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ON THE CENTROID OF THE PRIME GAMMA RINGS

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ABSTRACT. We define and study the extended centroid of a prime Γ-ring.

1. Introduction

N. Nobusawa [6] introduced the notion of a Γ-ring, more general than a ring. W. E. Barnes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. W. E. Barnes [1], J. Luh [3] and S. Kyuno [2] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. In this paper, we define and study the extended centroid of a prime Γ-ring.

2. Preliminaries

Let M and Γ be two abelian groups. If for all $x, y, z \in M$ and all α, $\beta \in \Gamma$ the conditions

(i) $x\alpha y \in M$,

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)z = x\alpha z + x\beta z$, $x\alpha(y + z) = x\alpha y + x\alpha z$,

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied, then we call M a *Γ-ring*. By a *right* (resp. *left*) *ideal* of a Γ-ring M we mean an additive subgroup U of M such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If U is both a right and a left ideal, then we say that U is an *ideal* of M. For each a of a Γ-ring M the smallest right ideal containing a is called the *principal right ideal generated* by a and

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is denoted by $< a >_r$. Similarly we define $< a >_l$ (resp. $< a >$), the *principal left* (resp. *two sided*) *ideal generated* by a. An ideal P of a Γ-ring M is said to be *prime* if for any ideals A and B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal Q of a Γ-ring M is said to be *semi-prime* if for any ideal U of M, $U\Gamma U \subseteq Q$ implies $U \subseteq Q$. A Γ-ring M is said to be *prime* (resp. *semi-prime*) if the zero ideal is prime (resp. semi-prime).

THEOREM 2.1. ([2, Theorem 4]) *If M is a Γ-ring, the following conditions are equivalent:*

(i) *M is a prime Γ-ring.*

(ii) *If $a, b \in M$ and $a\Gamma M\Gamma b = (0)$, then $a = 0$ or $b = 0$.*

(iii) *If $< a >$ and $< b >$ are principal ideals in M such that $< a > \Gamma < b >= (0)$, then $a = 0$ or $b = 0$.*

(iv) *If A and B are right ideals in M such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.*

(v) *If A and B are left ideals in M such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.*

3. Centroids

Let M be a Γ-ring. A mapping $D(\cdot,\cdot) : M \times M \rightarrow M$ is said to be *symmetric bi-additive* if it is additive in both arguments and $D(x, y) = D(y, x)$ for all $x, y \in M$. By the *trace* of $D(\cdot,\cdot)$ we mean a map $d : M \rightarrow M$ defined by $d(x) = D(x, x)$ for all $x \in M$. A symmetric bi-additive map is called a *symmetric bi-derivation* if $D(x\beta z, y) = D(x, y)\beta z + x\beta D(z, y)$ for all $x, y, z \in M$ and $\beta \in \Gamma$. Since a map $D(\cdot,\cdot)$ is symmetric bi-additive, the trace of $D(\cdot,\cdot)$ satisfies the relation $d(x+y) = d(x)+d(y)+2D(x, y)$ for all $x, y \in M$ and is an even function.

Let M be a prime Γ-ring such that $M\Gamma M \neq M$. Denote

$$\begin{aligned}\mathcal{M} := \{(U, f) \mid U(\neq 0) &\text{ is an ideal of } M \text{ and} \\ &f : U \rightarrow M \text{ is a right } M\text{-module homomorphism}\}.\end{aligned}$$

Define a relation $\sim$ on $\mathcal{M}$ by $(U, f) \sim (V, g) \iff \exists W(\neq 0) \subset U \cap V$ such that $f = g$ on W. Since M is a prime Γ-ring, it is possible to find a non-zero W and so "$\sim$" is an equivalence relation. This gives a chance for us to get a partition of $\mathcal{M}$. We then denote the equivalence class by $Cl(U,f) = \hat{f}$, where $\hat{f} := \{g : V \to M | (U,f) \sim (V,g)\}$, and denote by Q the set of all equivalence classes. Now we define an addition "+" on Q as follows:

$$\hat{f} + \hat{g} = Cl(U,f) + Cl(V,g) = Cl(U \cap V, f+g)$$

where $f+g : U \cap V \to M$ is a right M-module homomorphism. Assume that $(U_1, f_1) \sim (U_2, f_2)$ and $(V_1, g_1) \sim (V_2, g_2)$. Then $\exists W_1 (\neq 0) \subset U_1 \cap U_2$ such that $f_1 = f_2$; and $\exists W_2 (\neq 0) \subset V_1 \cap V_2$ such that $g_1 = g_2$. Taking $W = W_1 \cap W_2$. Then $W \neq 0$ and

$$W = W_1 \cap W_2 \subset (U_1 \cap U_2) \cap (V_1 \cap V_2) = (U_1 \cap V_1) \cap (U_2 \cap V_2).$$

For any $x \in W$, we have $(f_1 + g_1)(x) = f_1(x) + g_1(x) = f_2(x) + g_2(x) = (f_2+g_2)(x)$, and so $f_1+g_1 = f_2+g_2$ in W. Therefore $(U_1 \cap V_1, f_1+g_1) \sim (U_2 \cap V_2, f_2 + g_2)$, which means that the addition "+" is well-defined. Now we will prove that Q is an additive abelian group. Let $\hat{f} = Cl(U,f)$, $\hat{g} = Cl(V,g)$ and $\hat{h} = Cl(W,h)$ be elements of Q. Then

$$\begin{aligned}(\hat{f} + \hat{g}) + \hat{h} &= Cl(U \cap V, f+g) + Cl(W,h) \\ &= Cl((U \cap V) \cap W, (f+g)+h) \\ &= Cl(U \cap (V \cap W), f + (g+h)) \\ &= Cl(U,f) + Cl(V \cap W, g+h) \\ &= \hat{f} + (\hat{g} + \hat{h}).\end{aligned}$$

Taking $\hat{0} := Cl(M, 0)$ where $0 : M \to M$, $x \mapsto 0$, for all $x \in M$ we have $\hat{f} + \hat{0} = Cl(U,f) + Cl(M,0) = Cl(U \cap M, f+0) = Cl(U,f) = \hat{f}$, and similarly $\hat{0} + \hat{f} = \hat{f}$. Hence $\hat{0}$ is the additive identity in Q. For any element $\hat{f} = Cl(U,f)$ of Q, it is easy to show that $-\hat{f} = Cl(U, -f)$ is an additive inverse of $\hat{f} = Cl(U,f)$. Finally, for any elements $\hat{f} = Cl(U,f)$

and $\hat{g} = Cl(V, g)$ of Q, we have

$$\begin{aligned}\hat{f} + \hat{g} &= Cl(U, f) + Cl(V, g)\\ &= Cl(U \cap V, f + g)\\ &= Cl(V \cap U, g + f)\\ &= Cl(V, g) + Cl(U, f)\\ &= \hat{g} + \hat{f}.\end{aligned}$$

Therefore Q is an additive abelian group.

Since $M\Gamma M \neq M$ and since M is a prime Γ-ring, $M\Gamma M$ $(\neq 0)$ is an ideal of M. We can take the homomorphism $1_{M\Gamma} : M\Gamma M \to M$ as a unit M-module homomorphism. Note that $M\beta M \neq 0$ for all $0 \neq \beta \in \Gamma$ so that $1_{M\beta} : M\beta M \to M$ is non-zero M-module homomorphism. Denote

$$\mathcal{N} := \{(M\beta M, 1_{M\beta}) \mid 0 \neq \beta \in \Gamma\},$$

and define a relation "$\approx$" on $\mathcal{N}$ by $(M\beta M, 1_{M\beta}) \approx (M\gamma M, 1_{M\gamma}) \iff \exists W := M\alpha M (\neq 0) \subset M\beta M \cap M\gamma M$ such that $1_{M\beta} = 1_{M\gamma}$ on W. We can easily check that "$\approx$" is an equivalence relation on $\mathcal{N}$. Denote by $Cl(M\beta M, 1_{M\beta}) = \hat{\beta}$, the equivalence class containing $(M\beta M, 1_{M\beta})$ and by $\hat{\Gamma}$ the set of all equivalence classes of $\mathcal{N}$ with respect to $\approx$, that is,

$$\hat{\beta} := \{1_{M\gamma} : M\gamma M \to M \mid (M\beta M, 1_{M\beta}) \approx (M\gamma M, 1_{M\gamma})\}$$

and $\hat{\Gamma} := \{\hat{\beta} \mid 0 \neq \beta \in \Gamma\}$. Define an addition "+" on $\hat{\Gamma}$ as follows:

$$\begin{aligned}\hat{\beta} + \hat{\delta} &= Cl(M\beta M, 1_{M\beta}) + Cl(M\delta M, 1_{M\delta})\\ &= Cl(M\beta M \cap M\delta M, 1_{M\beta} + 1_{M\delta})\end{aligned}$$

for every $\beta(\neq 0)$, $\delta(\neq 0) \in \Gamma$. Then $(\hat{\Gamma}, +)$ is an abelian group. Now we define a mapping $(-,-,-) : Q \times \hat{\Gamma} \times Q \to Q$, $(\hat{f}, \hat{\beta}, \hat{g}) \mapsto \hat{f}\hat{\beta}\hat{g}$, as follows:

$$\begin{aligned}\hat{f}\hat{\beta}\hat{g} &= Cl(U, f)Cl(M\beta M, 1_{M\beta})Cl(V, g)\\ &= Cl(V\Gamma M\beta M\Gamma U, f1_{M\beta}g)\end{aligned}$$

where

$$V\Gamma M\beta M\Gamma U = \Big\{ \sum v_i\gamma_i m_i\beta n_i\alpha_i u_i \mid v_i \in V, u_i \in U, m_i, n_i \in M \text{ and } \alpha_i, \gamma_i \in \Gamma \Big\}$$

is an ideal of M and $f1_{M\beta}g : V\Gamma M\beta M\Gamma U \to M$ which is given by

$$f1_{M\beta}g\Big(\sum v_i\gamma_i m_i\beta n_i\alpha_i u_i\Big) = f\Big(\sum g(v_i)\gamma_i m_i\beta n_i\alpha_i u_i\Big)$$

is a right M-module homomorphism. Then it is routine to check that such mapping is well-defined. We will show that Q is a $\hat{\Gamma}$-ring with unity. Let $\hat{f}, \hat{g}, \hat{h} \in Q$ and $\hat{\beta}, \hat{\gamma} \in \hat{\Gamma}$, i.e., $\hat{f} = Cl(U, f)$, $\hat{g} = Cl(V, g)$, $\hat{h} = Cl(W, h)$, $\hat{\beta} = Cl(M\beta M, 1_{M\beta})$ and $\hat{\gamma} = Cl(M\gamma M, 1_{M\gamma})$. Then

$$\begin{aligned}
(\hat{f} + \hat{g})\hat{\beta}\hat{h} &= (Cl(U, f) + Cl(V, g))Cl(M\beta M, 1_{M\beta})Cl(W, h) \\
&= Cl(U \cap V, f + g)Cl(M\beta M, 1_{M\beta})Cl(W, h) \\
&= Cl(W\Gamma M\beta M\Gamma(U \cap V), (f + g)1_{M\beta}h) \\
&= Cl(W\Gamma M\beta M\Gamma U \cap W\Gamma M\beta M\Gamma V, f1_{M\beta}h + g1_{M\beta}h) \\
&= Cl(W\Gamma M\beta M\Gamma U, f1_{M\beta}h) + Cl(W\Gamma M\beta M\Gamma V, g1_{M\beta}h) \\
&= \hat{f}\hat{\beta}\hat{h} + \hat{g}\hat{\beta}\hat{h},
\end{aligned}$$

and the equalities $\hat{f}(\hat{\gamma} + \hat{\beta})\hat{g} = \hat{f}\hat{\gamma}\hat{g} + \hat{f}\hat{\beta}\hat{g}$ and $\hat{f}\hat{\beta}(\hat{g} + \hat{h}) = \hat{f}\hat{\beta}\hat{g} + \hat{f}\hat{\beta}\hat{h}$ are proved in an analogous way. Moreover we have

$$\begin{aligned}
(\hat{f}\hat{\gamma}\hat{g})\hat{\beta}\hat{h} &= (Cl(U, f)Cl(M\gamma M, 1_{M\gamma})Cl(V, g))Cl(M\beta M, 1_{M\beta})Cl(W, h) \\
&= Cl(V\Gamma M\gamma M\Gamma U, f1_{M\gamma}g)Cl(M\beta M, 1_{M\beta})Cl(W, h) \\
&= Cl(W\Gamma M\beta M\Gamma(V\Gamma M\gamma M\Gamma U), (f1_{M\gamma}g)1_{M\beta}h \\
&= Cl((W\Gamma M\beta M\Gamma V)\Gamma M\gamma M\Gamma U, f1_{M\gamma}(g1_{M\beta}h)) \\
&= Cl(U, f)Cl(M\gamma M, 1_{M\gamma})Cl(W\Gamma M\beta M\Gamma V, g1_{M\beta}h) \\
&= Cl(U, f)Cl(M\gamma M, 1_{M\gamma})(Cl(V, g)Cl(M\beta M, 1_{M\beta})Cl(W, h)) \\
&= \hat{f}\hat{\gamma}(\hat{g}\hat{\beta}\hat{h}).
\end{aligned}$$

Next we will show that Q has an identity. Let $\hat{f} \in Q$ and $\hat{\beta} \in \hat{\Gamma}$. Take $\hat{I} = Cl(M, I) \in Q$ where $I : M \to M$, $x \mapsto x$, is a M-module homomorphism. Then

$$\begin{aligned}\hat{f}\hat{\beta}\hat{I} &= Cl(U, f)Cl(M\beta M, 1_{M\beta})Cl(M, I)\\ &= Cl(M\Gamma M\beta M\Gamma U, f1_{M\beta}I)\\ &= Cl(U, f) = \hat{f},\end{aligned}$$

and similarly we have $\hat{I}\hat{\beta}\hat{f} = \hat{f}$. Hence Q is a $\hat{\Gamma}$-ring with identity. Noticing that the mapping $\varphi : \Gamma \to \hat{\Gamma}$ defined by $\varphi(\beta) = \hat{\beta}$ for every $0 \neq \beta \in \Gamma$ is an isomorphism, we know that the $\hat{\Gamma}$-ring Q is a Γ-ring. Finally we prove that M is a subring of Q. For a fixed element a in M and every element $\gamma \in \Gamma$, consider a mapping $\lambda_{a\gamma} : M \to M$ defined by $\lambda_{a\gamma}(x) = a\gamma x$ for all $x \in M$. It is easy to prove that the mapping $\lambda_{a\gamma}$ is a right M-module homomorphism, so that $\lambda_{a\gamma}$ is an element of Q. Define a mapping $\psi : M \to Q$ by $\psi(a) = \hat{a} = Cl(M, \lambda_{a\gamma})$ for all $a \in M$ and $\gamma \in \Gamma$. Clearly ψ is well-defined. To prove ψ is one-to-one, it is enough to show that

$$\ker\psi = \{a \in M \mid \psi(a) = \hat{0}\} = \{0_M\}.$$

Let $a \in \ker\psi$. Then $\psi(a) = \hat{0}$, i.e., $Cl(M, \lambda_{a\gamma}) = Cl(M, 0)$. It follows that $0_M = \lambda_{a\gamma}(M) = a\gamma M$. Since M is a prime Γ-ring, we have $a = 0_M$ and so $\ker\psi = \{0_M\}$. In order to prove ψ is a homomorphism, let $\gamma, \beta \in \Gamma$ and $a, b \in M$. Then

$$\begin{aligned}\lambda_{(a+b)\gamma}(x) &= (a+b)\gamma x = a\gamma x + b\gamma x\\ &= \lambda_{a\gamma}(x) + \lambda_{b\gamma}(x) = (\lambda_{a\gamma} + \lambda_{b\gamma})(x)\end{aligned}$$

and

$$\begin{aligned}\lambda_{(a\beta b)\gamma}(x) &= (a\beta b)\gamma x = a\beta(b\gamma x) = \lambda_{a\beta}(b\gamma x)\\ &= \lambda_{a\beta}(1_{M\beta}(b\gamma x)) = \lambda_{a\beta}(1_{M\beta}(\lambda_{b\gamma}(x)))\\ &= (\lambda_{a\beta}1_{M\beta}\lambda_{b\gamma})(x)\end{aligned}$$

for all $x \in M$. It follows that $\lambda_{(a+b)\gamma} = \lambda_{a\gamma} + \lambda_{b\gamma}$ and $\lambda_{(a\beta b)\gamma} =$

$\lambda_{a\beta} 1_{M\beta} \lambda_{b\gamma}$. Hence

$$\begin{aligned}
\psi(a+b) &= \widehat{a+b} = Cl(M, \lambda_{(a+b)\gamma}) \\
&= Cl(M \cap M, \lambda_{a\gamma} + \lambda_{b\gamma}) \\
&= Cl(M, \lambda_{a\gamma}) + Cl(M, \lambda_{b\gamma}) \\
&= \hat{a} + \hat{b} = \psi(a) + \psi(b)
\end{aligned}$$

and

$$\begin{aligned}
\psi(a\beta b) &= \widehat{a\beta b} = Cl(M, \lambda_{(a\beta b)\gamma}) \\
&= Cl(M\Gamma M\beta M\Gamma M, \lambda_{a\beta} 1_{M\beta} \lambda_{b\gamma}) \\
&= Cl(M, \lambda_{a\beta}) Cl(M\beta M, 1_{M\beta}) Cl(M, \lambda_{b\gamma}) \\
&= \hat{a}\hat{\beta}\hat{b} \\
&= \psi(a)\beta\psi(b). \quad [\Gamma \text{ is isomorphic to } \hat{\Gamma}].
\end{aligned}$$

Therefore M is a subring of Q, and in such case we call Q the *quotient* Γ*-ring* of M.

Let M be any Γ-ring (in the sense of Barnes) and let $E(M,\Gamma)$ be the set of endomorphisms of the additive group of M. We can easily check that $E(M,\Gamma)$ is a Γ-ring. For $a \in M$, define maps $R_a : M \to M$ and $L_a : M \to M$ by $R_a(m) = m\gamma a$ and $L_a(m) = a\gamma m$, respectively, for all $m \in M$ and $\gamma \in \Gamma$. Then $R_a, L_a \in E(M,\Gamma)$. Let $B(M,\Gamma)$ be the subring of $E(M,\Gamma)$ generated by all R_a and L_a for $a \in M$.

DEFINITION 3.1. The set of elements in $E(M,\Gamma)$ which commute elementwise with $B(M,\Gamma)$ is called the *centroid* of M.

For purposes of convenience, we use q instead of $\hat{q} \in Q$.

LEMMA 3.2. *Let M be a prime Γ-ring. For each non-zero $q \in Q$, there is a non-zero ideal U of M such that $q(U) \subset M$.*

Proof. Straightforward. □

LEMMA 3.3. *Let M be a prime Γ-ring. Then the quotient Γ-ring Q of M is a prime Γ-ring.*

Proof. Let $p, q \in Q$ be such that $p\Gamma Q\Gamma q = 0$. If $p \neq 0 \neq q$, then there exist non-zero ideals U and V of M such that $p(U) \subset M$ and $q(V) \subset M$. Since $p \neq 0 \neq q$, there exist non-zero elements $u \in U$ and $v \in V$ such that $p(u) \neq 0 \neq q(v)$. Noticing that M is a subring of Q, we have

$$p(u)\Gamma M\Gamma q(v) \subset p(u)\Gamma Q\Gamma q(v) = 0$$

and so $p(u)\Gamma M\Gamma q(v) = 0$. This is a contradiction. Hence $p = 0$ or $q = 0$, ending the proof. □

DEFINITION 3.4. The set

$$C_\Gamma := \{g \in Q \mid g\gamma f = f\gamma g \text{ for all } f \in Q \text{ and } \gamma \in \Gamma\}$$

is called the *extended centroid* of a Γ-ring M.

Let M be a prime Γ-ring and let C_Γ be the extended centroid of M. Note that if a_i and b_i are non-zero elements of M such that $\sum a_i\gamma_i x\beta_i b_i = 0$ for all $x \in M$ and $\beta_i, \gamma_i \in \Gamma$, then the a_i's (also b_i's) are linearly dependent over C_Γ. Moreover, if $a\gamma x\beta b = b\gamma x\beta a$ for all $x \in M$ and $\beta, \gamma \in \Gamma$ where $a(\neq 0), b \in M$ are fixed, then there exists $\lambda \in C_\Gamma$ such that $b = \lambda\alpha a$ for $\alpha \in \Gamma$.

LEMMA 3.5. *Let M be a 2-torsion free prime Γ-ring, $D(\cdot,\cdot)$ the symmetric bi-derivation of M and d the trace of $D(\cdot,\cdot)$. If*

$$a\gamma d(x) = 0 \tag{1}$$

for all $x \in M$ and $\gamma \in \Gamma$ where a is a fixed element of M, then $a = 0$ or $D = 0$.

Proof. Let $x, y, z \in M$ and $\beta, \gamma \in \Gamma$. Replacing x by $x + y$ in (1), we get

$$a\gamma D(x, y) = 0. \tag{2}$$

If we substitute $z\beta x$ for x in (2), then

$$a\gamma z\beta D(x, y) = 0. \tag{3}$$

Since M is a prime Γ-ring, it follows that $a = 0$ or $D = 0$. □

LEMMA 3.6. *Let M be a 2-torsion free prime Γ-ring, $D_1(\cdot,\cdot)$ and $D_2(\cdot,\cdot)$ the symmetric bi-derivations of M and d_1 and d_2 the traces of $D_1(\cdot,\cdot)$ and $D_2(\cdot,\cdot)$, respectively. If*

$$d_1(x)\gamma d_2(y) = d_2(x)\gamma d_1(y) \tag{4}$$

for all $x, y \in M$ and $\gamma \in \Gamma$ and $d_1 \neq 0$, then there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda\alpha d_1(x)$ for $\alpha \in \Gamma$, where C_Γ is the extended centroid of M.

Proof. Let $x, y, z \in M$ and $\beta, \gamma \in \Gamma$. Substituting $y + z$ for y in (4), we have

$$d_1(x)\gamma D_2(y,z) = d_2(x)\gamma D_1(y,z). \tag{5}$$

Replacing z by $z\beta y$ in (5), we have

$$d_1(x)\gamma z\beta d_2(y) = d_2(x)\gamma z\beta d_1(y). \tag{6}$$

Now if we replace y by x in (6), then

$$d_1(x)\gamma z\beta d_2(x) = d_2(x)\gamma z\beta d_1(x). \tag{7}$$

If $d_1(x) \neq 0$ then $d_2(x) = \lambda(x)\alpha d_1(x)$ for all $\alpha \in \Gamma$ and for some $\lambda(x) \in C_\Gamma$. Thus if $d_1(x) \neq 0 \neq d_1(y)$, then it follows from (6) that

$$(\lambda(y) - \lambda(x))\alpha d_1(x)\gamma z\beta d_1(y) = 0. \tag{8}$$

Since M is a prime Γ-ring, by using Lemma 3.5 we conclude that $\lambda(x) = \lambda(y)$. Hence we have proved that there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda\alpha d_1(x)$ for all $\alpha \in \Gamma$ and $x \in M$ with $d_1(x) \neq 0$. On the other hand, if $d_1(x) = 0$ then $d_2(x) = 0$ as well. Therefore $d_2(x) = \lambda\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$. □

THEOREM 3.7. *Let M be a 2-torsion free prime Γ-ring, $D_1(\cdot,\ \cdot)$, $D_2(\cdot,\ \cdot)$, $D_3(\cdot,\ \cdot)$ and $D_4(\cdot,\ \cdot)$ the symmetric bi-derivations of M and d_1, d_2, d_3 and d_4 the traces of $D_1(\cdot,\ \cdot)$, $D_2(\cdot,\ \cdot)$, $D_3(\cdot,\ \cdot)$ and $D_4(\cdot,\ \cdot)$ respectively. If*

$$d_1(x)\gamma d_2(y) = d_3(x)\gamma d_4(y) \tag{9}$$

for all $x, y \in M$ and $\gamma \in \Gamma$ and $d_1 \neq 0 \neq d_4$, then there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda\alpha d_4(x)$ and $d_3(x) = \lambda\alpha d_1(x)$ for $\alpha \in \Gamma$ where C_Γ is the extended centroid of M.

Proof. Let $x, y, z, w \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Replacing y by $y+z$ in (9), we get

$$d_1(x)\gamma D_2(y,z) = d_3(x)\gamma D_4(y,z). \tag{10}$$

If we substitute $z\beta x$ for z in (10), then

$$d_1(x)\gamma z\beta d_2(y) = d_3(x)\gamma z\beta d_4(y). \tag{11}$$

Substituting $z\alpha d_4(w)$ for z in (11), we have

$$d_1(x)\gamma z\alpha d_4(w)\beta d_2(y) = d_3(x)\gamma z\alpha d_4(w)\beta d_4(y). \tag{12}$$

By (11), we know that $d_1(x)\gamma z\alpha d_2(w) = d_3(x)\gamma z\alpha d_4(w)$ and so

$$d_1(x)\gamma z\alpha(d_4(w)\beta d_2(y) - d_2(w)\beta d_4(y)) = 0$$

which implies that $d_4(w)\beta d_2(y) = d_2(w)\beta d_4(y)$ since $d_1 \neq 0$ and M is a prime Γ-ring. It follows from $d_4 \neq 0$ and Lemma 3.6 that $d_2(y) = \lambda\alpha d_4(y)$ for some $\lambda \in C_\Gamma$. Hence, by (11), we conclude that

$$(\lambda\alpha d_1(x) - d_3(x))\gamma z\beta d_4(y) = 0,$$

and so $d_3(x) = \lambda\alpha d_1(x)$. This completes the proof. □

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