A STUDY ON BAER AND P.P. NEAR-RINGS

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Abstract. Baer rings were introduced by Kaplansky [3] to abstract various properties of rings of operators on a Hilbert spaces. On the other hand, P.P. rings were introduced by A. Hattori [2] to study the torsion theory. In this paper we introduce Baer near-rings and P.P. near-rings and study some properties and give some examples.

1. Introduction

In this paper we introduce Baer near-rings and P.P. near-rings and study some properties and give some examples. Let $G$ be an additively written (but not necessarily abelian) group with zero $0$ and $M_0(G) = \{f : G \to G \mid f(0) = 0\}$, the near-ring of all zero respecting mappings on $G$. We show that $M_0(G)$ is a Baer near-ring. As a corollary we show that every zero-symmetric near-ring can be embedded into a Baer near-ring. Let $R$ be a commutative ring with identity. It is well known that $R$ is a Baer (resp. P.P.) ring if and only if the polynomial ring $R[x]$ is a Baer (resp. P.P.) ring (See e.g., Armendariz [1]). Corresponding to this result, we will prove that the zero-symmetric part of $R[x]$ is a Baer (resp. P.P.) near-ring if and only if $R$ is a Baer (resp. P.P.) ring. We also study the structure of the near-ring $R \oplus M$, where $R$ is an associative ring with identity and $M$ is a unital left $R$-module. Then $R \oplus M$ is a P.P. near-ring if and only if $R$ is a P.P. ring.

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2. Baer near-rings and p.p. near-rings

A (right) near-ring is a set $N$ with two binary operations $+$ and $\cdot$ such that $(N, +)$ is a not necessarily abelian group with identity $0$, $(N, \cdot)$ is a semigroup and $(x + y)z = xz + yz$ for all $x, y, z \in N$. Some basic definitions and concepts in near-ring theory can be found in Meldrum [4] and Pilz [5].

For a subset $S$ of a near-ring $N$, the set $\{ n \in N \mid nS = 0 \}$ is called the annihilator of $S$ in $N$ and is denoted by $\text{Ann}_N(S) = \text{Ann}(S)$.

A near-ring is called a Baer near-ring if, for any subset $S$ of $N$, $\text{Ann}(S) = \text{Ann}(e)$ for some idempotent $e \in N$. The following propositions is obvious.

**Proposition 1.** Let $N_i (i \in I)$ be a family of near-rings. Then the direct product $\prod_{i \in I} N_i$ is a Baer near-ring if and only if $N_i$ is a Baer near-ring for each $i \in I$.

**Example 1.** (1) Every integral near-ring with identity is a Baer near-ring

(2) Every constant near-ring is a Baer near-ring.

(3) A direct product of integral near-rings with identity is a Baer near-ring.

Let $G$ be an additively written (but not necessarily abelian) group with zero $0$ and $M_0(G) = \{ f : G \to G \mid f(0) = 0 \}$, the near-ring of all zero respecting mappings on $G$.

**Theorem 2.** The near-ring $M_0(G)$ is Baer.

**Proof.** Let $S$ be a subset of $M_0(G)$ and let $H = \{ s(y) \mid s \in S, y \in G \}$. Let $e$ be a mapping on $G$ such that $e(x) = x$ for each $x \in H$, and $e(y) = 0$ for $y \in G - H$. Then $e$ is an idempotent of $M_0(G)$ and $\text{Ann}(S) = \text{Ann}(e)$. This implies that $M_0(G)$ is a Baer near-ring.

**Corollary 3.** Every zero-symmetric near-ring can be embedded into a Baer near-ring.

**Proof.** By [5, 1, 102], every zero-symmetric near-ring can be embedded into a zero-symmetric near-ring with identity. Let $N$ be a zero-symmetric near-ring with identity. By Theorem 2, $M_0(N)$ is a
Baer near-ring. For \( r \in N \), the mapping \( f_r : t \in N \to rt \in N \) is an element of \( M_0(N) \). Since \( N \) contains an identity it follows that the mapping \( f : N \to M_0(N) \), defined by \( f(r) = f_r \), is a near-ring monomorphism.

An associative ring \( R \) called a (left) p.p. ring if every principal left ideal of \( R \) is projective. This is equivalent to the condition that, for any \( a \in R \), \( \text{Ann}(a) = \text{Ann}(e) \) for some idempotent \( e \in R \). Similarly we can define for near-rings. \( N \) is called a p.p near ring if for any \( a \in N \), \( \text{Ann}(a) = \text{Ann}(e) \) for some idempotent \( e \in N \).

EXAMPLE 2 Recall that a near-ring \( N \) is called regular if, for any \( x \in N \), there exists \( y \in N \) such that \( xyx = x \). Then \( xy \) is an idempotent and \( \text{Ann}(x) = \text{Ann}(xy) \). Hence every regular near-ring is a p.p. near-ring.

Let \( R \) be a commutative ring with identity and let \( R[x] \) denote the set of all polynomials in one indeterminate over \( R \). Under usual addition +, and substitution \( \circ \) of polynomials, \( (R[x], +, \circ) \) becomes a near-ring. Following Pilz [5], \( R_0[x] \) denote the zero symmetric part of \( R[x] \), that is \( R_0[x] = \{ \sum_{i=1}^{n} a_i x^i \mid n \geq 1, a_i \in R \} \).

A ring (or near-ring) without non-zero nilpotent element is called reduced.

THEOREM 4 Let \( R \) be a commutative ring with identity. The the following conditions are equivalent:

1) \( R_0[x] \) is a p.p. near-ring.
2) \( R \) is a p.p. ring.

PROOF. 1) \( \Rightarrow \) 2) First we claim that \( R \) is reduced. Suppose that \( a \in R \) with \( a^2 = 0 \). By hypothesis, there exists an idempotent \( f \in R_0[x] \) such that \( \text{Ann}(ax) = \text{Ann}(f) \). Let \( f = a_1 x - a_2 x^2 + \cdots + a_n x^n \) with \( a_i \in R \). Since \( f \) is an idempotent, we have \( a_1^2 = a_1 \). Since \( ax \in \text{Ann}(ax) \), \( ax \circ f = af = 0 \). In particular, \( aa_1 = 0 \). Since \( a - f \in \text{Ann}(f) \), \( 0 = (x - f) \circ ax = ax - f(ax) \). Hence \( ax = a_1 ax = 0 \), that is \( a = 0 \). This proves that \( R \) is reduced.

Since \( R \) is reduced, the set of idempotents of \( R_0[x] \) is just \( \{ ex \mid e^2 = e \in R \} \). Now let \( r \) be an arbitrary element of \( R \). By hypothesis there
exists an idempotent \( e \in R \) such that \( \text{Ann}(rx) = \text{Ann}(ex) \). Clearly this implies that \( \{ s \in R \mid sr = 0 \} = R(1 - e) \). Hence \( R \) is a p.p. ring.

2) \( \Rightarrow \) 1). Let \( f = a_1x + \cdots + a_nx^n \in R_0[x] \) and \( g = b_1x + \cdots + b_mx^m \in R_0[x] \). First we claim that \( f \circ g = 0 \) if and only if \( a_i b_j = 0 \) for all \( i, j \). It suffices to prove the 'only if' part. Let \( P \) be an arbitrary prime ideal of \( R \) and let \( \bar{f} \) and \( \bar{g} \) denote the image of \( f \) and \( g \) in \( (R/P)[x] \), respectively. Since \( R/P \) is an integral domain and since \( \bar{f} \circ \bar{g} = 0 \), we can easily see that either \( \bar{f} = 0 \) or \( \bar{g} = 0 \) holds. Hence \( a_i b_j \in P \) for all \( i, j \). Since a prime ideal \( P \) is arbitrary, this implies that \( a_i b_j \in \text{Rad}(R) \), where \( \text{Rad}(R) \) denote the prime radical of \( R \). Since \( R \) is a p.p. ring. \( R \) is reduced and hence \( \text{Rad}(R) = 0 \). This proves our claim. Therefore \( a_1, \ldots, a_n \in \text{Ann}_R(b_1, \ldots, b_m) \) Since \( R \) is a p.p. ring, there exist idempotents \( e_i \in R \) such that \( \text{Ann}(b_i) = \text{Ann}(e_i) \) for all \( i \). If \( n = 2 \), then \( f = e_1 + e_2 - e_1 e_2 \) is an idempotent and \( \text{Ann}_R(b_1, b_2) = \text{Ann}(f) \). Using induction on \( n \), we can find an idempotent \( e \) of \( R \) such that \( \text{Ann}_R(b_1, \ldots, b_m) = \text{Ann}(e) \). Then \( ex \) is an idempotent of \( R_0[x] \) and \( \text{Ann}(g) = \text{Ann}(ex) \). Therefore \( R_0[x] \) is a p.p. near-ring.

**Theorem 5.** Let \( R \) be a commutative ring with identity. Then the following conditions are equivalent:

1) \( R_0[x] \) is a Baer near-ring.
2) \( R \) is a Baer ring.

**Proof.** 1) \( \Rightarrow \) 2). Let \( T \) be a subset of \( R \) and consider the subset \( S = \{ tx \mid t \in T \} \) of \( R_0[x] \). As we have shown in the proof of 1) \( \Rightarrow \) 2) in Theorem 4, the set of idempotents of \( R_0[x] \) is just \( \{ ex \mid e^2 = e \in R \} \). Since \( R_0[x] \) is Baer, \( \text{Ann}(S) = \text{Ann}(ex) \) for some idempotent \( e \in R \). Now we can easily see that \( \text{Ann}_R(T) = \text{Ann}_R(e) \). Hence \( R \) is a Baer ring.

2) \( \Rightarrow \) 1). Let \( S \) be a subset of \( R_0[x] \) and consider the set \( T \) of all coefficients of \( g(x) \in S \). Let \( f = a_1x + \cdots + a_nx^n \in \text{Ann}(S) \). As in the proof of 2) \( \Rightarrow \) 1) in Theorem 4, \( a_i \in \text{Ann}_R(T) \) for all \( i \). Since \( R \) is a Baer ring, there exists an idempotent \( e \) such that \( \text{Ann}_R(T) = \text{Ann}_R(e) \). Now we can easily see that \( \text{Ann}(S) = \text{Ann}(ex) \). This proves that \( R_0[x] \) is a Baer near-ring.

Let \( R \) be an associative ring with identity and let \( M \) be a unital left
If we define a multiplication on the additive group \( R \oplus M \) by \((a, b) \cdot (c, d) = (ac, ad + b)\) for any \((a, b), (c, d) \in R \oplus M\), then \( R \oplus M \) becomes a near-ring with identity \((1, 0)\).

**Theorem 6.** Let \( R \) be an associative ring with identity and let \( M \) be a unital left \( R \)-module. Then the following conditions are equivalent:

1) \( R \oplus M \) is a p.p. near-ring.

2) \( R \) is a p.p. ring.

**Proof.** 2) \( \Rightarrow \) 1). We can easily see that, for \((c, d) \in R \oplus M\), \(\text{Ann}(c, d) = \{(a, -ad) \mid a \in \text{Ann}(a)\}\). Since \( R \) is a left p.p. ring, there is an idempotent \( e \in R \) such that \(\text{Ann}_R(e) = \text{Ann}(e)\). Then \((e, (1 - e)d)\) is an idempotent of \( R \oplus M \) and \(\text{Ann}(c, d) = \text{Ann}(e, (1 - e)d)\). Thus \( R \oplus M \) is a p.p. near-ring. 1) \( \Rightarrow \) 2). We first note that the set of idempotents of \( R \oplus M \) is equal to \(\{(e, (1 - e)x) \mid e = e^2 \in R, x \in M\}\). Hence, for any \( e \in R \), there exists idempotent \( e \in R \) and \( x \in M \) such that \(\text{Ann}(c, 0) = \text{Ann}(e, (1 - e)x)\). By the way, \(\text{Ann}(c, 0) = \{(a, 0) \mid a \in \text{Ann}(a)\}\). On the other hand, \((1 - e, -(1 - e)x) \in \text{Ann}(e, (1 - e)x)\). Hence \((1 - e)x = 0\), and so \(\text{Ann}(c, 0) = \text{Ann}(c, 0)\). Thus implies \(\text{Ann}(c) = \text{Ann}(c)\). Therefore \( R \) is a p.p. ring.

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**References**


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