Comm. Korean Math. Soc. **15** (2000), No. 3, pp. 493–498

ON DISTINGUISHED PRIME SUBMODULES

YONG HWAN CHO

ABSTRACT. In this paper we find some properties of distinguished prime submodules of modules and prove theorems about the dimension of modules.

1. Introduction

In this paper all rings are commutative with identity and all modules are unitary. Let R be a ring and M an R-module. A proper submodule P of M is said to be *prime* if $rm \in P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in (P : M)$. Specially prime submodule P is called $\mathcal{P}$-*prime* if $(P : M) = \mathcal{P}$. Clearly if P is a prime submodule of M, then $(P : M)$ is a prime ideal of R. A proper submodule Q of M is called a *primary* submodule if $rm \in Q$ for $r \in R$ and $m \in M$ implies that either $m \in Q$ or $r \in \sqrt{(Q : M)}$. A primary submodule Q of M is said to be $\mathcal{P}$-*primary* if $\sqrt{(Q : M)} = \mathcal{P}$. Clearly if Q is a $\mathcal{P}$-primary submodule, then $\mathcal{P}$ is a prime ideal of R. An R-module M is called *multiplication module* if every submodule N of M is of the form $\mathcal{A}M$ for some ideal $\mathcal{A}$ of R and an R-module M is said to be a *weak multiplication module* if every prime submodule N of M is of the form $\mathcal{A}M$ for some ideal $\mathcal{A}$ of R. It is clear that every multiplication module is a weak multiplication but the converse is not true; for example, the Z-module Q is a weak multiplication module which is not a multiplication module. Let N be a non-zero prime submodule of Q. Then $N \neq Q$. Therefore we can take $x \in Q - N$ and $y \in N - 0$. Let $x = k/l, y = r/s$ for some non-zero integers k, l, r, s. Hence $rlx = rk = (r/s)sk = (sk)y \in N$. But $x \notin N$ and N is a prime submodule. Thus $rl \in (N : Q)$. So $rlQ \subseteq N$. Now

Received October 29, 1999. Revised March 18, 2000.
2000 Mathematics Subject Classification: 13C11, 13C99.
Key words and phrases: distributive module, multiplication module, prime submodule, primary submodule, Krull dimension.

since $rlQ = Q$, we have $N = Q$, a contradiction. Hence 0 is the only prime submodule of Q and $0 = 0M$. This means that Q is a weak multiplication module. On the other hand, Z is a submodule of Q and $0 \neq Z \neq \mathcal{A}Q = Q$ for every non-zero ideal $\mathcal{A}$ of Z. Thus Q is not a multiplication module. In section 2 of this paper we consider some other conditions (Proposition 2.2, Proposition 2.3) which give results of Proposition 1.1 of [1], and prove a theorem (Theorem 2.4) about primary submodules which is very similar to Proposition 1.1 of [1]. In section 3, we prove that $dimM = cl.k.dimM$ (Theorem 3.5, Theorem 3.6) if M belongs to any of the following cases:

(1) M is a finitely generated distributive module.
(2) M is a distributive module with $\mathcal{M}M \neq M$ over a local ring $(R, \mathcal{M})$

Lastly, we prove that for every prime ideal $\mathcal{P}$ of R and for a finitely generated distributive R-module M, $dimM_{\mathcal{P}} = cl.k.dimM_{\mathcal{P}}$ (Theorem 3.7).

2. Distinguished Prime Submodules

Let N_1 and N_2 be submodules of an R-module M. Then we write $N_1 \sim N_2$ if and only if $N_1 : M = N_2 : M$. It is clear that $\sim$ is an equivalence relation on the set of all submodules of M. We denote each class by C_A where $A = N : M$ for each $N \in C_A$. Let M be an R-module, $\mathcal{P}$ a prime ideal of R, $S_{\mathcal{P}} = R - \mathcal{P}$ and $\mathcal{P}M(S_{\mathcal{P}}) = \{x \in M : sx \in \mathcal{P}M$ for some $s \in S_{\mathcal{P}}\}$. Then it is clear that $\mathcal{P}M(S_{\mathcal{P}})$ is a submodule of M containing $\mathcal{P}M$ and $\mathcal{P} \subseteq \mathcal{P}M : M \subseteq \mathcal{P}M(S_{\mathcal{P}}) : M$.

PROPOSITION 2.1 ([1]). *Let M be an R-module and $\mathcal{P}$ a prime ideal of R such that $\mathcal{P}M(S_{\mathcal{P}}) \neq M$. Then $\mathcal{P}M(S_{\mathcal{P}})$ is a $\mathcal{P}$-prime submodule of M and $\mathcal{P}M(S_{\mathcal{P}})$ is the intersection of all $\mathcal{P}$-prime submodules of $C_{\mathcal{P}}$.*

PROPOSITION 2.2. *Let M be an R-module containing a $\mathcal{P}$-prime submodule. Then $\mathcal{P}M(S_{\mathcal{P}})$ is a $\mathcal{P}$-prime submodule of M and $\mathcal{P}M(S_{\mathcal{P}})$ is the intersection of all $\mathcal{P}$-prime submodules of $C_{\mathcal{P}}$.*

PROOF. Let N be a $\mathcal{P}$-prime submodule of M. Then, $(N : M) = \mathcal{P}$ and let $m \in \mathcal{P}M(S_{\mathcal{P}})$. Then there exist $s \in S_{\mathcal{P}}$ such that $sm \in \mathcal{P}M$ and hence $sm \in N$. However since N is a $\mathcal{P}$-prime submodule and $s \notin \mathcal{P}$,

$m \in N$, i.e., $\mathcal{P}M(S_{\mathcal{P}}) \subseteq N \neq M$. The result follows from Proposition 2.1. $\square$

PROPOSITION 2.3. *Let M be an R-module and $\mathcal{P}$ a prime ideal of R such that $\mathcal{P} = (\mathcal{P}M : M)$ and $M/\mathcal{P}M$ is a finitely generated $R/\mathcal{P}$-module. Then $\mathcal{P}M(S_{\mathcal{P}})$ is a $\mathcal{P}$-prime submodule of M and $\mathcal{P}M(S_{\mathcal{P}})$ is the intersection of all $\mathcal{P}$-prime submodules of $C_{\mathcal{P}}$.*

PROOF. In view of Proposition 2.1, it suffices to prove that $\mathcal{P}M(S_{\mathcal{P}}) \neq M$. Now assume that $\mathcal{P}M(S_{\mathcal{P}}) = M$. Since $M/\mathcal{P}M$ is finitely generated, $M/\mathcal{P}M = (R/\mathcal{P})\bar{m}_1 + \cdots + (R/\mathcal{P})\bar{m}_k$. Hence $M = Rm_1 + \cdots + Rm_k + \mathcal{P}M$. However $m_i(i = 1, \cdots, k) \in M = \mathcal{P}M(S_{\mathcal{P}})$. So, there exists $s_i \in S_{\mathcal{P}}$ for each $i = 1, \cdots, k$, such that $s_i m_i \in \mathcal{P}M$. Therefore $s_1 s_2 \cdots s_k M \subseteq \mathcal{P}M$ and $s_1 s_2 \cdots s_k \subseteq (\mathcal{P}M : M) = \mathcal{P}$. Since $\mathcal{P}$ is a prime ideal there exists j such that $s_j \in \mathcal{P}$, a contradiction. Thus $\mathcal{P}M(S_{\mathcal{P}}) \neq M$. $\square$

Next, we have similar result for primary submodules.

THEOREM 2.4. *Let M be a finitely generated R-module and $\mathcal{Q}$ a $\mathcal{P}$-primary ideal of R containing $Ann_R M$. Then $\mathcal{Q}M(S_{\mathcal{P}}) = \{x \in M : sx \in \mathcal{Q}M$ for some $s \in S_{\mathcal{P}}\}$ is a $\mathcal{P}$-primary submodule of M and $\mathcal{Q}M(S_{\mathcal{P}})$ is the intersection of all $\mathcal{P}$-primary submodules of M in C_B where $B = \mathcal{Q}M(S_{\mathcal{P}}) : M$.*

PROOF. We first prove that $\mathcal{Q}M(S_{\mathcal{P}}) \neq M$. Assume that $\mathcal{Q}M(S_{\mathcal{P}}) = M$. Then since M is finitely generated, there exist $m_1, m_2, \cdots, m_n \in \mathcal{Q}M(S_{\mathcal{P}})$ and $s_1, s_2, \cdots, s_n \in S_{\mathcal{P}}$ such that $M = Rm_1 + Rm_2 + \cdots + Rm_n$ and $s_i m_i \in \mathcal{Q}M$ for each i. Consequently, for every i, there are $q_{ij} \in \mathcal{Q}$ such that $s_i m_i = \sum_{j=1}^{n} q_{ij} m_j$. Then it follows that $\sum_{j=1}^{n}(q_{ij} - s_i\delta_{ij})m_j = 0$ for each i. Hence $dm_j = 0$ for every j where $d = det(q_{ij} - s_i\delta_{ij}) = q \pm s_1 s_2 \cdots s_n$ and $q \in \mathcal{Q}$. Therefore $dM = 0$ and so $d \in Ann_R M \subseteq \mathcal{Q}$. Since $\sqrt{\mathcal{Q}} = \mathcal{P}$ and $\mathcal{P}$ is a prime ideal of R, there exist $j(1 \leq j \leq n)$ such that $s_j \in \mathcal{P}$, a contradiction. Thus $\mathcal{Q}M(S_{\mathcal{P}}) \neq M$. Now suppose that $r \in \sqrt{\mathcal{Q}M(S_{\mathcal{P}}) : M}$ and $r \notin \mathcal{P}$. Then there exist n such that $r^n M \subseteq \mathcal{Q}M(S_{\mathcal{P}})$. Hence for every $m \in M, r^n m \in \mathcal{Q}M(S_{\mathcal{P}})$ and so there exist $t \in S_{\mathcal{P}}$ such that $tr^n m \in \mathcal{Q}M$. Since $tr^n \in S_{\mathcal{P}}$ we have $m \in \mathcal{Q}M(S_{\mathcal{P}})$. So $M = \mathcal{Q}M(S_{\mathcal{P}})$, a contradiction, i.e., $\sqrt{QM(S_{\mathcal{P}}):M} \subseteq \mathcal{P}$. On the other hand, for every $p \in \mathcal{P} = \sqrt{Q}$ there exists n such that $p^n \in Q$. Hence $p^n M \subseteq QM \subseteq QM(S_{\mathcal{P}})$ and $p^n \in (QM(S_{\mathcal{P}}) : M)$. This means that $p \in \sqrt{QM(S_{\mathcal{P}}):M}$ and so $\mathcal{P} = \sqrt{QM(S_{\mathcal{P}}):M}$. Lastly, let $rm \in QM(S_{\mathcal{P}})$ for $r \notin \mathcal{P} = \sqrt{QM(S_{\mathcal{P}}):M}$ and $m \in M$. Then $r \in S_{\mathcal{P}}$ and there exists $s \in S_{\mathcal{P}}$ such that $s(rm) \in QM$. Since $sr \in S_{\mathcal{P}}$ we have $m \in QM(S_{\mathcal{P}})$. Therefore $QM(S_{\mathcal{P}})$ is a $\mathcal{P}$-primary submodule of M. Next, let N be a $\mathcal{P}$-primary submodule of M in C_B where $B = QM(S_{\mathcal{P}}) : M$. Supose that $m \in QM(S_{\mathcal{P}})$. Then there exists $s \in S_{\mathcal{P}}$ such that $sm \in QM$. However, $QM \subseteq QM(S_{\mathcal{P}})$ and so $Q \subseteq (QM(S_{\mathcal{P}}) : M)$. Since $N \in C_B$ and $B = (QM(S_{\mathcal{P}}) : M)$, $Q \subseteq (QM(S_{\mathcal{P}}) : M) = (N : M)$ and $sm \in QM \subseteq N$. Since N is $\mathcal{P}$-primary and $s \notin \mathcal{P}$ and $m \in M$, we have $m \in N$. By above discussion we know that $QM(S_{\mathcal{P}})$ is a $\mathcal{P}$-primary submodule of M. Therefore $QM(S_{\mathcal{P}})$ is the intersection of all $\mathcal{P}$-primary submodules of M in C_B, $B = QM(S_{\mathcal{P}}) : M$. □

A $\mathcal{P}$-prime submodule N of an R-module M is called a *distinguished $\mathcal{P}$-prime submodule* if and only if $N = \mathcal{P}M(S_{\mathcal{P}})$.

3. The Dimension of Modules

The classical Krull dimension of a ring R (cl.k.dim R) is either infinite or cl.k.dim $R = n$, where n is nonnegative integer such that R has a strict increasing chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of $n+1$ distinct prime ideals of R but no chain of $n+2$ distinct prime ideals.

We know that the classical Krull dimension of an R-module M is defined as the cl.k.dim $(R/ann_R M)$. On the other hand, Sadi Abu-Saymeh ([1]) defined the classical Krull dimension of a module in terms of lengths of chains of distinguished prime submodules and investigated the relation between these two dimensions.

The classical Krull dimension of an R-module M, $dimM$, is defined in terms of ascending chains of distinguished prime submodules. We set $dimM = n$ if there is a strictly increasing chain $N_0 \subset \cdots \subset N_n$ of $n+1$ distinguished prime submodules and there is no such chain of $n+2$ distinguished prime submodules and we set $dimM = \infty$ if there is a chain of the above kind for every value of n.

THEOREM 3.1 ([1]). *Let M be a finitely generated R-module. Then $dimM \leq cl.k.dimM$.*

THEOREM 3.2 ([1]). *Let M be a finitely generated R-module. Then $dimM = cl.k.dimM$ if M belongs to any of the following cases:*

(1) *M is a weak multiplication module.*

(2) *M is a content module such that $rc(x) \subseteq \sqrt{c(rx)}$ for every $r \in R$ and $x \in M$.*

(3) *M is a flat module.*

(4) *M is a serial module.*

A submodule N of M will be called a *distributive submodule* if the following equivalent conditions are satisfied : $(P+Q) \cap N = (P \cap N) + (Q \cap N)$; $(P \cap Q) + N = (P+N) \cap (Q+N)$ for all submodules P, Q, N of M. Thus a module M is *distributive* if every submodule of M is a distributive submodule ([3],[4],[6]).

PROPOSITION 3.3 ([4]). *Let R be a local ring and let M be an R-module. Then M is a distributive R-module if and only if the set of submodules of M is linearly ordered.*

PROPOSITION 3.4 ([3]). *Let R be a ring, M an R-module and S be a multiplicatively closed subset of R. Then if M is a distributive R-module, then $S^{-1}M$ is a distributive $S^{-1}R$-module.*

THEOREM 3.5. *Let M be a finitely generated distributive R-module. Then $dimM = cl.k.dimM$.*

PROOF. First take R to be a local ring and let M be a finitely generated distributive R-module. Then the set of submodules of M is linearly ordered by Proposition 3.3. Since M is finitely generated, M is cyclic. So, M is a multiplication module ([2]).

Now we go to the general case. Let R be any ring and N a submodule of M. Since M is finitely generated, we know that $(N : M)_P = (N_P : M_P)$ for each prime ideal P of R. By Proposition 3.4, we know that $M_{\mathcal{M}}$ is a finitely generated distributive $R_{\mathcal{M}}$-module. By the local case, $N_{\mathcal{M}} = (N_{\mathcal{M}} : M_{\mathcal{M}})M_{\mathcal{M}} = ((N : M)M)_{\mathcal{M}}$ for all maximal ideals $\mathcal{M}$ of R. Hence $N = (N : M)M$ and M is a multiplication module. Clearly, since any multiplication module is a weak multiplication module, by Theorem 3.2 $dimM = cl.k.dimM$. □

THEOREM 3.6. *Let R be a local ring with maximal ideal $\mathcal{M}$ and M an distributive R-module with $\mathcal{M}M \neq M$. Then $dimM = cl.k.dimM$.*

PROOF. Since M is distributive and $\mathcal{M}M \neq M$, we can easily show that $M/\mathcal{M}M$ is a non-zero distributive vector space over the field $R/\mathcal{M}$. So, we can take $0 \neq m \in M - \mathcal{M}M$. Then $Rm + \mathcal{M}M/\mathcal{M}M$ is a distributive submodule of $M/\mathcal{M}M$. However since we know that any module over a field has no non-trivial distributive submodule ([2]), $Rm + \mathcal{M}M/\mathcal{M}M = M/\mathcal{M}M$. Hence $Rm + \mathcal{M}M = M$. But by Proposition 3.3 and $Rm \not\subseteq \mathcal{M}M$ we have $\mathcal{M}M \subset Rm$. Therefore it follows that $M = Rm$. Thus M is cyclic and a weak multiplication module. Hence we have the result by Theorem 3.2. □

An R-module M is called a *serial module* if its submodules are linearly ordered with respect to inclusion.

THEOREM 3.7. *Let M be a finitely generated distributive R-module. Then for any prime ideal $\mathcal{P}$ of R, $dimM_{\mathcal{P}} = cl.k.dimM_{\mathcal{P}}$.*

PROOF. Let $\mathcal{P}$ be any prime ideal of R. Then, by Proposition 3.4, $M_{\mathcal{P}}$ is a finitely generated distributive $R_{\mathcal{P}}$-module and we know that $M_{\mathcal{P}}$ is a serial module from Proposition 3.3. Thus Theorem 3.2 gives the result. □

References

[1] Sadi Abu-Saymeh, *On dimension of finitely generated modules*, Comm. in Algebra **23** (1995), no. 3, 1131-1144.

[2] A. D. Barnard, *Multiplication Modules*, J. Algebra **71** (1981), 174-178.

[3] T. M. K. Davison, *Distributive homomorphisms of rings and modules*, J. Reine Angew. Math. **270** (1974), 28-34.

[4] V. Erdŏgdu, *Distributive Modules*, Canad. Math. Bull. **30** (1987), 248-254.

[5] Y. H. Cho, *On Multiplication Modules (II)*, Commun. Korean Math. Soc. **13** (1998), no. 4, 727-733.

[6] ———, *On Multiplication Modules (III)*, Korean Annals of Mathematics **16** (1999), 21-25.

Department of Mathematics Education
Chonbuk National University
Chonju, Chonbuk 561-756, Korea
E-mail: cyh@moak.chonbuk.ac.kr