SOME CHARACTERIZATIONS OF BEST APPROXIMATION ELEMENT FROM SUBSPACES IN LINEAR 2-NORMED SPACES

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Abstract. In this paper, we shall give new characterizations of best approximation element in linear 2-normed spaces in terms of bounded linear 2-functionals and 2-hyperplanes.

1. Introduction

Let X be a linear space of dimension greater than 1, and let \( ||\cdot,\cdot|| : X \times X \to \mathbb{R} \) be a function with the following conditions:
\[
\begin{align*}
(N_1) \quad & ||x, y|| = 0 \text{ if and only if } x \text{ and } y \text{ are linearly dependent,} \\
(N_2) \quad & ||x, y|| = ||y, x||, \\
(N_3) \quad & ||\alpha x, y|| = ||x, y||, \text{ where } \alpha \text{ is real,} \\
(N_4) \quad & ||x + y, z|| \leq ||x, z|| + ||y, z||.
\end{align*}
\]

\( ||\cdot,\cdot|| \) is called a 2-norm on X and \( (X, ||\cdot,\cdot||) \) a linear 2-normed space([6]).

Let \( A, C \) be a subspaces of X. A bilinear functional \( f : A \times C \to \mathbb{R} \) is called a bounded linear 2-functional if there is a real constant \( K > 0 \) such that \( |f(x, y)| \leq K ||x, y|| \) for \( x, y \in X \). For a bounded linear 2-functional we have

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\[ \|f\| = \inf \{ K : |f(x, y)| \leq K \|x, y\| \text{ for all } x, y \in X \}. \]

Additional properties of bounded linear 2-functionals may be found in \([4], [5], [9] \text{ and } [12]\).

Let \((X, \|\cdot, \cdot\|))\) be a linear 2-normed space and \(V(x_1, x_2, \ldots, x_n)\) be a subspace of \(X\) generated by \(x_1, x_2, \ldots, x_n\) in \(X\). For all \(x, y \in X\), define

\[ \rho_\pm(x, z)(y) = \lim_{t \to 0^\pm} \frac{\|x + ty, z\|^2 - \|x, z\|^2}{2t} \]

for any real \(t\) and \(z \in X \setminus V(x, y)\).

**Theorem 1.1**([1], [2]). We have some properties of \(\rho_\pm:\)

1. \(\rho_\pm(\alpha x, z)(\beta y) = \alpha \beta \rho_\pm(x, z)(y)\) for \(\alpha \beta > 0\).
2. \(\rho_\pm(x, z)(\alpha x + y) = \alpha \rho_\pm(x, z)(x) + \rho_\pm(x, z)(y)\) for all \(\alpha \in R\).
3. \(\rho_\pm(x, z)(y + y') \leq (\rho_\pm(x, z)(x))^{1/2}(\rho_\pm(y, z)(y))^{1/2} = \rho_\pm(x, z)(y').\)
4. \(\rho_+(x, z)(-y) = \rho_+(-x, z)(y) = -\rho_-(x, z)(y).\)
5. \(\rho_+(x, z)(x) = \rho_-(x, z)(x) = \|x, z\|^2.\)
6. \((X, \|\cdot, \cdot\|)\) is smooth at \(x_0 \in X \setminus \{0\}\) if and only if \(\rho_+(x, z)(y) = -\rho_-(x, z)(y).\)
7. \(x \perp z (\alpha x + y)\) if and only if \(\rho_-(x, z)(y) \leq -\alpha \|x, z\|^2 \leq \rho_+(x, z)(y)\) where \(\perp\) is orthogonality([7]), that is, \(x \perp z y\) means \(\|x + ty, z\| \geq \|x, z\|\) for all \(t \in R\).

Let \((X, \|\cdot, \cdot\|)\) be a linear 2-normed space. For a subspace \(G\) of \(X\), let \([x, G]\) be the subspace of \(X\) generated by \(x\) and \(G\), where \(x \in X \setminus G\). Then for \(z \in X \setminus [x, G]\), an element \(g_0 \in G\) is called the best approximation element of \(x\) by \(G\) (with respect to \(z\)) if

\[ \|x - g_0, z\| \leq \|x - g, z\| \]

for all \(g \in G([10])\). The set of all elements of best approximation of \(x\) by \(G\) with respect to \(z\) is denoted by \(P_{G, x}(z)\), that is,
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\[ P_{G,z}(x) = \{ g_o \in G : \| x - g_o, z \| \leq \| x, g, z \| \}. \]

The following theorem gives a relationship between orthogonality and best approximation in linear 2-normed spaces.

**Theorem 1.2.** ([4]) Let \((X, \| \cdot, \cdot \|)\) be a linear 2-normed space, \(G\) a linear subspace of \(X\), \(x \in X \setminus G\) and \(z \in X \setminus [x, G]\). Then \(g_o \in P_{G,z}(x)\) if and only if \((x - g_o) \perp z\).

In 1994 and 1990, I. Franic([4]) and S. Mabizela([9]) gave some characterizations of the best approximation in terms of bounded linear 2-functions, respectively. Also, some results on approximation theory in linear 2-normed spaces have been obtained by S.S. Kim, Y.J. Cho and T.D. Narang([8]), S. Elumalai, Y.J. Cho and S.S. Kim([3]) and R. Ravi([11]).

In this paper, new characterizations of best approximation in linear 2-normed spaces is given in terms of bounded linear 2-functionals and 2-hyperplanes.

**2. Characterizations of best approximation**

Let \(f\) be a non-zero linear 2-functional on \(X \times V(z)\). Then we define the 2-hyperplane \(H\) through the origin by

\[ H = \{ x \in X | f(x, z) = 0 \}. \]

**Theorem 2.1.** Let \((X, \| \cdot, \cdot \|)\) be a linear 2-normed space, \(f\) a non-zero bounded linear 2-functional on \(X \times V(z)\) and \(H\) a 2-hyperplane through the origin, \(x_o \in X \setminus H, z \in X \setminus [x, H]\) and \(g_o \in H\). Then the following statements are equivalent:

1. \(g_o \in P_{H,z}(x_o)\);
(2) (a) For all $x \in X$

$$
\rho_\ (-\left(\frac{f(x_0, z)(x_0 - g_0)}{\|x_0 - g_0, z\|^2}, z\right)(x)
\leq f(x, z) \leq \rho_\ (+\left(\frac{f(x_0, z)(x_0 - g_0)}{\|x_0 - g_0, z\|^2}, z\right)(x), \quad \text{and}
\rho_\ (-\left(\frac{f(x_0, z)(x_0 - g_0)}{\|x_0 - g_0, z\|^2}, z\right)(x)
\leq f(x, z) \leq \rho_\ (+\left(\frac{f(x_0, z)(x_0 - g_0)}{\|x_0 - g_0, z\|^2}, z\right)(x), \quad \text{and}
$$

(2.1)

(b) $\|f\| = \frac{|f(x_0, z)|}{\|x_0 - g_0, z\|}$.

**Proof.** (1) implies (2): Suppose that $g_0 \in P_{H, z}(x_0)$. By Theorem 1.2, $(x_0 - g_0) \perp_z H$. Let $w = x_0 - g_0$ and $x \in X$. Then we have $f(x, z)w - f(w, z)x$ belong to $H$ and so $w \perp_z (f(x, z)w - f(w, z)x)$. By Theorem 1.1,

$$
\rho_-(w, z)(f(x, z)w - f(w, z)x) \leq 0 \leq \rho_+(w, z)(f(x, z)w - f(w, z)x)
$$

for all $x \in X$ and $z \in X \setminus [x, H]$. Since

$$
\rho_\pm(w, z)(f(x, z)w - f(w, z)x)
= f(x, z)\|w, z\|^2 + \rho_\pm(w, z)(-f(w, z)x)
$$

and $w \perp_z H$, if $w$ is any non-zero element of $X$, then $f(w, z) \neq 0$. Now we will consider two cases: $f(w, z) > 0$ and $f(w, z) < 0$.

**Case 1.** Suppose that $f(w, z) > 0$. Then we have

$$
0 \leq f(x, z)\|w, z\|^2 + \rho_+(w, z)(-f(w, z)x)
= f(x, z)\|w, z\|^2 - \rho_-(f(w, z)w, z)(x)
$$

and so

$$
f(x, z) \geq \rho_-\left(\frac{f(w, z)w}{\|w, z\|^2}, z\right)(x).
$$

On the other hand, we have

$$
0 \geq f(x, z)\|w, z\|^2 + \rho_-(w, z)(-f(w, z)x)
= f(x, z)\|w, z\|^2 - \rho_+(f(w, z)w, z)(x)
$$
and so
\[ f(x, z) \leq \rho_+ \left( \frac{f(w, z)w}{\|w, z\|^2}, z \right)(x). \]

Therefore, it follows that
\[ \rho_- \left( \frac{f(w, z)w}{\|w, z\|^2}, z \right)(x) \leq f(x, z) \leq \rho_+ \left( \frac{f(w, z)w}{\|w, z\|^2}, z \right)(x). \]

Case 2. Suppose that \( f(w, z) < 0 \). For any \( x, y \in X \) and \( z \in X \setminus V(x, y) \),
\[ \rho_-(x, z)(y) = -\rho_+(x, z)(-y) = -\rho_+(-x, z)(y) \]
and
\[ \rho_-(x, z)(y) = -\rho_+(-x, z)(-y) = -\rho_+(x, z)(y) \]
hold. Since \( f(w, z) < 0 \), we have
\[ 0 \leq f(x, z)\|w, z\|^2 + \rho_+(w, z)(-f(w, z)x) \]
\[ = f(x, z)\|w, z\|^2 - \rho_-(f(w, z)w, z)(x) \]
and so
\[ f(x, z) \geq \rho_- \left( \frac{f(w, z)w}{\|w, z\|^2}, z \right)(x). \]

Also, by the similar method we have
\[ f(x, z) \leq \rho_+ \left( \frac{f(w, z)w}{\|w, z\|^2}, z \right)(x). \]

Therefore,
\[ \rho_- \left( \frac{f(w, z)w}{\|w, z\|^2}, z \right)(x) \leq f(x, z) \leq \rho_+ \left( \frac{f(w, z)w}{\|w, z\|^2}, z \right)(x). \]

Since \( g_o \in H, f(w, z) = f(x_o, z) \) and so we obtain (a).
Next, let \( u = f(x_0, z)(x_0 - g_0)/\|x_0 - g_0, z\|^2 \). Then, by (a)

\[
f(x, z) \leq \rho_+(u, z)(x) \leq \|x, z\|\|u, z\|
\]

and

\[
f(x, z) \geq \rho_-(u, z)(x) = -\rho_+(u, z)(-x) \geq \|x, z\|\|u, z\|.
\]

Therefore, \(-\|u, z\| \leq f(x, z)/\|x, z\| \leq \|u, z\|\) and hence \(\|f\| \leq \|u, z\|\).

On the other hand, we have

\[
\|f\| \geq \frac{f(u, z)}{\|u, z\|} \geq \frac{\rho_-(u, z)(u)}{\|u, z\|} = \|u, z\|
\]

and so we conclude that (b) holds.

(2) implies (1): From (a), for \( x \in H \)

\[
\rho_\left( \frac{f(x_0, z)(x_0 - g_0)}{\|x_0 - g_0, z\|^2}, z \right)(x) \leq 0 \leq \rho_+ \left( \frac{f(x_0, z)(x_0 - g_0)}{\|x_0 - g_0, z\|^2}, z \right)(x).
\]

Therefore, it follows that

\[
\frac{f(x_0, z)(x_0 - g_0)}{\|x_0 - g_0, z\|^2} \perp_z H
\]

and so since \( f(x_0, z) \neq 0, (x_0 - g_0) \perp_z H \). Therefore, by Theorem 1.2 we have \( g_o \in P_{H,z}(x_o) \).

By Theorem 2.1, we obtain easily the following corollaries:

**Corollary 2.2.** Let \((X, (\cdot, \cdot))\) be a 2-inner product space, \(f\) a non-zero bounded linear 2-functional on \( X \times V(z) \), \( H \) a \( 2 \)-hyperplane through the origin, \( x_0 \in X \setminus H \), and \( z \in X \setminus [x, H] \). Then there exists \( g_o \in H \) such that

\[
f(x, z) = \left( x, \frac{f(x_0, z)(x_0 - g_0)}{\|x_0 - g_0, z\|^2} \right) \quad \text{and} \quad \|f\| = \frac{|f(x_0, z)|}{\|x_0 - g_0, z\|}.
\]
Corollary 2.3. Let \((X, \| \cdot, \cdot \|)\) be a smooth linear 2-normed space, \(f\) a non-zero bounded linear 2-functional on \(X \times V(z)\), \(H\) a 2-hyperplane through the origin, \(x_o \in X \setminus H, z \in X \setminus [x, H]\) and \(g_o \in H\). Then the following statements are equivalent:

1. \(g_o \in P_{H,z}(x_o)\);

2. \(f(x, z) = \rho_+ \left( \frac{f(x_o, z)(x_o - g_o)}{\|x_o - g_o, z\|^2}, z \right)(x)\) and \(\|f\| = \frac{|f(x_o, z)|}{\|x_o - g_o, z\|}\).

Let \((X, \| \cdot, \cdot \|)\) be a linear 2-normed space, \(G\) a linear subspace of \(X, x \in X \setminus G\) and \(z \in X \setminus [x, G]\). If \(P_{G,z}(x)\) has at least one element for every \(x \in X\), then \(G\) is said to be proximinal ([10]).

Lemma 2.4. ([10]) Let \((X, \| \cdot, \cdot \|)\) be a linear 2-normed space and \(H\) be a 2-hyperplane through the origin. Then \(H\) is proximinal if and only if there exists a non-zero \(x \in X\) such that \(0 \in P_{H,z}(x)\).

From Theorem 2.1 and Lemma 2.4, we obtain easily the following:

Theorem 2.5. Let \((X, \| \cdot, \cdot \|)\) be a linear 2-normed space, \(f\) a non-zero bounded linear 2-functional on \(X \times V(z)\) and \(H\) a 2-hyperplane through the origin. Then the following statements are equivalent:

1. \(H\) is proximinal;

2. For non-zero \(u \in X\) and \(z \in X \setminus V(x, u), \rho_- (u, z)(x) \leq f(x, z) \leq \rho_+ (u, z)(x)\)

\((b)\) \(\|f\| = \|u, z\|\).

Corollary 2.6. Let \((X, \| \cdot, \cdot \|)\) be a smooth linear 2-normed space and \(H\) a 2-hyperplane through the origin. Then \(H\) is proximinal if and only if there exists a non-zero \(u \in X\) such that \(f(x, z) = \rho_+(u, z)(x)\) for all \(x \in X\) and \(\|f\| = \|u, z\|\).

3. A variational characterization of best approximation

In this section, we will give a variational characterizations of best approximation element.
Theorem 3.1. Let \((X, \|\cdot\|)\) be a linear 2-normed space, \(f\) be a non-zero bounded linear 2-functional on \(X \times V(z)\) and a non-zero element \(w \in X\). Then the following statements are equivalent:

1. The following inequality holds,

\[(3.1) \quad \rho_-(w, z)(x) \leq f(x, z) \leq \rho_+(w, z)(x) \quad \text{for all} \quad x \in X,\]

2. The element \(w\) minimize the quadratic functional \(F_{f_z} : X \to \mathbb{R}\) defined by

\[F_{f_z}(u) = \|u, z\|^2 - 2f(u, z).\]

Proof. (i) \(\Rightarrow\) (ii): If \(w\) satisfies the relation (3.1), then we have \(f(w, z) = \|w, z\|^2\) for \(x = w\). Now, let \(u \in X\). Then we have

\[
F_{f_z}(u) - F_{f_z}(w) = \|u, z\|^2 - 2f(u, z) + \|w, z\|^2 \\
\geq \|u, z\|^2 - 2\rho_+(w, z)(u) + \|w, z\|^2 \\
\geq \|u, z\|^2 - 2\|u, z\|\|w, z\| + \|w, z\|^2 \\
= (\|u, z\| - \|w, z\|)^2 \geq 0,
\]

and so \(w\) minimize the functional \(F_{f_z}\).

(ii) \(\Rightarrow\) (i): Suppose that \(w\) minimize the functional \(F_{f_z}\). Then we have

\[F_{f_z}(w + \lambda u) - F_{f_z}(w) \geq 0\]

for all \(u \in X\) and \(\lambda \in \mathbb{R}\). On the other hand, since \(F_{f_z}(w + \lambda u) - F_{f_z}(w) = \|w + \lambda u, z\|^2 - \|w, z\|^2 - 2\lambda f(u, z)\) we have

\[2\lambda f(u, z) \leq \|w + \lambda u, z\|^2 - \|w, z\|^2 (3.2)\]

for all \(u \in X\) and \(\lambda \in \mathbb{R}\). Now, we assume that \(\lambda > 0\). Then by (3.2) we have

\[f(u, z) \leq \frac{\|w + \lambda u, z\|^2 - \|w, z\|^2}{2\lambda} \quad \text{for all} \quad u \in X,
\]

which gives \(f(u, z) \leq \rho_+(w, z)(u)\) for \(\lambda \to 0^+\) and all \(u \in X\). Putting \((-u)\) instead of \(u\), we have \(f(u, z) \geq -\rho_+(w, z)(-u) = \rho_-(w, z)(u)\) for all \(u \in X\). Therefore, we have the relation (3.1).
By Theorem 3.1, we obtain the following:

**Corollary 3.2.** Let \((X, \| \cdot \|)\) be a linear 2-normed space and \(f\) a non-zero bounded linear 2-functional on \(X \times V(z)\) and a non-zero element \(w \in X\). Then \(w\) is an element of smoothness of \(X\) and it minimizes the functional \(F_f\) if and only if

\[
f(x, z) = \rho_4(w, z)(x) \quad \text{for all} \quad x \in X.
\]

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