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THE JACOBSON RADICAL OF THE ENDOMORPHISM RING, THE JACOBSON RADICAL, AND THE SOCLE OF AN ENDO-FLAT MODULE

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ABSTRACT. For any $S-$*flat* module ${}_RM$(which will be called *endo-flat*) with a commutative ring R with identity, where S is the endomorphism ring ${}_RM$, the fact that every epimorphism is an automorphism has been proved and the Jacobson Radical $Rad(S)$ of S is described as follows; $Rad(S) = \{ f \in S \mid Imf = Mf$ is *small* in $M \} = \{ f \in S \mid Imf \leq Rad(M) \}$. Additionally for any *quasi-injective endo-flat* module ${}_RM$, the fact that every monomorphism is an automorphism has been proved and the Jacobson Radical $Rad(S)$ for any *quasi-injective endo-flat* module has been studied too. Also some equivalent conditions for the semi-primitivity of any *faithful endo-flat* module ${}_RM$ with the *open* Jacobson Radical $Rad(M)$ and those for the semi-simplicity of of any *faithful endo-flat quasi-injective* module ${}_RM$ with the *closed* Socle $Soc(M)$ have been studied.

1. Introduction

Throughout this research, the ring R is assumed to be a commutative ring with an identity. In this paper, the author investigates the tools

$$I^L = Hom_R(M, L) = \{f \in End_R(M) \mid Imf \leq L \}$$
and
$$I_L = \{ f \in End_R(M) \mid L \leq \ker f \}$$

for each submodule ${}_RL \leq {}_RM$ of a left R–module ${}_RM$ in order to find the relationships between the submodules, the Jacobson Radical, the

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Socle of $_RM$, the ideals, and the Jacobson Radical of the endomorphism ring $End_R(M)$(briefly, denoted by S) on which $_RM$ is an $S-$*flat* module.

We will call the sum of images of endomorphisms which are elements of a left(or right, or two-sided) ideal J of the ring S the image of J, simply denoted by $ImJ = \sum_{f \in J} Imf$ and call the intersection of the kernels of endomorphisms in a left(or right, or two-sided) ideal J the kernel of J, denoted by $kerJ = \cap_{g \in J} \ker g$ which are studied in the section §2.

DEFINITION 1.1. A left $R-$module $_RM$ is said to be *endo-flat*(or $S-$*flat*) if for any left ideal J of S, we always have a $Z-$isomorphism

$$\mu_J : M \otimes_S J \rightarrow MJ$$

defined by $(m \otimes j)\mu_J = mj$ for all $m \in M$ and for all $j \in J$, and we have the commutative diagram below:

$$\begin{array}{ccc} M \otimes_S J & \xrightarrow{1_M \otimes \iota} & M \otimes_S S \\ \mu_J \downarrow & & \downarrow \mu \\ MJ & \xrightarrow{\hookrightarrow} & MS = M \end{array}$$

where $\hookrightarrow$ denotes the inclusion mapping.

For a commutative ring R, the abelian group $_RM \otimes S$ is an $R-$module and for each element r of the ring R, let $\rho(r)$ defined by $m\rho(r) = rm$ for all $m \in M$ denote the left scalar multiplication by r.

For any left $R-$module $_RM$ and for any left $S-$module $_SN$, the tensor product $M \otimes_S N$ is the quotient abelian group $Z^{(M \times N)}/K$ modulo the generated subgroup $K = \langle (m + m', n) - (m, n) - (m', n), (m, n + n') - (m, n) - (m, n'), (ms, n) - (m, sn) \mid m, m' \in M, n, n' \in N, \text{ and } s \in S \rangle$ where $Z^{(M \times S)}$ is the free abelian group of the basis $M \times S$.

REMARK 1.2. In this paper, we are concerning only the *endo-flat* modules, in order to avoid symbol troubles with the well-definedness of tensor product $M \otimes B$ of M and any subset B of S, $\underline{M \otimes_S B} = \langle m \otimes b \mid m \in M,\ b \in B \rangle$(is not always a tensor product $M \otimes_S B$ of M and B), just will denote the image of $M \times B$ under $\otimes_S(1_M \otimes \iota)$ where $1_M \otimes \iota : M \otimes_S SB \rightarrow M \otimes_S S$ is the tensor product of the identity mapping 1_M on M and the inclusion mapping $\iota : SB \hookrightarrow S$, for any subset $B \subseteq S$.

LEMMA 1.3 (p. 522, [7]). *A module $_RU$ is flat over its endomorphism ring if and only if it generates the kernel of each homomorphism*

$$d : U^{(n)} \to U \ \ (n = 1, 2, 3, \dots),$$

where $U^{(n)}$ denotes the direct product of n copies-U.

Hyman Bass had proved on page 474 in [2] that for a nonzero *unitary projective* module M, $MRad(R) \neq M$.

And Roger Ware [5] had shown that, for any *projective* module with the condition that $_RM$ is assumed to be a unitary module, $MRad(S) \neq Rad(M)$.

He also showed the relations between the Jacobson radical $Rad(S)$ of the endomorphism ring $S = End_R(M)$, the Jacobson radical $Rad(M)$ of a *projective* module $_RM$, and the *small* submodules of M. Like if $Rad(M)$ is *small*, then

$$Rad(S) \ = \ \{ \ f \in S \mid Imf \text{ is a } small \text{ submodule of } M \ \}.$$

2. Epimorphisms, Monomorphisms, and Automorphisms

In this paper, we use the composition of mappings in the direction of arrows:

$$fg \ : A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

because $S = End_R(M)$ acts on the right side of an R-module $_RM_S$.

THEOREM 2.1. *For an endo-flat module $_RM$ with a commutative ring R with an identity, every epimorphism is an automorphism.*

PROOF. It suffices to show that for an *endo-flat* module $_RM$ and for any epimorphism

For $g : {_RM} \to {_RM}$, there is an inverse $h \in S$ of g such that $hg = 1_M$ and $gh = 1_M$, where 1_M denotes the identity mapping on $_RM$. For an S-balanced mapping

$$\beta : M \times Sg^2 \to M \otimes Sg$$

defined by $(m, sg^2)\beta = m \otimes sg$ for every $m \in M$ and every $s \in S$, there is a unique S-homomorphism (and hence an R-homomorphism)

$$\tau : M \otimes Sg^2 \to M \otimes Sg \text{ such that } \otimes \tau = \beta.$$

Consider the following diagram :

$$\begin{array}{ccccccc}
 & & & & & & M \times Sg^2 \\
 & & & & & & \Big\downarrow \beta \\
M = Mg & \xrightarrow{g} & M = Mg^2 & \xrightarrow{{\mu_{Sg^2}}^{-1}} & M \otimes_S Sg^2 & \xrightarrow{\tau} & M \otimes_S Sg \\
 & & & & & & \Big\downarrow \mu_{Sg} \\
 & & & & & & Mg = M \\
 & & & & & & \Big\downarrow g \\
 & & & & & & Mg^2 = M .
\end{array}$$

Then $h = {\mu_{Sg^2}}^{-1}\tau\mu_{Sg}$ is the required inverse endomorphism of g such that $hg = 1_{Mg^2} = 1_M$ and $gh = 1_{Mg} = 1_M$. □

The following definition is well-known. A left module ${}_RM$ is said to be *quasi-injective* provided for any ${}_RK$, for any submodule ${}_RN \leq {}_RM$ and any homomorphism $f : N \to K$, there is an R-homomorphism $h : M \to K$ such that h is an extension of f(p. 22, [3]).

THEOREM 2.2. *If an endo-flat module ${}_RM$ is quasi-injective, then every monomorphism is an automorphism.*

PROOF. It follows easily from Theorem 2.1 and the definition of the quasi-injective module. □

REMARK 2.3. The *endo-flat*ness of the hypothesis of Theorem 2.1 is essential. For a prime number p, a scalar multiplication $\rho(p)$ on the non-*endo-flat* Z-module ${}_ZZ(p^\infty)$ is an epimorphism but not an automorphism.

The *endo-flat*ness and the *quasi-injective*ness of the hypotheses of Theorem 2.2 are essential.

For any prime number p, ${}_Z Z(p^\infty)$ is *quasi-injective* but not *endo-flat*. And the monomorphism $\rho(q)$ with $0 \leq q \leq p$, $(q,p) = 1$ is not an automorphism. A non-unitary module, the set of even integers ${}_Z E = {}_Z\{\ 2a \mid a \in Z\ \}$ is not *quasi-injective* but *endo-flat* on which the monomorphism $\rho(2)$ is not an automorphism.

On the lattice of all submodules of ${}_R M$, we define an operation "^{o}" by

$$A^o = \bigcap_\alpha \{L_\alpha \leq {}_R M \mid I^{L_\alpha} = I^A\},$$

for each submodule $A \leq {}_R M$ with which $I^A = \{\ f \in S \mid\ Imf \leq A\ \}$. Then we will say that a submodule $A \leq {}_R M$ is *open* if $A = A^o$.

Also we define an operation "-" on the lattice of all submodules of M by

$$\overline{B} = \sum_\alpha \{\ N_\alpha \leq {}_R M \mid I_{N_\alpha} = I_B\ \},$$

for each submodule $B \leq M$ with which $I_B = \{\ f \in S \mid\ B \leq \ker f\ \}$. Then we will say that a submodule $B \leq {}_R M$ is *closed* if $B = \overline{B}$.

The proofs of the following Propositions 2.4 and 2.5 are established easily from the definitions of the operators ^{o} and $^-$.

PROPOSITION 2.4. *For any endo-flat module ${}_R M$, we have the following properties:*

(1) *For open submodules $A_\alpha \leq M$, $(\bigcap_\alpha A_\alpha)^o = \bigcap_\alpha (A_\alpha)^o = \bigcap_\alpha A_\alpha$ for any index α in an indexed set.*

(2) *For open submodules $A_\alpha \leq {}_R M$,*

$$\underline{M \otimes_S I^{\sum_\alpha A_\alpha}} = \sum_\alpha \underline{(M \otimes_S I^{A_\alpha})} \ \text{ in } M \otimes_S S,$$

and thus $$(\sum_\alpha A_\alpha)^o = \sum_\alpha (A_\alpha)^o = \sum_\alpha A_\alpha.$$

Hence $$MI^{\sum_\alpha A_\alpha} = \sum_\alpha (MI^{A_\alpha}).$$

(3) *For every endomorphism $f : {}_R M \to {}_R M$, the image $Imf = Mf$ is always an open submodule of ${}_R M$.*

PROPOSITION 2.5. *For an endo-flat module* $_RM$*, we have the following properties:*

(1) *For any closed submodules* $A_\alpha \leq M$,

$$\overline{\sum_\alpha A_\alpha} = \sum_\alpha \overline{A_\alpha} = \sum_\alpha A_\alpha \quad \text{and} \quad \ker(\bigcap_\beta I_{B_\beta}) = \ker I_{\left(\sum_\beta B_\beta\right)}.$$

(2) *For every closed submodules* $B_\beta \leq {}_RM$,

$$\bigcap_\beta \ker I_{B_\beta} = \ker I_{\cap_\beta B_\beta} \text{ in } {}_RM$$

$$\text{and thus} \qquad \overline{\bigcap_\beta B_\beta} = \bigcap_\beta \overline{B_\beta} = \bigcap_\beta B_\beta.$$

(3) *For every endomorphism* $f : {}_RM \to {}_RM$*, the kernel* $\ker f$ *of* f *is always a closed submodule of* $_RM$*.*

REMARK 2.6. The items (2)'s of the Propositions 2.4 and 2.5 don't hold for the general left R-modules without *endo-flat*ness.

3. The Jacobson Radicals and the Socle

In this section, we discuss maximal submodules of an *endo-flat* module $_RM$ and maximal left(or right, or two-sided) ideals of the endomorphism ring $End_R(M)$. We also discuss *small*(*superfluous*), *large*(*essential*) submodules of an *endo-flat* module $_RM$ and *small* (*superfluous*), *large* (*essential*) left(or right, or two-sided) ideals of the endomorphism ring $End_R(M)$.

We firstly present some definitions from (p. 118, p. 120, [8]). The Jacobson Radical of a module M is defined by

$$Rad(M) = \bigcap_\alpha L_\alpha, \text{ for all maximal submodules } L_\alpha \text{ of } M.$$

And the Socle of a module M is defined by

$$Soc(M) = \sum_\alpha H_\alpha, \text{ for all minimal submodules } H_\alpha \text{ of } M.$$

On pages 57, 58 in [8] the Jacobson Radical of a ring T has been studied.

PROPOSITION 3.1. *For an endo-flat module ${}_RM$, we have the following properties:*

(1) *If a submodule $N \ll M$ is small(superfluous) in M, then the left ideal*

$$I^N \text{ is small in } End_R(M).$$

(2) *For the Jacobson Radical $Rad(S)$ of the endomorphism ring $S = End_R(M)$,*

$$Rad(S) \ = \ \{ \ f \in S \mid \ Imf \ \text{ is small(superfluous) in } M \ \}.$$

(3) *In addition, if ${}_RM$ is quasi-injective and if a submodule $N \leq M$ is large(essential) in M, then the right ideal I_N is small (superfluous) in $End_R(M)$.*

(4) *If ${}_RM$ is quasi-injective, then for the Jacobson Radical $Rad(S)$ of the endomorphism ring $S = End_R(M)$, we have that*

$$\begin{aligned} Rad(S) \ &= \ \{ \ f \in S \mid \ Imf \text{ is small in } M \ \} \\ &= \ \{ \ f \in S \mid \ \ker f \text{ is large in } \ M \ \}. \end{aligned}$$

PROOF. (1): Let $J \trianglelefteq_l S$ be any left ideal of $S = End_R(M)$ such that $I^N + J = S$. Then $1_M = f + j$ for some $f \in I^N$ and $j \in J$. Then $j = 1_M - f$ is an epimorphism from the *small* submodule $Imf \leq N \ll M$. Hence by Theorem 2.1 j is an automorphism implying that $J = S$. Thus I^N is a left *small* ideal of S.

(2): Now suppose that f is an endomorphism such that $f \in Rad(S)$ and if a submodule $K \leq M$ is such that $Imf + K = M$, then by 1) we have equalities

$$\underline{M \otimes_S I^K} \ + \ \underline{M \otimes_S I^{Imf}} \ = \ \underline{M \otimes_S I^{(K+Imf)}} \ = \ \underline{M \otimes_S I^M} = M \otimes_S S$$

having image $K^o + Imf = M$. Hence there is an endomorphism $j \in I^K$ such that $j + f = 1_M$. Then $j = 1_M - f$ is a unit from the *quasi-regular* endomorphism $f \in Rad(S)$. Hence $I^K = S$ follows. Therefore $M = K^o \leq K \leq M$ implies that $Imf \ll M$ is a *small* submodule of M.

It remains to show that $\{ \ f \in S \mid Imf \ll M \ \} \subseteq Rad(S)$. For any endomorphism $f : M \to M$ with the *small* submodule Imf, we have a *small* left ideal I^{Imf} containing f is contained in the largest *small* left ideal $Rad(S)$. Hence $f \in Rad(S)$ follows.

(3) follows from the dual proof of (1) with Theorem 2.2.

(4): By the Theorem 2.16 (p49, [9]), for a *quasi-injective* module ${}_RM$, $Rad(S) = \{ f \in S \mid \ker f \text{ is } large \text{ in } M \}$ has been proved. And from (2) it follows immediately that

$$\begin{aligned} Rad(S) &= \{ f \in S \mid Imf \text{ is } small \text{ in } M \} \\ &= \{ f \in S \mid \ker f \text{ is } large \text{ in } M \}. \end{aligned}$$ □

THEOREM 3.2. *For an endo-flat module ${}_RM$, we have the following:*

(1) *If $Rad(M) = \sum_\alpha N_\alpha$, for all small submodules N_α, then*

$$Rad(S) = I^{(Rad(M))^\circ} = I^{Rad(M)}.$$

(2) *If ${}_RM$ is quasi-injective and if $Soc(M) = \bigcap_\alpha K_\alpha$, for all large submodules K_α, then*

$$Rad(S) = I_{\overline{Soc(M)}} = I_{Soc(M)}.$$

PROOF. From the Proposition 3.1 and the largest *small* ideal $Rad(S)$, it follows immediately. □

REMARK 3.3. From the above Proposition 3.1 and Theorem 3.2, if ${}_RM$ is *endo-flat* and *quasi-injective*, then it follows that

$$Rad(S) = \sum_\alpha I^{N_\alpha} = \bigcap_\beta I_{K_\beta} = I^{Rad(M)} = I_{Soc(M)},$$

where $Rad(M) = \sum_\alpha N_\alpha$ with *small* submodules N_α of M and $Soc(M) = \bigcap_\beta K_\beta$ with *large* submodules K_β of M.

Recall that (15.18 Corollary, p. 171 [11]) for any left $R-$module ${}_RM$, and for any right $S-$module M_S, we always have $Rad(R)M \leq Rad(M)$ and $MRad(S) \leq Rad(M)$.

THEOREM 3.4. *For an endo-flat module $_RM$, we have the following:*

(1) *If $Rad(M)$ is open, then S is semi-primitive if and only if M is semi-primitive.*
(2) *If $_RM$ is quasi-injective module and if $Soc(M)$ is closed, then S is semi-primitive if and only if M is semi-simple.*

PROOF. By Theorem 3.2, the items (1) and (2) are proved easily. □

When two $R-$modules $_RA$, $_RB$ are $R-$isomorphic, then we will write this fact; $A \simeq B$ briefly. And the symbol $C \equiv D$ will indicate the identification of C with D.

REMARK 3.5. Both the *endo-flat*ness of R-module $_RM$ and the *open*ness of the Jacobson radical $Rad(M)$ are necessary for (1) of Theorem 3.4. With only the hypothesis of the it endo-flatness of $_RM$ we still have the result $(Rad(M))^o = MRad(S)$.

For example, take the commutative ring $R = Z[x]$, $M = {}_{Z[x]}Z[x]$ the polynomial ring of indeterminate x in which $Rad(M) = xZ[x]$ is <u>*not*</u> an *open* submodule of M.

For this *small* submodule $Rad(M) = xZ[x]$, we have that $0 = I^{Rad(M)} = I^{xZ[x]} = Rad(S)$. Thus $S = End(M) \simeq \rho(Z) \equiv Z$ is *semi-primitive* but M is not *semi-primitive* with the non-zero Jacobson Radical $Rad(M) = \bigcap_{p \in Z}(xZ[x]+pZ) = xZ[x] \neq 0$ and with $MRad(S) = (Rad(M))^o = 0$.

However if $R = Z$ and $M = {}_ZZ[x] \simeq \oplus Z$ without multiplication with $S \simeq End(\oplus Z)$, then $_ZZ[x]$ is not *endo-flat* however $Rad(M) = xZ[x]$ is *open closed.* It follows that

$$(xZ[x])^o \;=\; Rad(M) \neq\; MRad(S) \;=\; 0$$

saying that the non-*endo-flat* module $_ZZ[x]$ is not *semi-primitive* but the endomorphism ring S (i.e., $Rad(S) = 0$) is *semi-primitive.*

For a *faithful* module $_RM$, the ring R is identified with the ring of left scalar multiplications. In other words,

$$R \;\equiv\; \rho(R) \;=\; \{\, \rho(r) \mid r \in R \,\} \;\subseteq\; End_R(M).$$

Moreover, $J \cap \rho(R) \equiv J \cap R$ and $\rho^{-1}(J \cap \rho(R)) \equiv J \cap R$, for any left ideal $J \trianglelefteq_l S$ of S.

THEOREM 3.6. *For a faithful R-module $_RM$, if $_RM$ is endo-flat, then $_RM$ is R-flat.*

PROOF. It is elementary. □

REMARK 3.7. However the converse of the Theorem 3.6 doesn't hold. For a counter-example, let's take $_{Z_4}Z_4 \oplus Z_4$ which is *R-flat* but not *endo-flat*. Because Theorem 2.1 implies that $_{Z_4}Z_4 \oplus Z_4$ is not *endo-flat* by the fact that $\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix}$ is an epimorphism in S but has no left inverse in S.

PROPOSITION 3.8. *For a faithful endo-flat module $_RM$, we have the following:*

(1) *If $J \trianglelefteq_l S$ is any left ideal of S, we have an $R-$isomorphism*

$$\eta : M \otimes_R J \to M \otimes_S J \simeq \underline{M \otimes_S J}.$$

(2) *For any left ideal $J \trianglelefteq_l S$ of S, there is an R-isomorphism*

$$\phi : M \otimes_R (J \cap \rho(R)) \equiv M \otimes_R (J \cap R)) \to \underline{(M \otimes_S J)} \cap \underline{(M \otimes_S R)}.$$

(3) *For any left ideal $J \trianglelefteq_l S$ of $S, (J \cap R)M \simeq MJ$.*

PROOF. Since $_RM$ is *faithful*, we can identify $\rho(R)$ with a commutative ring R.

(1): For an $R-$balanced mapping $\otimes_S : M \times J \to M \otimes_S J$, there is a unique $Z-$homomorphism, in fact, an R-homomorphism $\eta : M \otimes_R J \to M \otimes_S J$ such that $\otimes_R \eta = \otimes_S$. Then it follows that η is clearly an epimorphism. Define $\beta : MJ \to M \otimes_R J$ by $(\sum_1^n m_j f_j)\beta = \sum_1^n m_j \otimes_R f_j$ for each $\sum_1^n m_j f_j \in MJ$. Then we obtain an $R-$balanced mapping $\otimes_R \eta \mu_J \beta$ and $\otimes_R\ \eta\ \mu_J\ \beta = \otimes_R 1_{M \otimes_R J}$ follows from the unique existence of $1_{M \otimes_R J}$, where $\mu_J : M \otimes_S J \to MJ$ is the restriction of the $R-$isomorphism $\mu : M \otimes_S S \to M$.

$$\begin{array}{ccc} M \times J & = & M \times J \\ \otimes_R \downarrow & & \downarrow \otimes_S \\ M \otimes_R J \equiv M \otimes_{\rho(R)} J & \xrightarrow{\ \eta\ } & M \otimes_S J \\ \beta \uparrow & & \downarrow \mu_J \\ MJ & = & MJ \end{array}$$

From the mapping $\otimes_R$ and $\eta\mu_J\beta =$ the identity mapping on $M \otimes_R J$, it follows that η is a monic mapping. Hence η is an $R-$isomorphism.

(2): From $M \times (J \cap R) = (M \times J) \cap (M \times R)$, (2) follows easily.

(3) follows from (2). □

LEMMA 3.9. *For a faithful endo-flat module* $_RM$, $\rho(R) \equiv R$ *is a large(essential) subring of* $End_R(M)$. *In other words,* $J \cap \rho(R) \equiv J \cap R \neq 0$ *for every non-zero left ideal* J *of* S.

PROOF. From the (3) of the above Proposition 3.8, it follows immediately. □

PROPOSITION 3.10. *For a faithful endo-flat module* $_RM$,

$$Rad(S) \cap R = Rad(R)$$

and

$$Rad(R)M \simeq MRad(S) = (Rad(M))^o = MI^{Rad(M)}.$$

PROOF. It follows easily that

$$Rad(S) \cap \rho(R) \equiv Rad(S) \cap R = Rad(R).$$

By the following isomorphisms (by Proposition 3.8) :

$$\begin{aligned} MRad(R) &\simeq M \otimes_R Rad(R) \simeq M \otimes_R (Rad(S) \cap R) \\ &\simeq \underline{(M \otimes_S Rad(S))} \cap \underline{(M \otimes_S R)} \\ &\simeq \underline{(M \otimes_S Rad(S))} \cap (M \otimes_S SR) \\ &\simeq M \otimes_S Rad(S) \simeq MRad(S) = (Rad(M))^o, \end{aligned}$$

the proof has been completed. □

REMARK 3.11. According to the Proposition 2.7 of Hyman Bass [2], for any ring R with an identity and for any unitary module $_RM$, $Rad(R)M \neq M$ and so $MRad(S) \neq M$ follows.

Thus for any unitary module $_RM$ which is *endo-flat*, we also have that $MRad(S) \neq M$ which implies that there is at least one maximal submodule in $_RM$.

THEOREM 3.12. *For a unitary faithful endo-flat module ${}_RM$, we have the following:*

(1) *${}_RM$ has at least one maximal submodule of ${}_RM$.*
(2) *If ${}_RM$ is quasi-injective, then ${}_RM$ has at least one minimal submodule of ${}_RM$.*

PROOF. From the above Remark 3.11, (1) and (2) follow easily. □

COROLLARY 3.13. *For a faithful endo-flat module ${}_RM$ with the open $Rad(M)$, the following are equivalent:*

(1) *R is semi-primitive.*
(2) *M is semi-primitive.*
(3) *S is semi-primitive.*

PROOF. Using the equations

$$Rad(S) = I^{(Rad(M))^o},\ (Rad(M))^o = MRad(S),$$

and $Rad(R)M \simeq MRad(S)$, the proof is completed easily. □

COROLLARY 3.14. *For a faithful endo-flat quasi-injective module ${}_RM$, if $Soc(M)$ is closed, then the following are equivalent:*

(1) *R is semi-primitive.*
(2) *M is semi-primitive.*
(3) *S is semi-primitive.*

PROOF. For the *closed* submodule $Soc(M)$, $Rad(S) = I_{Soc(M)}$ and $Rad(R) = Rad(S) \cap R$. And we have that $\ker(Rad(S)) = \overline{Soc(M)} = Soc(M)$ from Theorem 3.2 and Proposition 3.8.

(1) $\Longleftrightarrow$ (3): This follows immediately from Proposition 3.10.
(2) $\Longleftrightarrow$ (3): This follows immediately from (2) of Theorem 3.1. □

The following definition is from [1] which is equivalent to the definition on page 94([11]).

Let $(M_\alpha)_{\alpha \in A}$ be an indexed set of left $R-$modules and let $\prod_\alpha M_\alpha$ be the direct product of $(M_\alpha)_{\alpha \in A}$. Then M is a subdirect product of

$(M_\alpha)_{\alpha\in A}$ in case there is a monomorphism $k : M \to \prod_\alpha M_\alpha$ such that $k\pi_\alpha : M \to M_\alpha$ is an epimorphism for each $\alpha \in A$.

Here is a result (which will have a main role) of the *subdirect product* of $(R/J_\alpha)_\alpha$ and $(S/J'_\alpha)_\alpha$ for ideals J_α of R or for left ideals J'_α of S.

REMARK 3.15. Let $(J_\alpha)_{\alpha\in A}$ be an indexed set of ideals of a ring R. Then the image of the natural map $\phi : R \to \prod_A (R/J_\alpha)$ defined coordinately by $(r)\phi\pi_\alpha = r + J_\alpha$, for every α is a subdirect product of $(R/J_\alpha)_\alpha$ with $\ker\phi = \cap_A J_\alpha$. $M/\ker J$ is a subdirect product of $(M/\ker f_\alpha)$ where $\{f_\alpha \in S\} = J$. Since we have an R−monomorphism $k : M/\ker J \to \prod_\alpha M/\ker f_\alpha$ defined by $(m + \ker J)k = (m + \ker f_\alpha)_\alpha$ for every $m \in M$, $M/\ker J$ is a subdirect product of $(M/\ker f_\alpha)$ where $\{f_\alpha\}_\alpha = J$.

According to Proposition 7 (p. 58, [8]) a ring R is a *semi-primitive* ring if and only if it is a subdirect product of *primitive*(or *simple*) rings.

THEOREM 3.16. *For a faithful endo-flat ${}_RM$ with the open $Rad(M)$, the following are equivalent:*

(1) *R is semi-primitive.*
(2) *R is a subdirect product of primitive rings.*
(3) *R is a subdirect product of simple rings.*
(4) *${}_RM$ is semi-primitive.*
(5) *${}_RM$ is a subdirect product of simple modules.*
(6) *S is semi-primitive.*
(7) *S is a subdirect product of primitive rings.*
(8) *S is a subdirect product of simple rings.*

PROOF. Since for any commutative ring T, the Jacobson Radical of T is calculated below:

$$\begin{aligned} Rad(T) &= \bigcap I_\alpha, \text{ where } I_\alpha \text{ is a maximal ideal of } T \\ &= \bigcap I_\alpha, \text{ where } I_\alpha \text{ is a \textit{primitive} ideal of } T. \end{aligned}$$

(1) $\Longleftrightarrow$ (2), (1) $\Longleftrightarrow$ (3), (6) $\Longleftrightarrow$ (7), (6) $\Longleftrightarrow$ (8), and (4) $\Longleftrightarrow$ (5) follow from Proposition 7 on p. 58 ([8]).

(1) $\Longleftrightarrow$ (4) and (4) $\Longleftrightarrow$ (6) are established by Corollary 3.13. □

THEOREM 3.17. *If a faithful endo-flat $_RM$ is quasi-injective with the closed $Soc(M)$ of M, then the following are equivalent:*

(1) *R is semi-primitive.*
(2) *R is asubdirect product of simple rings.*
(3) *R is a subdirect product of primitve rings.*
(4) *$_RM$ is semi-simple.*
(5) *$_RM$ is a direct sum of simple modules.*
(6) *S is semi-primitive.*
(7) *S is a subdirect product of simple rings.*
(8) *S is a subdirect product of primitive rings.*

PROOF. (1) $\Longleftrightarrow$ (2), (1) $\Longleftrightarrow$ (3), (6) $\Longleftrightarrow$ (7), (6) $\Longleftrightarrow$ (8), and (4) $\Longleftrightarrow$ 5) follow from Proposition 7 on p. 58 ([8]) and 15.14 Proposition on p169 ([11]).

(1) $\Longleftrightarrow$ (4) and (4) $\Longleftrightarrow$ (6) are established by Corollary 3.14. □

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