{\epsilon}{2} \geq 0.$$

Now it is easy to find $C_2 > 0$ with $\frac{3}{2} < \epsilon < 2$.

THEOREM. *For $u, v \in S_{h^2,1}$, there is a positive constant C_3, independent of h, such that*

$$|\ell_{N^2}(u, v)| \leq C_3 ||u||_1 ||v||_1. \tag{2.4}$$

PROOF. We have, from (2.3),

$$\begin{aligned} &|\ \ell_{N^2}(u, v)| \\ &\leq h\sum_{j=1}^{N}[|g(u_{1,j}, u_{2,j})| + \sum_{i=2}^{N-2}\frac{|u_{i+1,j} - u_{i,j}||\bar{v}_{i+1,j} - \bar{v}_{i,j}|}{h_i} \\ &+ |g(-u_{N,j}, -u_{N-1,j})|] \\ &+ h\sum_{i=1}^{N}[|g(u_{i,1}, u_{i,2})| + \sum_{j=2}^{N-2}\frac{|u_{i,j+1} - u_{i,j}||\bar{v}_{i,j-1} - \bar{v}_{i,j}|}{h_j} \\ &+ |g(-u_{i,N}, -u_{i,N-1})|]. \end{aligned}$$

By (2.2), we have $|g(u_1,u_2)| \le Cf(u_1,u_2)$ where C is an absolute positive constant. Therefore, using the fact $u \in S_{h,1}$ and the boundary condition, we have

$$
\begin{aligned}
&|\ell_{N^2}(u,v)| \\
&\le Ch\sum_{j=1}^{N}\{\left(\sum_{i=0}^{N}\frac{1}{h_i}|u_{i+1,j}-u_{i,j}|^2\right)^{\frac12}\left(\sum_{i=0}^{N}\frac{1}{h_i}|\bar v_{i+1,j}-\bar v_{i,j}|^2\right)^{\frac12}\} \\
&+ Ch\sum_{i=1}^{N}\{\left(\sum_{j=0}^{N}\frac{1}{h_j}|u_{i,j+1}-u_{i,j}|^2\right)^{\frac12}\left(\sum_{j=0}^{N}\frac{1}{h_j}|\bar v_{i,j+1}-\bar v_{i,j}|^2\right)^{\frac12}\} \\
&= Ch\sum_{j=1}^{N}\left(\sum_{i=0}^{N}\int_{\xi_i}^{\xi_{i+1}}|u_x(\cdot,\xi_j)|^2dx\right)^{\frac12}\left(\sum_{i=0}^{N}\int_{\xi_i}^{\xi_{i+1}}|v_x(\cdot,\xi_j)|^2dx\right)^{\frac12} \\
&+ Ch\sum_{i=1}^{N}\left(\sum_{j=0}^{N}\int_{\xi_j}^{\xi_{j+1}}|u_y(\xi_i,\cdot)|^2dy\right)^{\frac12}\left(\sum_{j=0}^{N}\int_{\xi_j}^{\xi_{j+1}}|v_y(\xi_i,\cdot)|^2dy\right)^{\frac12} \\
&= Ch\sum_{j=1}^{N}\left(\int_0^1|u_x(\cdot,\xi_j)|^2dx\right)^{\frac12}\left(\int_0^1|v_x(\cdot,\xi_j)|^2dx\right)^{\frac12} \\
&+ Ch\sum_{i=1}^{N}\left(\int_0^1|u_y(\xi_i,\cdot)|^2dy\right)^{\frac12}\left(\int_0^1|v_y(\xi_i,\cdot)|^2dy\right)^{\frac12}.
\end{aligned}
$$

If we use Cauchy Schwarz inequality, the boundary condition, and Lemma, we have

$$
\begin{aligned}
|\ell_{N^2}(u,v)| \le dh&\left(\sum_{j=1}^{N}\int_0^1|u_x(\cdot,\xi_j)|^2dx\right)^{\frac12}\left(\sum_{j=1}^{N}\int_0^1|v_x(\cdot,\xi_j)|^2dx\right)^{\frac12} \\
&+ dh\left(\sum_{i=1}^{N}\int_0^1|u_y(\xi_i,\cdot)|^2dy\right)^{\frac12}\left(\sum_{i=1}^{N}\int_0^1|v_y(\xi_i,\cdot)|^2dy\right)^{\frac12}
\end{aligned}
$$

$$\leq d\left(\int_0^1 \sum_{j=0}^{N} \frac{h}{2}[|u_x(\cdot,\xi_j)|^2 + |u_x(\cdot,\xi_{j+1})|^2]dx\right)^{\frac{1}{2}}$$

$$\cdot\left(\int_0^1 \sum_{j=0}^{N} \frac{h}{2}[|v_x(\cdot,\xi_j)|^2 + |v_x(\cdot,\xi_{j+1})|^2]dx\right)^{\frac{1}{2}}$$

$$+ d\left(\int_0^1 \sum_{i=0}^{N} \frac{h}{2}[|u_y(\xi_i,\cdot)|^2 + |u_y(\xi_{i+1},\cdot)|^2]dy\right)^{\frac{1}{2}}$$

$$\cdot\left(\int_0^1 \sum_{i=0}^{N} \frac{h}{2}[|v_y(\xi_i,\cdot)|^2 + |v_y(\xi_{i+1},\cdot)|^2]dy\right)^{\frac{1}{2}}$$

$$\leq C\left(\int_\Omega u_x^2\right)^{\frac{1}{2}}\left(\int_\Omega v_x^2\right)^{\frac{1}{2}} + C\left(\int_\Omega u_y^2\right)^{\frac{1}{2}}\left(\int_\Omega v_y^2\right)^{\frac{1}{2}}$$

$$\leq C\{||u_x||^2 + ||u_y||^2\}^{\frac{1}{2}}\{||v_x||^2 + ||v_y||^2\}^{\frac{1}{2}} \leq C||u||_1||v||_1,$$

where C is a constant.

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