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ON PRESERVING rg-CLOSED SETS

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ABSTRACT Weak forms of regular continuity and regular closure are introduced and used to strengthen some results concerning the preservation of rg-closed sets.

1. Introduction

Levine [10] introduced the concept of generalized closed (briefly, g-closed) sets in a topological space and a class of topological spaces called $T_{1/2}$ spaces. Since then several authors [2, 3, 4, 6, 7, 8, 10] have obtained many interesting results on these sets. Balachandran et al [3] introduced the notion of g-continuous mappings by using g-closed sets and obtained some of their properties. Recently, Paraniappan and Rao [12] introduced the concept of regular generalized closed (briefly, rg-closed) sets, which is weaker than that of g-closed sets, and the notions of rg-continuous and rg-irresolute mappings. Arokiarani and Balachandran [1] further studied rg-continuous mappings in topological spaces and introduced classes of topological spaces namely T_{rg}, $T^*_{1/2}$ spaces Park et al [13] investigated some properties of $T^*_{1/2}$ spaces and rg-continuous mappings using the notion of rg-closure operator [1]

The purpose of this note is to introduce new weak forms of regular continuity and regular closure (which we call ar-continuity and ar-closure) and to use these forms to strengthen the result of Paraniappan and Rao [12]. We also characterize $T^*_{1/2}$ spaces in terms of ar-continuity

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and ar-closure. Finally some of the basic properties of ar-continuous mappings and ar-closed mappings are investigated.

2. Some characterizations

The symbols X, Y and Z denote topological spaces with no separation axioms assumed unless explicitly stated. The closure and interior of a subset A of a space X are denoted by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$, respectively.

DEFINITION 2.1. [10,12] A subset A of a space X is called g-closed (resp. rg-closed) if $\mathrm{cl}(A) \subset G$ whenever $A \subset G$ and G is open (resp. regular open) in X. A subset A is called g-open (resp. rg-open) in X if its complement is g-closed (resp. rg-closed).

DEFINITION 2.2. [1,10] A space X is called $T_{1/2}$ (resp. $T^*_{1/2}$) if every g-closed (resp. rg-closed) set in X is closed in X.

Every $T^*_{1/2}$ space is $T_{1/2}$ but the converse is not true in general.

EXAMPLE 2.3. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is $T_{1/2}$ but not $T^*_{1/2}$.

THEOREM 2.4. *X is a $T^*_{1/2}$ space if and only if for each $x \in X$, either $\{x\}$ is open or regular closed.*

PROOF. Let $x \in X$. If $\{x\}$ is not regular closed, then X is the only regular open set containing $X \setminus \{x\}$ and thus $X \setminus \{x\}$ is rg-closed in X. By the $T^*_{1/2}$ property of X, $X \setminus \{x\}$ is closed and hence $\{x\}$ is open.

Conversely, let A be rg-closed in X and $x \in \mathrm{cl}(A)$. If $\{x\}$ is open, then $\{x\} \cap A \neq \phi$ and thus $x \in A$. Otherwise, if $\{x\}$ is regular closed, then $x \in A$. Indeed, if $x \notin A$, then $X \setminus \{x\}$ is regular open set containing A and thus $\mathrm{cl}(A) \subset X \setminus \{x\}$. Hence $x \notin \mathrm{cl}(A)$. This contradicts $x \in \mathrm{cl}(A)$. Therefore, in any case, A is closed.

DEFINITION 2.5. The intersection of all rg-closed (resp. g-closed) sets containing A is called the rg-closure [1] (resp. g-closure [7]) of A and denoted by $\mathrm{cl}^*_r(A)$ (resp. $\mathrm{cl}^*(A)$).

Theorem 2.6. *[13] For any subsets A, B of (X,τ), the following statements are valid:*
(i) $A \subset cl_r^*(A) \subset cl^*(A) \subset cl(A)$.
(ii) $cl_r^*(\phi) = \phi$ and $cl_r^*(X) = X$.
(iii) $cl_r^*(A \cup B) = cl_r^*(A) \cup cl_r^*(B)$.
(iv) $cl_r^*(cl_r^*(A)) = cl_r^*(A)$.

Remark 2.7. (i) The containment relation in Theorem 2.6 (i) may be proper (see Example 2.8 below).
(ii) $\tau_r^* = \{G : \mathrm{cl}_r^*(X \setminus G) = X \setminus G\}$ is a topology on X [13].

Example 2.8. Consider $X = \{a,b,c\}$ with topology $\tau = \{X, \phi, \{a\}, \{a,b\}\}$. Then $\mathrm{cl}_r^*\{a\} = \{a\}$ since the rg-closed supersets of $\{a\}$ are $\{a\}$, $\{a,b\}$, $\{a,c\}$ and X. But since the g-closed supersets of $\{a\}$ are $\{a,c\}$ and X, $\mathrm{cl}^*\{a\} = \{a,c\}$. That is, $\mathrm{cl}_r^*\{a\} \subsetneqq \mathrm{cl}^*\{a\} \subsetneqq \mathrm{cl}\{a\}$.

Theorem 2.9 *For any space (X,τ), $x \neq y$ implies $cl_r^*\{x\} \neq cl_r^*\{y\}$.*

Proof. If $\{x\}$ is closed, then $y \notin \{x\} = \mathrm{cl}_r^*\{x\}$ from Theorem 2.6. Otherwise, since $X \setminus \{x\}$ is rg-closed set containing $\{y\}$, $\mathrm{cl}_r^*\{y\} \subset X \setminus \{x\}$ and hence $x \notin \mathrm{cl}_r^*\{y\}$.

The previous result shows that (X, τ_r^*) is always a T_0 space. In fact, we can establish strong result:

Theorem 2.10. *(X, τ_r^*) is a $T_{1/2}$ space.*

Proof. If $\{x\}$ is regular closed in (X,τ), then $\{x\}$ is rg-closed in (X,τ) and hence it is closed in (X,τ_r^*). Otherwise, since $X \setminus \{x\}$ is rg-closed in (X,τ), $\mathrm{cl}_r^*(X \setminus \{x\}) = X \setminus \{x\}$, which implies that $\{x\}$ is open in (X,τ_r^*). By Theorem 2.6 in [7], (X,τ_r^*) is $T_{1/2}$ space.

Corollary 2.11. *For any topology τ on X, $(\tau_r^*)^* = (\tau_r^*)_r^* = \tau_r^*$.*

Theorem 2.12. *For any space (X,τ) the following are equivalent:*
(i) For each $x \in X$, $X \setminus \{x\}$ is rg-closed in X.
(ii) If $\{x\}$ is regular closed in X, then $\{x\}$ is regular open in X.

PROOF. (i)⇒(ii): Let $\{x\}$ be regular closed in X. Since $X \setminus \{x\}$ is regular open in X, we have $\mathrm{cl}(\mathrm{int}(X \setminus \{x\})) = X \setminus \{x\}$, which implies that $\{x\}$ is regular open in X.

(ii)⇒(i): obvious.

3. ar-continuous mappings and ar-closed mappings

DEFINITION 3.1. [3,12] A mapping $f : X \to Y$ is called g-continuous (resp. rg-continuous) if the inverse image of every closed set in Y is g-closed (resp. rg-closed) in X.

THEOREM 3.2. *Let $f : (X,\tau) \to (Y,\sigma)$ be a mapping.*

(i) If $f : (X,\tau) \to (Y,\sigma)$ is rg-continuous, then $f(cl^*_\tau(A)) \subset cl(f(A))$ for every subset A of X.

(ii) The following statements are equivalent:

(a) For each $x \in X$ and each open set V containing $f(x)$, there exists a rg-open set U containing x such that $f(U) \subset V$.

(b) For each $A \subset X$, $f(cl^*_\tau(A)) \subset cl(f(A))$.

(c) For each $B \subset Y$, $cl^*_\tau(f^{-1}(B)) \subset f^{-1}(cl(B))$.

(d) The mapping $f : (X,\tau^*_\tau) \to (Y,\sigma)$ is continuous.

PROOF. (i): Let A be any subset of X. Then $A \subset f^{-1}(f(A)) \subset f^{-1}(\mathrm{cl}(f(A)))$. Since f is rg-continuous, $\mathrm{cl}^*_\tau(A) \subset \mathrm{cl}^*_\tau(f^{-1}(\mathrm{cl}(f(A)))) = f^{-1}(\mathrm{cl}(f(A)))$. Hence $f(\mathrm{cl}^*_\tau(A)) \subset \mathrm{cl}(f(A))$.

(ii): (a)⇒(b): Let $y \in f(\mathrm{cl}^*_\tau(A))$ and let V be any open neighborhood of y. Then there exists a $x \in X$ such that $f(x) = y$ and $x \in \mathrm{cl}^*_\tau(A)$, and by (a) there exists a rg-open set U containing x such that $f(U) \subset V$. Since $x \in \mathrm{cl}^*_\tau(A)$, $U \cap A \neq \phi$ and then $V \cap f(A) \neq \phi$. Hence $y = f(x) \in \mathrm{cl}(f(A))$.

(b)⇒(a): Let $x \in X$ and V be any open set containing $f(x)$. Put $A = f^{-1}(Y \setminus V)$. Then $x \notin A$. Since $f(\mathrm{cl}^*_\tau(A)) \subset \mathrm{cl}(f(A)) \subset Y \setminus V$. $\mathrm{cl}^*_\tau(A) \subset f^{-1}(Y \setminus V) = A$ and so $\mathrm{cl}^*_\tau(A) = A$. Since $x \notin \mathrm{cl}^*_\tau(A)$, there exists a rg-open set U containing x such that $U \cap A = \phi$ and hence $f(U) \subset f(X \setminus A) \subset V$.

(b)⇔(c): Straightforward.

(b)⇔(d): Since the closure of A in (X, τ^*_τ) coincide with $\mathrm{cl}^*_\tau(A)$ in (X,τ). the equivalence is easily proved.

The converse of the above theorem (i) need not be true as seen from the following example.

EXAMPLE 3.3. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping defined by $f(a) = b, f(b) = a$ and $f(c) = c$. Then for every subset A of X, $f(\mathrm{cl}^*_\tau(A)) \subset \mathrm{cl}(f(A))$ holds but f is not rg-continuous.

Arokiarani and Balachandran, in [1], showed that if $f : X \to Y$ and $g : Y \to Z$ are rg-continuous mappings and Y is a $T^*_{1/2}$-space, then $g \circ f : X \to Z$ is also rg-continuous. The following example shows that the condition, i.e. Y is a $T^*_{1/2}$ space can not be removed

EXAMPLE 3.4. Let $X = Y = Z = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$ and $\omega = \{Z, \phi, \{a, c\}\}$ Let $(X, \tau) \to (Y, \sigma)$ be a mapping defined by $f(a) = c, f(b) = b$ and $f(c) = a$ Let $g : (Y, \sigma) \to (Z, \omega)$ be the identity mapping Then f and g are rg-continuous but $g \circ f$ is not rg-continuous

DEFINITION 3 5 [12] Let $f \cdot X \to Y$ be a mapping. Then f is called

(i) regular continuous if the inverse image of every closed set of Y is regular closed in X,

(ii) regular closed if the image of every closed set of X is regular closed in Y

DEFINITION 3.6. Let $f : X \to Y$ be a mapping. Then f is called

(i) approximately regular continuous (briefly, ar-continuous) provided that $\mathrm{cl}(A) \subset f^{-1}(G)$ whenever G is open subset of Y, A is a rg-closed subset of X and $A \subset f^{-1}(G)$,

(ii) approximately regular closed (briefly, ar-closed) provided that $f(F) \subset \mathrm{int}(B)$ whenever F is a closed subset of X, B is a rg-open subset of Y and $f(F) \subset B$

Clearly, every regular continuous (resp. regular closed) mapping is ar-continuous (resp. ar-closed) The following examples show the converse implications do not hold.

EXAMPLE 3.7 (i) Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$ and let $f \cdot (X, \tau) \to (X, \tau)$ be a mapping defined by $f(a) = f(b) = f(c) =$

a. Then since $\{b,c\}$ is closed in (X,τ), $f(\{b,c\}) = \{a\} \subset \{a\}$ and $f(\{b,c\}) \subset \text{int}\{a\}$ since $\{a\}$ is open. Thus f is ar-closed, but since $\text{cl}(\text{int}(f(\{b,c\}))) \neq f(\{b,c\}) = \{a\}$, f is not regular closed.

(ii) Let $X = \{a,b,c\}$, $\tau = \{X, \phi, \{a\}, \{b,c\}\}$ and let $f : (X,\tau) \to (X,\tau)$ be a mapping defined by $f(a) = f(c) = a$ and $f(b) = b$. Since $\{a\}$ is rg-closed in (X,τ) and $\{a\}$ is open and closed in (Y,σ), $\{a\} \subset f^{-1}(\{a\}) = \{a,c\}$ and $\text{cl}\{a\} = \{a\} \subset f^{-1}(\{a\}) = \{a,c\}$. Thus f is ar-continuous. But since $f^{-1}(\{a\}) = \{a,c\}$ is not closed in (X,τ), f is not regular continuous.

THEOREM 3.8. *If $f : X \to Y$ is g-continuous and ar-closed mapping, then the inverse image of every rg-closed (resp. rg-open) subset of Y is rg-closed (resp. rg-open) in X.*

PROOF. Let B be a rg-closed subset of Y and $f^{-1}(B) \subset U$, where U is a regular open subset of X. Then $X \setminus U \subset f^{-1}(Y \setminus B)$, which implies $f(X \setminus U) \subset Y \setminus B$. Since f is ar-closed, $f(X \setminus U) \subset \text{int}(Y \setminus B) = Y \setminus \text{cl}(B)$. Hence $f^{-1}(\text{cl}(B)) \subset U$. Since f is g-continuous, $f^{-1}(\text{cl}(B))$ is g-closed. Thus we have $\text{cl}(f^{-1}(B)) \subset \text{cl}(f^{-1}(\text{cl}(B))) \subset U$ and consequently $f^{-1}(B)$ is rg-closed. A similar argument shows that inverse image of rg-open set of Y is rg-open in X.

COROLLARY 3.9. *If $f : X \to Y$ is g-continuous and regular closed mapping, then the inverse image of every rg-closed subset of Y is rg-closed in X.*

COROLLARY 3.10. *[12] If $f : X \to Y$ is continuous and regular closed mapping, then the inverse image of every rg-closed subset of Y is rg-closed in X.*

THEOREM 3.11. *If $f : X \to Y$ is ar-continuous and closed mapping, then the image of every rg-closed subset of X is rg-closed in Y.*

PROOF. Let A be a rg-closed subset of X and $f^{-1}(A) \subset V$, where V is regular open subset of Y. Then $A \subset f^{-1}(V)$, and since f is ar-continuous, $\text{cl}(A) \subset f^{-1}(V)$ which implies $f(\text{cl}(B)) \subset V$. Since f is closed, we have $\text{cl}(f(A)) \subset \text{cl}(f(\text{cl}(A))) = f(\text{cl}(A)) \subset V$. Hence $f(A)$ is rg-closed.

THEOREM 3.12. *A space X is $T^*_{1/2}$ if and only if for every space Y and every mapping $f : X \to Y$, f is ar-continuous.*

PROOF. Let A be a nonempty rg-closed subset of X and Y be the set with the topology $\{Y, \phi, A\}$ and $f : X \to Y$ be the identity mapping. By assumption, f is ar-continuous Since A is rg-closed in X and open in Y and $A \subset f^{-1}(A)$, it follows that $\mathrm{cl}_X(A) \subset f^{-1}(A) = A$ Then A is closed in X and hence X is $T^*_{1/2}$. The converse can be easily shown from the definition of ar-continuity.

THEOREM 3.13. *A space Y is $T^*_{1/2}$ if and only if for every space X and every mapping $f : X \to Y$, f is ar-closed.*

PROOF The proof is similar to that of Theorem 3.12.

THEOREM 3 14. *If the open and closed sets of Y coincide, then a mapping $f : X \to Y$ is ar–closed if and only if $f(A)$ is open for every closed subset A of X.*

PROOF Let f be an ar-closed mapping. By Theorem 2.10 in [10] and Theorem 3.8 in [12], all subsets of Y are rg-closed (and hence all are rg-open). Thus for any closed subset A of X. $f(A)$ is rg-open in Y. Since f is ar-closed, $f(A) \subset \mathrm{int}(f(A))$ and then $f(A) = \mathrm{int}(f(A))$. Hence $f(A)$ is open.

Conversely, let F be a closed subset of X and let B be a rg-open subset of Y and $f(F) \subset B$. By hypothesis, $f(F)$ is open in Y and then $f(F) = \mathrm{int}(f(F)) \subset \mathrm{int}(B)$. Hence f is ar-closed.

COROLLARY 3.15 *If the open and closed subsets of Y coincide, the a mapping f $X \to Y$ is ar-closed if and only if it is regular closed.*

PROOF. Let A be a closed subset of X Then by Theorem 3.14. $f(A)$ is open and closed and hence f is regular closed.

Conversely, let F be a closed subset of X and let B be a rg-open subset of Y and $f(F) \subset B$. By hypothesis, $f(F)$ is regular closed in Y and thus $f(F)$ is regular open Then $f(F) = \mathrm{int}(f(F)) \subset \mathrm{int}(B)$ and hence f is ar-closed.

The proofs of the following results are analogous and are omitted

THEOREM 3.16. *If the open and closed sets of X coincide, then a mapping $f : X \to Y$ is ar-continuous if and only if $f^{-1}(B)$ is closed for every open subset B of Y.*

COROLLARY 3.17. *If the open and closed subsets of Y coincide, the a mapping $f : X \to Y$ is ar-closed if and only if it is regular continuous.*

THEOREM 3.18. *For $f : X \to Y$ and $g : Y \to Z$, the following statements are hold:*

(i) If f is closed and g is ar-closed, then $g \circ f$ is ar-closed.

(ii) If f is ar-closed, g is open and inversely preserves rg-open sets, then $g \circ f$ is ar-closed.

(iii) If f is ar-continuous and g is continuous, then $g \circ f$ is ar-continuous.

PROOF. (i): Let F be a closed subset of X and B be a rg-open subset of Z such that $(g \circ f)(F) \subset B$. Since f is closed, $f(F)$ is closed in Y. Hence $g(f(F)) \subset \text{int}(B)$ because g is ar-closed.

(ii): Let F be a closed subset of X and B be a rg-open subset of Z such that $(g \circ f)(F) \subset B$. Then $f(F) \subset g^{-1}(B)$ and since $g^{-1}(B)$ is rg-open and f is ar-closed, $f(F) \subset \text{int}(g^{-1}(B))$. Hence $(g \circ f)(F) \subset g(\text{int}(g^{-1}(B))) \subset \text{int}(g(g^{-1}(B))) \subset \text{int}(B)$.

(iii): Let A be a rg-closed subset of X and V be a closed subset of Z such that $A \subset (g \circ f)^{-1}(V)$. Since g is continuous, $g^{-1}(V)$ is open. Hence $\text{cl}(A) \subset f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ because f is ar-continuous.

COROLLARY 3.19. *Let $f_\alpha : X \to Y_\alpha$ be a mapping for each $\alpha \in \Lambda$ and $f : X \to \prod Y_\alpha$ be the product mapping given by $f(x) = (f_\alpha(x))$. If f is ar-continuous, then f_α is ar-continuous for each $\alpha \in \Lambda$.*

PROOF. Let $p_\beta : \prod Y_\alpha \to Y_\beta$ be the projection mapping for each $\beta \in \Lambda$. Since p_β is continuous, by Theorem 3.18(iii), $f_\beta = p_\beta \circ f$ is ar-continuous.

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