A CLASS OF SERIES INVOLVING
THE ZETA FUNCTION

HYERIM LEE, YOUNG JOON CHO,
KEUMSIK LEE AND TAE YOUNG SEO

Abstract. The authors apply the theory of multiple Gamma functions, which was recently revived in the study of the determinants of the Laplacians, in order to present a class of closed-form evaluations of series involving the Zeta function by appealing only to the definitions of the double and triple Gamma functions.

1. Introduction and Preliminaries

The multiple Gamma functions were defined and studied by Barnes (cf. [3] and [4]) and others in about 1900. Although these functions did not appear in the tables of the most well-known special functions, yet the double Gamma function was cited in the exercises by Whittaker and Watson [26, p. 264] and recorded also by Gradshteyn and Ryzhik [14, p. 661, Entry 6.441(4); p. 937, Entry 8.333]. Recently, these functions were revived in the study of the determinants of the Laplacians on the $n$-dimensional unit sphere $S^n$ (see [7], [10], [11], [18], [23], and [25]). More recently, Choi et al. ([8], [9], [11], and [12]) used these functions to evaluate the sums of several classes of series involving the Riemann Zeta function, the subject of which can be traced back to an over two-century old theorem of Christian Goldbach (1690-1764) noted in the
work of Srivastava [20, p. 1] who investigated this subject in a rather systematic and unified manner among many other authors. Before their investigation by Barnes, these functions had been introduced in a different form by several authors. The recent uses besides the above-mentioned subjects have been made of the theory of the double Gamma function by many authors for their own purposes.

In this note we aim at providing a class of closed-form evaluations of series involving the Zeta function by appealing only to the definitions of the double and triple Gamma functions.

To do so, we start with recalling the Barnes G-function \((1/G = \Gamma_2\) is so-called the double Gamma function) which has several equivalent forms one of which is

\[
\{\Gamma_2(z+1)\}^{-1} = G(z+1)
\]

\[
= (2\pi)^{\frac{3}{2}} e^{-\frac{1}{2}[(1+\gamma)z^2+z]} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k} \right)^k e^{-z + \frac{z^2}{2k}} \right\},
\]

where \(\gamma\) denotes the Euler-Mascheroni constant given by

\[
\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) \equiv 0.5772156649015325 \ldots
\]

For sufficiently large real \(x\) and \(a \in \mathbb{C}\), we have the Stirling formula for the G-function:

\[
\log G(x + a + 1) = \frac{x + a}{2} \log(2\pi) - \log A + \frac{1}{12} - \frac{3x^2}{4} - ax
\]

\[
+ \left( \frac{x^2}{2} - \frac{1}{12} + \frac{a^2}{2} + ax \right) \log x + O(1/x) \ (x \to \infty),
\]

where \(A\) is the Glaisher-Kinkelin constant defined by

\[
\log A = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} k \log k - \left( \frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \log n + \frac{n^2}{4} \right]
\]

\[
\equiv 1.282427130 \ldots
\]
The $G$-function satisfies the following fundamental functional relationships:

\[(5) \quad G(1) = 1 \quad \text{and} \quad G(z + 1) = \Gamma(z) G(z) \quad (z \in \mathbb{C}),\]

where $\Gamma$ denotes the familiar Gamma function which also satisfies the basic relationships:

\[(6) \quad \Gamma(1) = 1 \quad \text{and} \quad \Gamma(z + 1) = z \Gamma(z) \quad (z \in \mathbb{C} \setminus \{-1, -2, -3, \ldots\}).\]

Vignéras [24] obtained a recurrence formula of the multiple Gamma functions $\Gamma_n$ ($n \in \mathbb{N}$) which, for $n = 3$, readily yields the explicit Weierstrass canonical product form of the triple Gamma function $\Gamma_3$ as follows (see Choi and Srivastava [11]):

\[\Gamma_3(1 + z) = G_3(1 + z) = \exp \left[ -\frac{1}{6} \left( \gamma + \frac{\pi^2}{6} + \frac{3}{2} \right) z^3 \right.\]
\[+ \frac{1}{4} \left( \gamma + \log(2\pi) + \frac{1}{2} \right) z^2 + \left( \frac{3}{8} - \frac{1}{4} \log(2\pi) - \log A \right) z \]
\[\times \prod_{k=1}^{\infty} \left\{ \left( 1 + \frac{z}{k} \right)^{-\frac{1}{k}(k+1)} \exp \left[ \frac{1}{2} (k + 1) z \right] \right.\]
\[\left. \left. - \frac{1}{4} \left( 1 + \frac{1}{k} \right) z^2 + \frac{1}{6k} \left( 1 + \frac{1}{k} \right) z^3 \right] \right\}.\]

The triple Gamma function $\Gamma_3$ also satisfies the following fundamental functional relationships:

\[(8) \quad \Gamma_3(1) = 1 \quad \text{and} \quad \Gamma_3(z + 1) = G(z) \Gamma_3(z) \quad (z \in \mathbb{C}).\]

Note that another form of the triple Gamma function $\Gamma_3$ appeared in the work of Choi [7] who expressed, in terms of the double and triple Gamma functions, the analogous Weierstrass canonical product of the shifted sequence of the eigenvalues of the Laplacian on the unit sphere $S^3$ with standard metric which was indispensable to evaluate the determinant of the Laplacian on $S^3$ there.
The Riemann Zeta function $\zeta(s)$ defined by

\[
\zeta(s) := \begin{cases} 
\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\
(1-2^{1-s})^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} & (\Re(s) > 0; \ s \neq 1),
\end{cases}
\]

which can, except for a simple pole only at $s = 1$ with its residue 1, be continued analytically to the whole $s$-plane by the contour integral representation (cf. Whittaker and Watson [26, p. 266]) or many other integral representations (cf. Erdélyi et al. [13, p. 33]) and satisfies the functional equation (see [22, p. 13]):

\[
\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \zeta(1-s) \sin \frac{\pi s}{2}.
\]

2. Review of some known mathematical constants

In addition to the Glaisher-Kinkelin constant $A$ defined by (4), by means of the Euler-Maclaurin summation formula (cf. Hardy [16, p. 318]), Choi and Srivastava [11] introduced two mathematical constants:

\[
\log B = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} k^2 \log k - \left( \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \log n + \frac{n^3}{9} - \frac{n}{12} \right],
\]

and

\[
\log C = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} k^3 \log k - \left( \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \log n + \frac{n^4}{16} - \frac{n^2}{12} \right],
\]

where the approximate numerical values of $B$ and $C$ are given by

\[
B \cong 1.030 \, 888 \, 116 \, 106 \, 913 \ldots
\]
and

\[(14) \quad C \cong 301,393,393,412,467,84 \times 10^{-2714341} \ldots .\]

The constant \(B\) was also considered by Choi and Srivastava [9, p. 102]. Just as the constant \(A\), the strangely-looking constants \(B\) and \(C\) are readily seen to be involved naturally in the study of the multiple Gamma functions (cf. Choi and Srivastava [9, 11]).

Voros [25] gave a connection between the Glaisher-Kinkelin constant \(A\) and the Riemann Zeta function \(\zeta(s)\):

\[(15) \quad \log A = \frac{1}{12} - \zeta'(-1),\]

which was also proved in a different way by Vardi [23]. By making use of the Voros’s technique, Choi and Srivastava [11] showed that

\[(16) \quad \log B = -\zeta'(-2)\]

and

\[(17) \quad \log C = -\zeta'(-3) - \frac{11}{720}.\]

We also obtain the relationship:

\[(18) \quad \log B = \frac{\zeta(3)}{4 \pi^2},\]

which immediately follows by applying the special case \((n = 1)\) of the general identity which is derivable from (10) (see Srivastava [21, p. 387, Eq. (1.15)]):

\[(19) \quad \zeta(2n + 1) = (-1)^n \frac{2(2n)^{2n}}{(2n)!} \zeta'(-2n) \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots \})\]

to (16). Choi and Srivastava [11] also showed the relationship (18) in a markedly different way by recalling a result of Choi and Srivastava [9, p. 111, Eq. (4.34)] in the following corrected form:

\[(20) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k)}{(k + 1)2^{2k}} = \frac{1}{2} - \log 2 + 14 \log B\]
and comparing (20) with another known result (cf., e.g., [6, p. 191, Eq. (3.19)]):

\begin{equation}
(21) \quad \zeta(3) = \frac{2\pi^2}{7} \left( \log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+1)2^{2k}} \right) .
\end{equation}

Just as the well-known result

\begin{equation}
(22) \quad \Gamma \left( \frac{1}{2} \right) = \pi^{\frac{1}{2}} ,
\end{equation}

it is known from the work of Barnes [3, p. 288, Section 17] that

\begin{equation}
(23) \quad G \left( \frac{1}{2} \right) = 2^{\frac{3}{4}} \cdot \pi^{-\frac{1}{4}} \cdot e^{\frac{1}{8}} \cdot A^{-\frac{3}{8}} .
\end{equation}

Choi and Srivastava [11] also expressed the value of \( \Gamma_3 \left( \frac{1}{2} \right) \) in terms of the mathematical constants \( \pi, \, e, \, A, \) and \( B \):

\begin{equation}
(24) \quad \Gamma_3 \left( \frac{1}{2} \right) = 2^{-\frac{1}{4}} \cdot \pi^{\frac{3}{16}} \cdot e^{-\frac{1}{8}} \cdot A^{\frac{3}{8}} \cdot B^{\frac{1}{8}} .
\end{equation}

The Catalan's constant \( G \) is defined by

\begin{equation}
(25) \quad G = \frac{1}{2} \int_{0}^{1} K(k) \, dk = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \approx 0.915965 \ldots ,
\end{equation}

where \( K \) is the complete elliptic integral of the first kind, given by

\begin{equation}
(26) \quad K(k) := \int_{0}^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} .
\end{equation}

The constant \( G \) becomes a special case of many other functions one of which will be considered in the next section and has, among other things, been used to evaluate integrals, for example (see [14, p. 526, Entry 4.224]):

\begin{equation}
(27) \quad \int_{0}^{\pi/4} \log(\sin t) \, dt = -\frac{\pi}{4} \log 2 - \frac{1}{2} G ,
\end{equation}
and to give closed-form evlauations of a certain class of series involving
the Zeta function (cf. Choi and Srivastava [9]).

By recalling the special value of the Gamma function (see Spiegel
[19, p. 1]):
\[ \Gamma \left( \frac{1}{4} \right) \cong 3.625609908241908 \ldots \]

and making use of a duplication formula for the G-function (cf. Barnes
[3, p. 291] for the general case), Choi and Srivastava [9] showed that

\[ G \left( \frac{1}{4} \right) = e^{\frac{3}{8} - \frac{\pi}{4\sqrt{2}}} \cdot A^{-\frac{3}{8}} \left\{ \Gamma \left( \frac{1}{4} \right) \right\}^{-\frac{3}{4}} \cong 0.293756 \ldots ; \]

or, equivalently,

\[ G \left( \frac{3}{4} \right) = 2^{-\frac{3}{8}} \cdot \pi^{-\frac{1}{2}} \cdot e^{\frac{3}{8} + \frac{3\pi}{4\sqrt{2}}} \cdot A^{-\frac{3}{8}} \left\{ \Gamma \left( \frac{3}{4} \right) \right\}^{\frac{1}{2}} \cong 0.848718 \ldots . \]

**Remark.** While writing this note, we have found that the con-
stants \( A, B, \) and \( C \) were considered by Bendersky [5] and Adamcik
[1] including their Riemann Zeta function representations (15), (16),
and (17).

### 3. Series involving the Zeta function

Taking logarithms on both sides of (1) with the Maclaurin series
expansion of \( \log(1 + z) \) yields

\[
\sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{k}{k+1} z^{k+1} = [1 - \log(2\pi)] \frac{z}{2} + (1 + \gamma) \frac{z^2}{2} + \log G(z + 1) \quad (|z| < 1),
\]

which, upon replacing \( z \) by \( -z \), gives

\[
\sum_{k=2}^{\infty} \frac{\zeta(k)}{k+1} z^{k+1} = [1 - \log(2\pi)] \frac{z}{2} - (1 + \gamma) \frac{z^2}{2} - \log G(1 - z) \quad (|z| < 1).
\]
Adding and subtracting (31) and (32), we obtain
\[
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k+1} z^{2k+1} = [1 - \log(2\pi)] \frac{z}{2} + \frac{1}{2} \log \frac{G(1+z)}{G(1-z)} \quad (|z| < 1)
\]
and
\[
\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{k+1} z^{2k+2} = -(1 + \gamma) z^2 - \log[G(1+z)G(1-z)] \quad (|z| < 1).
\]

If we take the limit on both sides of (31) as \( z \to 1 \) with (5) and (6), we find that
\[
\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k+1} = 1 + \frac{\gamma}{2} - \frac{1}{2} \log(2\pi),
\]
which was considered by several earlier authors (cf. Srivastava [20]).

Now setting \( z = \frac{1}{2}, \ z = \frac{3}{4}, \) and \( z = \frac{3}{4} \) in (31) through (34) and making use of (5), (6), (23), (29), and (30), we readily obtain a class of closed-form evaluations of series involving the Riemann Zeta function:

\[
\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{(k+1) 2^k} = 1 + \gamma - \frac{5}{12} \log 2 - 3 \log A;
\]

\[
\sum_{k=2}^{\infty} \frac{\zeta(k)}{(k+1) 2^k} = - \frac{\gamma}{4} - \frac{7}{12} \log 2 + 3 \log A;
\]

\[
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1) 2^{2k}} = \frac{1}{2} - \frac{1}{2} \log 2;
\]

\[
\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1) 2^{2k}} = -2 - \gamma - \frac{1}{3} \log 2 + 12 \log A;
\]
(40) \[ \sum_{k=2}^{\infty} \frac{(-1)^{k} \zeta(k)}{(k+1)2^{2k}} = 1 + \frac{\gamma}{8} - \frac{G}{\pi} + \log \left[ 2^{-\frac{1}{2}} \cdot \pi^{-\frac{1}{2}} \cdot A^{\frac{3}{2}} \cdot \Gamma \left( \frac{1}{4} \right) \right] ; \]

(41) \[ \sum_{k=2}^{\infty} \frac{\zeta(k)}{(k+1)2^{2k}} = -\frac{\gamma}{8} - \frac{G}{\pi} + \log \left[ \pi^\frac{1}{2} \cdot A^{\frac{3}{2}} \cdot \{\Gamma \left( \frac{1}{4} \right) \}^{-1} \right] ; \]

(42) \[ \sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1)2^{4k}} = \frac{1}{2} - \frac{G}{\pi} - \frac{1}{4} \log 2 ; \]

(43) \[ \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(2k+1)2^{4k}} = -4 - \gamma + \log \left[ 2^2 \cdot \pi^2 \cdot A^{36} \cdot \{\Gamma \left( \frac{1}{4} \right) \}^{-8} \right] ; \]

(44) \[ \sum_{k=2}^{\infty} (-1)^{k} \zeta(k) \left( \frac{3}{4} \right)^{k} \left( \frac{k}{k+1} \right) = 1 + \frac{3\gamma}{8} + \frac{G}{3\pi} + \log \left[ \pi^\frac{1}{2} \cdot A^{-\frac{3}{2}} \cdot \{\Gamma \left( \frac{1}{4} \right) \}^{-1} \right] ; \]

(45) \[ \sum_{k=2}^{\infty} \frac{\zeta(k)}{k+1} \left( \frac{3}{4} \right)^{k} = -\frac{3\gamma}{8} + \frac{G}{3\pi} + \log \left[ 2^{-\frac{1}{2}} \cdot \pi^{-\frac{1}{2}} \cdot A^{\frac{3}{2}} \cdot \Gamma \left( \frac{1}{4} \right) \right] ; \]

(46) \[ \sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k+1} \left( \frac{3}{4} \right)^{2k} = \frac{1}{2} + \frac{G}{3\pi} - \frac{1}{4} \log 2 ; \]

(47) \[ \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{k+1} \left( \frac{3}{4} \right)^{2k} = -\frac{4}{3} - \gamma + \log \left[ 2^{-\frac{3}{2}} \cdot \pi^{-\frac{3}{2}} \cdot A^4 \cdot \{\Gamma \left( \frac{1}{4} \right) \}^{\frac{9}{4}} \right] . \]
Taking logarithms on both sides of (7), we readily have

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{k+2} [\zeta(k) + \zeta(k+1)] z^{k+2}$$

$$= \left[ -\frac{3}{4} + \frac{1}{2} \log(2\pi) + 2 \log A \right] z$$

$$- \frac{1}{2} \left[ \gamma + \log(2\pi) + \frac{1}{2} \right] z^2 + \frac{1}{3} \left( \gamma + \frac{\pi^2}{6} + \frac{3}{2} \right) z^3$$

$$+ 2 \log \Gamma_3(1+z) \quad (|z| < 1),$$

which, upon letting $z \to 1$, yields

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{k+2} [\zeta(k) + \zeta(k+1)] = -\frac{1}{2} + \frac{\pi^2}{18} - \frac{\gamma}{6} + 2 \log A.$$

In view of $\zeta(2) = \frac{\pi^2}{6}$ and (35), we find that

$$\sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k+2} = -1 - \frac{\gamma}{2} + \frac{\pi^2}{18} + \frac{1}{2} \log(2\pi),$$

which, upon combining with (49), yields

$$\sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k+2} = \frac{1}{2} + \frac{\gamma}{3} - \frac{1}{2} \log(2\pi) + 2 \log A.$$

Replace $z$ by $-z$ in (48) and we have

$$\sum_{k=2}^{\infty} \frac{\zeta(k) + \zeta(k+1)}{k+2} z^{k+2}$$

$$= \left[ \frac{3}{4} - \frac{1}{2} \log(2\pi) - 2 \log A \right] z$$

$$- \frac{1}{2} \left[ \gamma + \log(2\pi) + \frac{1}{2} \right] z^2 + \frac{1}{3} \left( \gamma + \frac{\pi^2}{6} + \frac{3}{2} \right) z^3$$

$$+ 2 \log \Gamma_3(1+z) \quad (|z| < 1).$$
Adding and subtracting (48) and (52), we obtain
\[ \sum_{k=1}^{\infty} \zeta(2k) + \frac{\zeta(2k+1)}{k+1} z^{2k+2} = - \left[ \gamma + \log(2\pi) + \frac{1}{2} \right] z^2 \]
\[ + 2 \log \left[ \Gamma_3(1+z) \Gamma_3(1-z) \right] \quad (|z| < 1) \]
and
\[ \sum_{k=1}^{\infty} \frac{\zeta(2k+1) + \zeta(2k+2)}{2k+3} z^{2k+3} = \left[ \frac{3}{4} - \frac{1}{2} \log(2\pi) - 2 \log A \right] z \]
\[ - \frac{1}{3} \left( \gamma + \frac{\pi^2}{6} + \frac{3}{2} \right) z^3 + \log \left[ \frac{\Gamma_3(1-z)}{\Gamma_3(1+z)} \right] \quad (|z| < 1). \]

Setting \( z = \frac{1}{2} \) in (48) with the aid of (23) and (24), and combining (36) with the resulting identity, we readily obtain
\[ \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{(k+2)2^k} = \frac{1}{2} + \frac{\gamma}{6} + \log \left( 2^{-\frac{3}{2}} \cdot A^{-2} \cdot B^7 \right). \]

Similarly if we set \( z = \frac{1}{2} \) in (52), (53), and (54), we readily find that
\[ \sum_{k=2}^{\infty} \frac{\zeta(k)}{(k+2)2^k} = -\frac{\gamma}{6} + \log \left( 2^{-\frac{3}{2}} \cdot A^2 \cdot B^7 \right); \]
\[ \sum_{k=1}^{\infty} \frac{\zeta(2k)}{(k+1)2^{2k}} = \frac{1}{2} + \log \left( 2^{-1} \cdot B^{14} \right); \]
\[ \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(2k+3)2^{2k}} = -\frac{1}{2} - \frac{\gamma}{3} + \log \left( 2^{-\frac{3}{2}} \cdot A^4 \right). \]

We conclude this note by remarking that these closed-form evaluations of series associated with the Riemann Zeta function may have a capability of diverse applications to compute (or evaluate) other mathematical subjects (cf. [10] and [11]).
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Department of Mathematics
Pusan National University
Pusan 609-735, Korea

E-mail: tyseo@hyowon.pusan.ac.kr