UNIFORM $N$-DICHOTOMY FOR EVOLUTIONARY PROCESS IN BANACH SPACES

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Abstract. We study some properties of $N$-dichotomy for evolutionary process and generalize the theory of the uniform $N$-equistability using these properties.

1. Introduction

Throughout this paper, $X$ is a real or complex Banach space and $L(X)$ is the set of all bounded linear operators from $X$ into itself. Let $T$ be the set defined by $T = \{(t, s) : 0 \leq s \leq t < \infty\}$. A mapping $P : T \to L(X)$ is called an evolutionary process ([1], [6]) if the following are satisfied:

(i) $P(t, s)P(s, t_0) = P(t, t_0)$ for all $0 \leq t_0 \leq s \leq t$,
(ii) $P(t, t)x = x$ for all $x \in X$,
(iii) $P(t, s)$ is strongly continuous in $s$ on $[0, t]$ and in $t$ on $[s, \infty)$,
(iv) there is a nondecreasing function $p : [0, \infty) \to (0, \infty)$ such that

$$\|P(t, s)\| \leq p(t - s) \quad \text{for all } (t, s) \in T.$$ 

Let $L^\infty_{t_0}(X)$ be the space of $X$-valued functions $f$ defined almost everywhere on $[t_0, \infty)$ such that $f$ is strongly measurable and essentially bounded, and let $X_1(t_0)$ be the set $X_1(t_0) = \{x \in X : P(\cdot, t_0)x \in \ldots$
$L^\infty_t(X_1)$. If $X_2(t_0)$ is a complementary subspace of $X_1(t_0)$ then we denote by $P_1(t_0)$ the projection along $X_2(t_0)$ and $P_2(t_0) = I - P_1(t_0)$ the projection along $X_1(t_0)$.

We also denote:

\[ P_1(t, t_0) = P(t, t_0)P_1(t_0) \quad \text{and} \quad P_2(t, t_0) = P(t, t_0)P_2(t_0). \]

In what follows we denote by $N$ the set of all functions $N : R_+ \rightarrow R_+$ satisfying the following conditions:

(i) $N$ is nondecreasing on $[0, \infty)$,

(ii) $N$ is continuous on $[0, \infty)$ and $N(0) = 0$,

(iii) $N(uv) \leq N(u)N(v)$ for all $u \geq 0$ and $v \geq 0$.

EXAMPLE. Let $N : R_+ \rightarrow R_+$, $N(u) = u$ for $u \in [0, 1]$ and $N(u) = u^2$ for $u > 1$. Then we know that $N \in N$.

DEFINITION 1.1. (cf. [2], [3], [4]) An evolutionary process $P$ is said to be:

(i) uniformly exponentially dichotomic (and we write $P$ is u.e.d) if there are $M_1, M_2, \nu_1, \nu_2 > 0$ such that

\[ \|P_1(t, t_0)x\| \leq M_1 \cdot \exp[-\nu_1(t - s)] \cdot \|P_1(s, t_0)x\| \]

and

\[ \|P_2(t, t_0)x\| \geq M_2 \cdot \exp[\nu_2(t - s)] \cdot \|P_2(s, t_0)x\|, \]

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$,

(ii) $N$-uniformly exponentially dichotomic (and we write $P$ is $N$-u.e.d) if there are $M_1, M_2, \nu_1, \nu_2 > 0$ such that

\[ N(\|P_1(t, t_0)x\|) \leq M_1 \cdot \exp[-\nu_1(t - s)] \cdot N(\|P_1(s, t_0)x\|), \]

and

\[ N(\|P_2(t, t_0)x\|) \geq M_2 \cdot \exp[\nu_2(t - s)] \cdot N(\|P_2(s, t_0)x\|), \]

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$.

It is clear that the uniform exponential dichotomy is a particular case (when $N(u) = u$) of the $N$-uniformly exponential dichotomy.
Lemma 1.1. Let \( \varphi : T \to R_+ \) be a function. If there exist positive numbers \( H, \delta, \eta \) with \( \eta > 1 \) such that

\[
\eta \varphi(s + \delta, t_0) \leq \varphi(s, t_0) \quad \text{and} \quad \varphi(t, t_0) \leq H\varphi(s, t_0),
\]

for all \( t \geq s \geq t_0 \geq 0 \), then there are \( K, \nu > 0 \) such that

\[
\varphi(t, t_0) \leq K \cdot \exp[-\nu(t - s)]\varphi(s, t_0),
\]

for all \( t \geq s \geq t_0 \geq 0 \).

Proof. Let \( t \geq s \geq t_0 \geq 0 \) and \( n = [(t - s) \cdot \delta^{-1}] \). Then we have

\[
\varphi(t, t_0) \leq H\varphi(s + n\delta, t_0) \leq \eta^{-n} H\varphi(s, t_0) = K \cdot \exp[-\nu(t - s)] \cdot \varphi(s, t_0),
\]

where \( K = \eta H \) and \( \nu = \delta^{-1} \cdot \ln \eta \). This completes the proof.

By the same method, we can also prove the following lemma.

Lemma 1.2. Let \( \Psi : T \to [0, \infty) \) be a function. If there exist positive numbers \( H, \delta, \eta > 0 \), with \( \eta \in (0, 1) \) such that

\[
\eta \Psi(s + \delta, t_0) \geq \Psi(s, t_0) \quad \text{and} \quad \Psi(t, t_0) \geq H\Psi(s, t_0),
\]

for all \( t \geq s \geq t_0 \geq 0 \), then there are \( K, \nu > 0 \) such that

\[
\Psi(t, t_0) \geq K \cdot \exp[\nu(t - s)]\Psi(s, t_0).
\]

Lemma 1.3. Let \( g : R_+ \to R_+^* \) be a continuous function on \( R_+ \) with \( \inf\{g(u) : u \geq 0\} < 1 \) and \( x \in X \) such that

\[
N(\|P_1(t, t_0)x\|) \leq g(t - s) \cdot N(\|P_1(s, t_0)x\|),
\]

for all \( t \geq s \geq t_0 \geq 0 \). Then there exist \( M, \nu > 0 \) such that

\[
N(\|P_1(t, t_0)x\|) \leq M \cdot \exp[-\nu(t - s)] \cdot N(\|P_1(s, t_0)x\|).
\]
Proof. Since \( \inf \{ g(u) : u \geq 0 \} < 1 \), there is \( \delta > 0 \) such that \( g(\delta) < 1 \). Let \( n = \lceil (t-s) \cdot \delta^{-1} \rceil \in N \). It is clear that there is \( r \in [0, \delta) \) such that \( t = s + n\delta + r \). Hence we have

\[
N(||P_1(t, t_0)x||) = N(||P(t, s + n\delta)P_1(s + n\delta, t_0)x||)
\leq N(P(t - s - n\delta)) \cdot N(||P_1(s + n\delta, t_0)||)N(P(\delta))
\cdot g(\delta) \cdot N(||P_1(s + (n-1)\delta, t_0)||)
\leq \cdots
\leq N(P(\delta)) \cdot (g(\delta))^n \cdot N(||P_1(s, t_0)x||)
= N(P(\delta)) \cdot \exp(-\nu n\delta) \cdot N(||P_1(s, t_0)x||)
= N(P(\delta)) \cdot \exp(-\nu(t-s)) \cdot \exp(\nu r) \cdot N(||P_1(s, t_0)x||)
= M \cdot \exp(-\nu(t-s)) \cdot N(||P_1(s, t_0)x||),
\]
for all \( t \leq s \leq t_0 \leq 0 \), where \( \nu = \delta^{-1} \cdot \ln(g(\delta)) > 0 \) and \( M = N(P(\delta)) \times \exp(\nu r) > 0 \). This completes the proof.

And we have the following corresponding lemma.

Lemma 1.4. Let \( h : [0, \infty) \to (0, \infty) \) be a continuous function on \([0, \infty)\) with \( \sup \{ h(u) : u \geq 0 \} > 1 \) and \( x \in X \) such that

\[
(1.11) \quad N(||P_2(t, t_0)x||) \geq h(t-s)N(||P_2(s, t_0)x||),
\]
for all \( t \geq s \geq t_0 \geq 0 \). Then there exist \( M', \nu' > 0 \) such that

\[
(1.12) \quad N(||P_2(t, t_0)x||) \geq M' \cdot \exp[\nu'(t-s)] \cdot N(||P_2(s, t_0)x||).
\]

2. \( N \)-dichotomy for evolutionary process

Theorem 2.1. The following statements are equivalent:

(a) There exists \( N \in \mathcal{N} \) such that \( P \) is \( N \)-u.e.d.;
(b) The evolutionary process \( P \) is u.e.d.;
(c) For every \( N \in \mathcal{N} \) the evolutionary process \( P \) is \( N \)-u.e.d.
PROOF. (a) ⇒ (b): Let \( s \geq t_0 \geq 0, \ u \geq 0 \) and \( x \in X \). From (1.3) we obtain

\[
M_1^{-1} \exp(\nu_1 u) \cdot N(\|P_1(s + u, t_0)x\|) \leq N(\|P_1(s, t_0)x\|).
\]

Since

\[
\lim_{u \to \infty} M_1^{-1} \exp(\nu_1 u) = \infty,
\]

there exists \( \delta > 0 \) such that

\[
N(2) \cdot N(\|P_1(s + u, t_0)x\|) \leq N(\|P_1(s, t_0)x\|),
\]

for all \( s \geq t_0 \geq 0, \ u \geq \delta \), and consequently

\[
2\|P_1(s + u, t_0)x\| \leq \|P_1(s, t_0)x\|
\]

for all \( s \geq t_0 \geq 0, \ u \geq \delta \) and \( x \in X \). On the other hand, if \( s \leq t \leq s + \delta \), then

\[
\|P_1(t, t_0)x\| = \|P(t, s)P_1(s, t_0)x\|
\]

\[
\leq M \exp[\omega(t - s)] \cdot \|P_1(s, t_0)x\|
\]

\[
\leq M \exp(\omega \delta) \cdot \|P_1(s, t_0)x\|.
\]

From (2.3), (2.4) and Lemma 1.1, we obtain that there exist \( M'_1, \nu'_1 > 0 \) such that

\[
\|P_1(t, t_0)x\| \leq M'_1 \exp[-\nu'(t - s)] \cdot \|P_1(s, t_0)x\|
\]

for all \( t \geq s \geq t_0 \geq 0 \) and \( x \in X \).

Let \( t \geq s \geq t_0 \geq 0 \) and \( \eta > 0 \) such that \( N(\eta) \leq M_2 \). Then

\[
N(\|P_2(t, t_0)x\|) \geq N(\eta) \cdot N(\|P_2(s, t_0)x\|)
\]

\[
\geq N(\eta \cdot \|P_2(s, t_0)x\|),
\]

and hence

\[
\|P_2(t, t_0)x\| \geq \eta \|P_2(s, t_0)x\|
\]
for all \( t \geq s \geq t_0 \geq 0 \). Since \( \lim_{u \to \infty} M_2 \cdot \exp(\nu_2 u) = \infty \), there exists \( \tau > 0 \) such that

\[
N(\|P_2(s+\delta,t_0)x\|) \geq N(2) \cdot N(\|P_2(s,t_0)x\|),
\]

and hence

(2.6) \[ \|P_2(s+\delta,t_0)x\| \geq 2\|P_2(s,t_0)x\| \]

for all \( s \geq t_0 \geq 0 \) and \( x \in X \). From (2.5), (2.6) and Lemma 1.2, we obtain that there exist \( M'_2, \nu'_2 > 0 \) such that

\[
\|P_2(t,t_0)x\| \geq M'_2 \exp[\nu'_2(t-s)] \cdot \|P_2(s,t_0)x\|
\]

for all \( t \geq s \geq t_0 \geq 0 \) and \( x \in X \).

(b) \Rightarrow (c): Let \( t \geq s \geq t_0 \geq 0 \) and \( N \in \mathcal{N} \). From (1.1) it follows that

(2.7) \[ N(\|P_1(t,t_0)x\|) \leq N(M_1 \exp[-\nu_1(t-s)]) \cdot N(\|P_1(s,t_0)x\|) \]

and from \( \lim_{u \to 0} N(u) = 0 \), there exists \( \delta_1 > 0 \) such that

(2.8) \[ N(\|P_1(s+\delta_1,t_0)x\|) \leq \frac{1}{2} N(\|P_1(s,t_0)x\|) \]

for all \( s \geq t_0 \geq 0 \) and \( x \in X \). On the other hand, it follows from (1.1) that

\[
\|P_1(t,t_0)x\| \leq M_1\|P_1(s,t_0)x\|
\]

for all \( t \geq s \geq t_0 \geq 0 \) and \( x \in X \). Using Lemma 1.1, we obtain that there exist \( M''_1, \nu''_1 > 0 \) such that

\[
N(\|P_1(t,t_0)x\|) \leq M''_1 \exp[-\nu''_1(t-s)] \cdot N(\|P_1(s,t_0)x\|)
\]

for all \( t \geq s \geq t_0 \geq 0 \) and \( x \in X \). From (1.2) we obtain

\[
M_2^{-1} \exp[-\nu_2(t-s)] \cdot \|P_2(t,t_0)x\| \geq \|P_2(s,t_0)x\|
\]
for all $t \geq s \geq t_0 \geq 0$, and hence

$$N(M_2^{-1}\exp[-\nu_2(t-s)]) \cdot N(||P_2(t,t_0)x||) \geq N(||P_2(s,t_0)x||).$$

Hence $\lim_{u \to 0} N(u) = 0$. Therefore, there exists $\tau_1 > 0$ such that

$$\frac{1}{2}N(||P_2(s+\tau_1,t_0)x||) \geq N(||P_2(s,t_0)x||)$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$.

By (1.2) it follows

$$M_2^{-1}||P_2(t,t_0)x|| \geq ||P_2(s,t_0)x||$$

for all $t \geq s \geq t_0 \geq 0$, and consequently

$$N(||P_2(t,t_0)x||) \geq [N(M_2^{-1})]^{-1} \cdot N(||P_2(s,t_0)x||)$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$.

(c) $\Rightarrow$ (a) is obvious.

Seo-Nam [5] proved the following theorem.

**Theorem 2.2.** The evolutionary process $P$ is u.e.d. if and only if there exist $M, m > 0$ such that

\begin{align*}
(2.9) \quad & \int_t^\infty ||P_1(u,t_0)x|| \, du \leq M \cdot ||P_1(t,t_0)x||, \\
(2.10) \quad & \int_{t_0}^t ||P_2(u,t_0)x|| \, du \leq M \cdot ||P_2(t,t_0)x||,
\end{align*}

and

\begin{equation}
(2.11) \quad m \cdot ||P_2(t,t_0)x|| \leq ||P_2(t+1,t_0)x||
\end{equation}

for all $(t,t_0) \in T$ and $x \in X$.

Now, we are in a position to prove the main theorem in this paper.
Theorem 2.3. The evolutionary process $P$ is $N$-u.e.d. if and only if there exist $N \in \mathbb{N}$ and $n, m > 0$ such that

$$
(2.12) \quad \int_t^\infty N(\|P_1(u, t_0)x\|)du \leq M \cdot N(\|P_1(t, t_0)x\|),
$$

$$
(2.13) \quad \int_{t_0}^t N(\|P_2(u, t_0)x\|)du \leq M \cdot N(\|P_2(t, t_0)x\|),
$$

and

$$
(2.14) \quad m \cdot N(\|P_2(t, t_0)x\|) \leq N(\|P_2(t + 1, t_0)x\|)
$$

for all $(t, t_0) \in T$ and $x \in X$.

Proof. The necessity is obvious.

For the sufficiency, let $t \geq s + 1 > s \geq t_0 \geq 0$. Then we have

$$
N(\|P_1(t, t_0)x\|) \cdot \int_0^1 (N(P(u)))^{-1}du
$$

$$
= \int_0^1 N(\|P(t, \tau)P_1(\tau, t_0)x\|) \cdot [N(P(u))]^{-1}du
$$

$$
\leq \int_{t-1}^t N(P(t - \tau)) \cdot N(\|P_1(\tau, t_0)x\|) \cdot (N(P(t - \tau)))^{-1}d\tau
$$

$$
= \int_{t-1}^t N(\|P_1(\tau, t_0)x\|)d\tau
$$

$$
\leq \int_s^\infty N(\|P_1(\tau, t_0)x\|)d\tau
$$

$$
\leq M \cdot N(\|P_1(s, t_0)x\|).
$$

Therefore

$$
N(\|P_1(t, t_0)x\|) \leq M(\int_0^1 (N(P(u)))^{-1}du)^{-1} \cdot N(\|P_1(s, t_0)x\|)
$$

for all $t \geq s + 1 > s \geq t_0 \geq 0$. 

If \( t_0 \leq s \leq t < s + 1 \), then
\[
N(\|P_1(t, t_0)x\|) \leq N(P(t - s)) \cdot N(\|P_1(s, t_0)x\|) \\
\leq N(P(1)) \cdot N(\|P_1(s, t_0)x\|),
\]
and hence
\[
(2.15) \quad N(\|P_1(t, t_0)x\|) \leq H \cdot N(\|P_1(s, t_0)x\|)
\]
for all \( t \geq s \geq t_0 \geq 0 \) and \( x \in X \), where
\[
H = \max\{M \cdot (\int_0^1 N(P(u))du)^{-1}, N(P(1))\}.
\]
Integrating (2.15) from \( s \) to \( t \) we obtain
\[
(2.16) \quad (t - s)N(\|P_1(t, t_0)x\|) \leq H \int_s^t N(\|P_1(u, t_0)x\|)du \\
\leq H \int_s^\infty N(\|P_1(u, t_0)x\|)du \\
\leq H \cdot M \cdot N(\|P_1(s, t_0)x\|).
\]
Combining this and (2.15), we obtain
\[
(2.17) \quad N(\|P_1(t, t_0)x\|) \leq M(H + 1) \cdot (t - s + 1)^{-1} \cdot N(\|P_1(s, t_0)x\|)
\]
for all \( t \geq s \geq t_0 \geq 0 \).

It follows from Lemma 1.3, there are \( M_1 > 0 \) and \( \nu_1 > 0 \) such that
\[
(2.18) \quad N(\|P_1(t, t_0)x\|) \leq M_1 \exp[-\nu_1(t - s)] \cdot N(\|P_1(s, t_0)x\|)
\]
for all \( t \geq s \geq t_0 \geq 0 \).

Let \( x \in X \) and \( t \geq s \geq t_0 + 1 > t_0 \geq 0 \). Then we have
\[
N(\|P_2(s, t_0)x\|) \cdot \int_0^1 (N(P(u)))^{-1}du \\
= N(\|P_2(s, t_0)x\|) \cdot \int_{s-1}^a (N(P(s - \tau)))^{-1}d\tau \\
\leq \int_{t_0}^t N(\|P_2(\tau, t_0)x\|)d\tau \\
\leq M \cdot N(\|P_2(t, t_0)x\|).
\]
Therefore, for $t \geq v + 1 > v \geq t_0$

$$N(||P_2(t, t_0)x||) \geq M^{-1} \int_0^1 (N(P(u)))^{-1} du \cdot N(||P_2(v + 1, t_0)x||)K \cdot N(||P_2(v, t_0)x||),$$

where

$$K = M^{-1} \cdot m \cdot \int_0^1 (N(P(u)))^{-1} du.$$

Integrating (2.19) from $v + 1$ to $t$, we obtain

$$\int_{v+1}^t N(||P_2(r, t_0)x||) dr \geq K(t - v - 1) \cdot N(||P_2(v, t_0)x||)$$

for all $t \geq v + 1 > v \geq t_0 \geq 0$.

Therefore,

$$K(t - v - 1) \cdot N(||P_2(v, t_0)x||) \leq \int_v^t N(||P_2(\tau, t_0)x||) d\tau \leq M \cdot N(||P_2(t, t_0)x||)$$

for all $t \geq v + 1 > v \geq t_0 \geq 0$. Hence from (2.19) we obtain

$$N(||P_2(t, t_0)x||) \geq K(t - v) \cdot (M + 1)^{-1} \cdot N(||P_2(v, t_0)x||) \geq K_1(t - v + 1) \cdot N(||P_2(t, t_0)x||)$$

for all $t \geq v + 1 > v \geq t_0 \geq 0$ and $x \in X$, where $K_1 = K \cdot [2(M + 1)]^{-1}$.

If $v \leq t < v + 1$, then

$$N(||P_2(v + 1, t_0)x||) = N(||P(v + 1, t)P_2(t, t_0)x||) \leq N(P(v + 1 - t)) \cdot N(||P_2(t, t_0)x||).$$

Therefore,

$$N(||P_2(t, t_0)x||) \geq (N(P(1)))^{-1} \cdot N(||P_2(v + 1, t_0)x||) \geq m \cdot N(P(1))^{-1} \cdot N(||P_2(v, t_0)x||),$$
and

\[(2.24) \quad N(\|P_2(t, t_0)x\|) \geq m \cdot N(P(1))^{-1} \cdot N(\|P_2(v, t_0)x\|) \cdot (t - v)\]

for all \(0 \leq t_0 \leq v \leq t < v + 1\).

Combining (2.23) and (2.24), we have

\[(2.25) \quad N(\|P_2(t, t_0)x\|) \geq K_2(t - v + 1) \cdot N(\|P_2(v, t_0)x\|)\]

for all \(0 \leq t_0 \leq v \leq t < v + 1\), where

\[K_2 = (m + 1) \cdot [2N(P(1))]^{-1} > 0.\]

From (2.24) and (2.25), we have

\[N(\|P_2(t, t_0)x\|) \geq K'(t - v + 1) \cdot N(\|P_2(v, t_0)x\|)\]

for all \(t \geq s \geq t_0 \geq 0\) and \(x \in X\), where \(K' = \min\{K_1, K_2\}\).

From Lemma 1.4, we know that there are \(M_2 > 0\) and \(v_2 > 0\) such that

\[N(\|P_2(t, t_0)x\|) \geq M_2 \exp[v_2(t - s)] \cdot N(\|P_2(s, t_0)x\|)\]

for all \(t \geq s \geq t_0 \geq 0\) and \(x \in X\). This completes the proof.

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