BOUNDARY DISTORTION OF CERTAIN DOMAINS IN $\mathbb{R}^n$

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1. Introduction

Suppose that $D$ is a domain in the extended complex plane $\mathbb{C}$. For each $z_0 \in \mathbb{C}$ and $0 < r < \infty$, we let $B(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$, $S(z_0, r) = \partial B(z_0, r)$, and $B^*(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| > r \}$.

For non-empty sets $A, B \subset \mathbb{C}$, $diam(A)$ is the diameter of $A$ and $d(A, B)$ is the distance of $A$ and $B$.

A domain $D$ in $\mathbb{C}$ is a $K$-quasiball, $1 \leq K < \infty$, if it is the image of the unit ball $B(0,1)$ under a $K$-quasiconformal self mapping $f$ of $\mathbb{C}$. The boundary $S$ of a $K$-quasiball $D$ is called a $K$-quasisphere. For $n = 2$ we call the domain a quasisphere, and call the boundary of a $K$-quasisphere $D$ a $K$-quasicircle([1, 6]).

Next we say that a Jordan curve $C$ in $\mathbb{R}^2$ has circular distortion $c$, $1 \leq c < \infty$, if for each Möbius transformation $\phi$, either $\phi(C)$ separates the boundary circles of an annulus

$$A = A(z_0; r, s) = \{ z \in \mathbb{R}^2 : r \leq |z - z_0| \leq s \}$$

with radii ratio $\frac{s}{r} = c$ or $\phi(C)$ contains the point $\infty$. The circular distortion is a Möbius invariant which measures how far a Jordan curve differs from being a circle or line. In particular, $C$ has circular distortion 1 if and only if it is a circle or line.


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R. Kühnau established the following relation between these two concepts.

**Lemma 1.1.** [5] If $C$ is a $K$-quasicircle in $\overline{C}$, then $C$ has circular distortion $c$, where $c$ depends only on $K$.

R. Kühnau found sharp bounds for the constant $c$ in terms of $K$ and then asked if the converse of lemma 1.1 is true, that is, if $C$ is a curve with circular distortion $c$, then it is a $K$-quasicircle where $K$ depends only on $c$. F. W. Gehring and C. Pommerenke [3] answered this question as follows.

**Lemma 1.2.** [3] If $C$ is a Jordan curve in $\mathbb{R}^2$ with circular distortion $c < \sqrt{2}$, then $C$ is a $K$-quasicircle where $K$ depends only on $c$.

Their proof was based on elementary classical properties of the exterior conformal mapping $g : B^*(0, 1) \to \text{ext}(C)$, defined by

$$g(z) = z + \sum_{j=0}^{\infty} b_j z^{-j}.$$  

Next lemma plays an important role in proving Lemma 1.2.

**Lemma 1.3.** [3] If $C$ is a Jordan curve in $\mathbb{R}^2$ which separates the boundary circles of an annulus $A$ with radii ratio $c$ and if $g$ maps $B^*(0, 1)$ onto $\text{ext}(C)$, then

$$|b_1| \leq \frac{c^2 - 1}{c^2 + 1}.$$

**Remark 1.5.** [3] The mapping

$$g(z) = z + \frac{c - 1}{c + 1} \frac{1}{z}$$

shows that one cannot replace the upper bound in (1.4) by anything less than $\frac{c-1}{c+1}$.
One of the main purposes of this paper is to give some other extreme examples for the constant $|b_1|$ in Lemma 1.3 (see Section 2).

The bound $c < \sqrt{2}$ in Lemma 1.2 is not sharp [3]. While we looked for the sharp bound of $c$ and another example of a Jordan curve $C$ in $\mathbb{C}$ with finite circular distortion $c$ which is not a quasicircle, K. Kim found one more geometric condition, so-called the double disk property, which is in between a quasicircle and circular distortion [4].

We say that a topological $n$-sphere $S$ in $\mathbb{R}^n$ has the double ball property if there exists a constant $b$, $1 \leq b < \infty$, such that for each $z_0 \in S$ and $0 < r \leq \text{diam}(S)$, there exist open balls $B_z$ and $B_e$ in $\mathbb{R}^n$ with

$$B_z \subset \text{int}(S), \quad B_e \subset \text{ext}(S), \quad B_z \cup B_e \subset B(z_0, r),$$

(1.6)

$$b \text{diam}(B_z) \geq r, \quad b \text{diam}(B_e) \geq r,$$

where $\text{int}(S)$ and $\text{ext}(S)$ are interior and exterior of $S$, respectively.

If we replace a topological $n$-sphere $S$ in $\mathbb{R}^n$ by a Jordan curve $C$ in $\mathbb{R}^2$ and replace open balls by open disks, then we say that a Jordan curve $C$ in $\mathbb{R}^2$ has the double disk property.

It is also equivalent to asking that for $0 < r \leq \text{diam}(S)$ (or $\text{diam}(C)$), each point $z$ of $S$ (or $C$) should subtend balls (or disks) of a fixed visual angle in each complementary domain of $S$ (or $C$) within distance $r$ of $z$. Again the constant $b$ in (1.6) measures how far $S$ (or $C$) differs from being a $n$-sphere (or circle), respectively. In particular, $b = 1$ if and only if $S$ (or $C$) is $n$-sphere (or circle), respectively.

In [4] K. Kim established relations between the double disk property, quasicircle and circular distortion.

**Lemma 1.7.** [4] If $C$ is a $K$-quasicircle in $\mathbb{R}^2$, then $C$ has the double disk property with constant $b$, where $b$ depends only on $K$. If $C$ has the double disk property with constant $b$, then $C$ has circular distortion $c = 16b$.

We say that a topological $n$-sphere $S$ in $\mathbb{R}^n$ has spherical distortion $c$, $1 \leq c < \infty$, if for each Möbius transformation $\phi$, either $\phi(S)$ separates the boundary spheres of a spherical annulus

$$A = A(z_0, r, s) = \{z \in \mathbb{R}^n : r \leq |z - z_0| \leq s\}$$
with radii ratio \( \frac{r}{R} = \text{c or } \phi(S) \) contains the point \( \infty \). It is well known fact that \( S \) is a 1-quasisphere if and only if \( S \) is a plane or \( n \)-sphere if and only if \( S \) has spherical distortion 1.

In Section 3, we give higher dimensional analogues of Lemmas 1.1 and 1.7 to the case of quasisphere as follows. If \( S \) is a \( K \)-quasisphere in \( \mathbb{R}^n \), then \( S \) has spherical distortion \( c \), where \( c \) depends only on \( K \). If \( S \) is a \( K \)-quasisphere in \( \mathbb{R}^n \) then \( S \) has the double ball property with constant \( b \), where \( b \) depends only on \( K \).

Next we say that a domain \( D \) in \( \mathbb{R}^n \) is called an \((\alpha, \beta)\)-John domain, \( 0 < \alpha \leq \beta < \infty \), if there is \( z_0 \in D \) such that for each \( z \in D \), \( z \) has a rectifiable curve \( \gamma : [0, \ell] \to D \), with arc length as parameter \( \gamma(0) = z, \gamma(\ell) = z_0, \ell \leq \beta \), and \( d(\gamma(t), \partial D) \geq \frac{\alpha}{\beta} t \), for all \( t \in [0, \ell] \). We call \( z_0 \) a John center (see [7]).

A simply connected John domain \( D \) in \( \mathbb{R}^2 \) is called an \((\alpha, \beta)\)-John disk. John disks can be thought of "one-sided quasidisk" [9, 10]. For example, a Jordan domain in the plane is a quasidisk if and only if \( D \) and \( D^* = \mathbb{R}^2 \setminus \overline{D} \) are John disks [9]. In [4] K. Kim showed that a John disk satisfies the one-sided analogue of the double disks property of quasidisks.

**Lemma 1.8.** [4] If a Jordan curve \( C \) in \( \mathbb{R}^2 \) is the boundary of a \((\alpha, \beta)\)-John disk \( D \), then there exists a constant \( b \), \( 1 \leq b < \infty \), such that for each \( w \in C \) and \( 0 < r \leq \text{diam}(C) \), there is an open disk \( B \) with

\[
B \subset \text{int}(C), \quad B \subset B(w, r), \quad b \text{diam}(B) \geq r,
\]

where \( b \) depends only on \( \alpha \) and \( \beta \).

In Section 4, we also give higher dimensional analogue of Lemma 1.8 to the case of John domain in \( \mathbb{R}^n \).

2. Some extreme examples for the constant \(|b_1|\) in lemma 1.3

**Example 2.1.** Suppose that \( C \) is a boundary curve of a simply connected domain \( D = B(0, 1) \setminus \{[-1, -1/c] \cup [1/c, 1]\} \), \( 1 \leq c < \infty \). Suppose also that \( f \) maps \( B(0, 1) \) conformally onto \( D \) with \( f(0) = 0, f'(0) > 0 \).
Then the mapping

\[ g(\zeta) = \frac{f'(0)}{f(\frac{1}{\zeta})} \]

shows that one can not replace the upper bound in (1.4) by anything less than \((\frac{c^2 - 1}{c^2 + 1})^2\).

PROOF. Let

\[ h : \mathbb{R}^2 \setminus [-1, 1] \to \mathbb{R}^2 \setminus [-\frac{1}{2}(c + \frac{1}{c}), \frac{1}{2}(c + \frac{1}{c})], \quad h(w) = \frac{1}{2}(c + \frac{1}{c})w, \]

and let

\[ k_1 : B(0, 1) \to \mathbb{R}^2 \setminus [-1, 1], \quad k_2 : D \to \mathbb{R}^2 \setminus [-\frac{1}{2}(c + \frac{1}{c}), \frac{1}{2}(c + \frac{1}{c})], \quad k_i(w) = \frac{1}{2}(w + \frac{1}{w}), \quad i = 1, 2. \]

Then \( g \) maps \( B^*(0, 1) \) onto \( \text{ext}(\frac{f'(0)}{f(z)}) \) and \( h \circ k_1 = k_2 \circ f \). Therefore

\[ S_{h \circ k_1}(z) = S_{k_2 \circ f}(z) \]

for each \( z \in B(0, 1) \), where \( S_f(z) = (\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2 \) is the Schwarzian derivative of \( f \). Thus

\[ S_h(k_1(z))(k'_1(z))^2 + S_{k_1}(z) = S_{k_2}(f(z))(f'(z))^2 + S_f(z). \]

Since \( S_h(k_1(z)) = 0 \), we get

\[ \frac{-6}{(z^2 - 1)^2} = \frac{-6}{(f(z)^2 - 1)^2}(f'(z))^2 + S_f(z). \]

Hence

\[ (2.2) \quad S_f(z) = \frac{-6}{(z^2 - 1)^2} + \frac{6}{(f(z)^2 - 1)^2}(f'(z))^2. \]
Since
\[ f(z) = k_2^{-1} \circ h \circ k_1(z) = \frac{1}{4}(c + \frac{1}{c})(z + \frac{1}{z}) - \sqrt{\frac{1}{4^2}(c + \frac{1}{c})^2(z + \frac{1}{z})^2 - 1}, \]
by L'Hospital's rule we have
\[ \lim_{z \to 0} \left( \frac{f'(z)}{(f(z))^2 - 1} \right)^2 = 4\left( \frac{c}{c^2 + 1} \right)^2. \]
Thus by (2.2)
\[ S_f(0) = -6 + 24\left( \frac{c}{c^2 + 1} \right)^2. \]
Let \( g(\zeta) = \zeta + \sum_0^\infty b_j \zeta^{-j} \). Then by [2] and [3],
\[ b_1 = -\frac{1}{6} S_f(0) = \left( \frac{c^2 - 1}{c^2 + 1} \right)^2. \]
Therefore one can not replace the upper bound in (1.4) by anything less than \( \left( \frac{c^2 - 1}{c^2 + 1} \right)^2 \).

**Example 2.3.** Suppose that \( C \) is a boundary curve of a simply connected domain \( D = B(0, 1) \setminus [\frac{1}{c}, 1], \) \( 1 \leq c < \infty \). Suppose also that \( f \) maps \( B(0, 1) \) conformally onto \( D \) with \( f(0) = 0, f'(0) > 0 \). Then the mapping
\[ g(\zeta) = \frac{f'(0)}{f(\frac{1}{c})} \]
also shows that one can not replace the upper bound in (1.4) by anything less than \( \left( \frac{c-1}{c+1} \right)^2 \left( 1 + \frac{4c}{(c+1)^2} \right) \).

**Proof.** Let
\[ h: \mathbb{R}^2 \setminus [\frac{1}{4}, \infty) \to \mathbb{R}^2 \setminus [\frac{c}{(c+1)^2}, \infty), \quad h(w) = \frac{4c}{(c+1)^2} w. \]
Let
\[ k_1 : B(0, 1) \to \mathbb{R}^2 \setminus \left[ \frac{1}{4}, \infty \right) \quad k_2 : D \to \mathbb{R}^2 \setminus \left[ \frac{c}{(c + 1)^2}, \infty \right), \]
and
\[ k_i(w) = \frac{w}{(1 + w)^2}, \quad i = 1, 2. \]
Then \( g \) maps \( B^*(0, 1) \) onto \( \text{ext} \left( \frac{f'(0)}{f(c)} \right) \) and \( h \circ k_1 = k_2 \circ f \). Therefore
\[ S_{h \circ k_1}(z) = S_{k_2 \circ f}(z) \]
for each \( z \in B(0, 1) \). With the same procedure as we have done for the identity (2.2) and the following equality,
\[ \frac{f(z)}{(1 + f(z))^2} = \frac{4c}{(c + 1)^2} \frac{z}{(1 + z)^2}, \]
we obtain
\[ f(z)(1 + z)^2 = \frac{4c}{(c + 1)^2} z(1 + f(z))^2. \]
By (2.4) and by \( f(0) = 0 \), we get
\[ \frac{f'(0)}{1 - (f(0))^2} = \frac{4c}{(c + 1)^2}. \]
Thus
\[ S_f(0) = -6 + 6 \left( \frac{4c}{(c + 1)^2} \right)^2. \]
Let \( g(\zeta) = \zeta + \sum_0^\infty b_j \zeta^{-j} \). Then
\[ b_1 = -\frac{1}{6} S_f(0) \]
\[ = 1 - \left( \frac{4c}{(c + 1)^2} \right)^2 \]
\[ = \left( \frac{c - 1}{c + 1} \right)^2 \left( 1 + \frac{4c}{(c + 1)^2} \right). \]
Therefore one can not replace the upper bound in (1.4) by anything less than \( \left( \frac{c - 1}{c + 1} \right)^2 \left( 1 + \frac{4c}{(c + 1)^2} \right) \).
REMARK 2.5. By Example 2.1, Example 2.3 and Remark 1.5 one can not replace the upper bound in (1.4) by anything less than \((\varepsilon^2 / c+1)^2\), \((\varepsilon^2 / c+1)^2(1 + \frac{4c}{(c+1)^2})\) and \(\frac{c-1}{c+1}\).

3. Quasisphere, the double ball property and spherical distortion

THEOREM 3.1. If a topological \(n\)-sphere \(S\) is a \(K\)-quasisphere in \(\mathbb{R}^n\), then \(S\) has spherical distortion \(c\), where \(c\) depends only on \(K\).

PROOF. Suppose that \(S\) is a \(K\)-quasisphere in \(\mathbb{R}^n\). Then there is a \(K\)-quasiconformal self mapping \(f\) of \(\mathbb{R}^n\) which maps \(S(0,1)\) onto \(S\) and \(f(\infty) = \infty\). Let \(w_0 = f(0)\) and let

\[
s = \max_{|z|=1} |f(z) - w_0|, \quad r = \min_{|z|=1} |f(z) - w_0|.
\]

Then \(\frac{s}{r} \leq \lambda(K)\), where \(\lambda(K) = \frac{1}{16} e^{\pi K} - \frac{1}{2} + O(e^{-\pi K})\) [6]. Thus \(S\) separates boundary spheres of \(A(w_0;r,cr)\), \(c = \lambda(K)\). If \(\phi\) is any Möbius transformation with \(\phi(S) \subset \mathbb{R}^n\), then \(g = \phi \circ f\) is also a \(K\)-quasiconformal mapping. Let \(\zeta_0 = \phi(w_0)\). We have an annulus \(A(\zeta_0; s, cs)\) whose boundary spheres are separated by \(\phi(S)\). Therefore \(S\) has spherical distortion \(c\), where \(c\) depends only on \(K\).

THEOREM 3.2. If \(S\) is a \(K\)-quasisphere in \(\mathbb{R}^n\), then \(S\) has the double ball property with constant \(b\), where \(b\) depends only on \(K\).

However the proof is similar to that of Lemma 1.7 in [4], but for the completeness we give the proof.

PROOF. Fix \(z_0 \in S\) and \(0 < r \leq \text{diam}(S)\). By hypothesis, there exists a \(K\)-quasiconformal self mapping \(f\) of \(\mathbb{R}^n\) which maps \(S\) onto an \(n\)-sphere \(S'\). By composing \(f\) with an auxiliary Möbius transformation we may further assume that \(f(\infty) = \infty\) and hence

\[f(\text{int}(S)) = \text{int}(S').\]
Let $w_0 = f(z_0)$, $B' = f(B(z_0, r))$ and let $w_t$, $w_e$ and $t_t$, $t_e$ denote the centers and radii of the largest balls in $B' \cap \text{int}(S')$, $B' \cap \text{ext}(S')$ which are tangent to $S'$ at $w_0$, respectively. Next set
\[ z_t = g(w_t), \quad z_e = g(w_e), \]
where $g = f^{-1}$, and let
\[ s_t = \max_{|w - w_t| = t_t} |g(w) - g(w_t)|, \quad r_t = \min_{|w - w_t| = t_t} |g(w) - g(w_t)|, \]
\[ s_e = \max_{|w - w_e| = t_e} |g(w) - g(w_e)|, \quad r_e = \min_{|w - w_e| = t_e} |g(w) - g(w_e)|. \]
Then by [6, Theorem 9.3] we have
\[ (3.3) \quad s_t \leq \lambda(K) r_t, \quad s_e \leq \lambda(K) r_e, \]
where $\lambda(K)$ is an increasing function of $K$ with $\lambda(1) = 1$. Finally let $B_t = B(z_t, r_t)$ and $B_e = B(z_e, r_e)$. Then by (3.3)
\[ \lambda(K) \text{diam}(B_t) \geq 2 s_t \geq r, \quad B_t \subset \text{int}(S) \cap B(z_0, r), \]
\[ \lambda(K) \text{diam}(B_e) \geq 2 s_e \geq r, \quad B_e \subset \text{ext}(S) \cap B(z_0, r), \]
and hence $B_t$ and $B_e$ satisfy (1.6) with $b = \lambda(K)$.

4. John domains and the double ball property

**Lemma 4.1.** [8] Suppose that $D \subset \mathbb{R}^n$ is an $(\alpha, \beta)$-John domain. If $0 < t \leq \alpha$ and $z_0 \in \partial D$, then
\[ d(z, \partial D) \geq \frac{\alpha}{\beta} t \]
for some $z \in S(z_0, t) \cap D$. 
Theorem 4.2. If a topological $n$-sphere $S$ in $\mathbb{R}^n$ is the boundary of a $(\alpha, \beta)$-John domain $D$, then there exists a constant $b$, $1 \leq b < \infty$, and for each $w \in S$ and $0 < r \leq \text{diam}(S)$, there exists an open disk $B$ with

$$B \subset \text{int}(S), \quad B \subset B(w, r), \quad b \text{diam}(B) \geq r,$$

where $b$ depends only on $\alpha$ and $\beta$.

However the proof is similar to that of Lemma 1.8 in [4], but for the completeness we give the proof.

Proof. Let $z_0 \in S$. First we consider the case $0 < r \leq \alpha \leq \text{diam}(S)$. Then by Lemma 4.1 with $t = \frac{r}{2}$,

$$d(z, S) \geq \frac{\alpha}{\beta} \cdot \frac{r}{2}$$

for some $z \in S(z_0, \frac{r}{2}) \cap D$. Hence there exists an open disk $B = B(z, \frac{2\beta}{\alpha}r)$ such that $B \subset D \cap B(z_0, r)$ and

$$\frac{2\beta}{\alpha} \text{diam}(B) \geq r.$$

Secondly if $0 < \alpha \leq r \leq \text{diam}(S)$, then by what we proved above, we can choose an open disk $B_\alpha$ for $\alpha$ such that $B_\alpha \subset D \cap B(z_0, \alpha)$ and $\frac{2\beta}{\alpha} \text{diam}(B_\alpha) \geq \alpha$. Thus

$$B_\alpha \subset D \cap B(z_0, r), \quad \text{diam}(S) \frac{2\beta}{\alpha} \text{diam}(B_\alpha) \geq r\alpha.$$

Since $\text{diam}(S) \leq 2\beta$, we have

$$\frac{(2\beta)^2}{\alpha} \text{diam} B_\alpha \geq r.$$

Therefore by (4.4) and (4.5), we obtain (4.3) with $b = \frac{(2\beta)^2}{\alpha}$. 
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