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EXTREMALLY RICH GRAPH C^*-ALGEBRAS

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ABSTRACT. Graph C^*-algebras $C^*(E)$ are the universal C^*-algebras generated by partial isometries satisfying the Cuntz-Krieger relations determined by directed graphs E, and it is known that a simple graph C^*-algebra is extremally rich in a sense that it contains enough extreme partial isometries in its closed unit ball. In this short paper, we consider a sufficient condition on a graph for which the associated graph algebra (possibly nonsimple) is extremally rich. We also present examples of nonextremally rich prime graph C^*-algebras with finitely many ideals and with real rank zero.

1. Introduction

Recall that a projection p in a C^*-algebra A is said to be *infinite* if it is Murray-von Neumann equivalent to its proper subprojection. We call a unital C^*-algebra A *infinite* if the unit projection is infinite, and *finite* otherwise. If a unital C^*-algebra A has stable rank one ($sr(A) = 1$, see [13]), that is, the set A^{-1} of all invertible elements is dense in A, then one can see that A should be finite. All AF-algebras([13]), irrational rotation algebras([12]) are those known to have stable rank one.

As an attempt to extend notions and results for finite C^*-algebras to infinite cases Brown and Pedersen [2] considered the quasi-invertible elements A_q^{-1} in a unital C^*-algebra A and call A *extremally rich* if the set A_q^{-1} is dense in A since it turns out in [2] that this condition is equivalent to say that the closed unit ball A_1 contains enough extreme points so that the convex hull of its extreme points coincides with the whole A_1;

$$conv(\mathcal{E}(A)) = A_1,$$

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where $\mathcal{E}(A)$ denotes the extreme points of A_1. Since $A^{-1} \subset A_q^{-1}$ for any unital C^*-algebra A we see that a unital C^*-algebra A with $sr(A) = 1$ is always extremally rich. On the other hand it is a nontrivial fact that purely infinite simple C^*-algebras (for example, Cuntz algebras) are also extremally rich (see [10],[11]). The Cuntz-algebra $\mathcal{O}_n (n \geq 2)$ is the universal C^*-algebra generated by n isometries having orthogonal ranges. More generally, Cuntz-Krieger algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by n partial isometries S_i satisfying the relation:

$$(*) \qquad S_i^* S_i = \sum_{j=1}^{n} A(i,j) S_j S_j^*$$

where A is an $n \times n$ $\{0,1\}$-matrix with no zero row or column (see [6]).

As a generalization of Cuntz-Krieger algebras, a class of C^*-algebras generated by partial isometries subject to the relations determined by directed graphs has been studied in [9], [8] and later in [4].

Since the graph algebras are basically generated by partial isometries they are considered to have sufficiently many extreme partial isometries in their closed unit balls and thus one may expect that most of them should be extremally rich. In fact, this is true for simple graph C^*-algebras.

In this paper we first consider a sufficient condition on a directed graph E for which the associated graph C^*-algebra (possibly nonsimple) is extremally rich and also present some examples of nonextremally rich prime graph C^*-algebras that have only finitely many ideals and have real rank zero. For C^*-algebras having real rank zero, refer to [1].

2. Preliminaries

We recall definitions and results from [8], [9], and [4] on directed graphs and graph C^*-algebras. A *directed graph* $E = (E^0, E^1, r, s)$ (or simply $E = (E^0, E^1)$) consists of countable sets E^0 of vertices and E^1 of edges, and the range, source maps $r, s : E^1 \to E^0$. E is *row finite* if each vertex $v \in E^0$ emits at most finitely many edges, and a row finite graph is *locally finite* if each vertex receives only finitely many edges. If $e_1, \ldots, e_n$ $(n \geq 2)$ are edges with $r(e_i) = s(e_{i+1})$, $1 \leq i \leq n-1$, then one can form a (finite) path $\alpha = (e_1, \ldots, e_n)$ of *length* $|\alpha| = n$, and extend the maps r, s by $r(\alpha) = r(e_n), s(\alpha) = s(e_1)$.

Let E^n be the set of all finite paths of length n (so vertices in E^0 are regarded as finite paths of length zero) and let E^* be the set of all finite paths. Similarly one can consider the set E^∞ of infinite paths. A vertex $v \in E^0$ with $s^{-1}(v) = \emptyset$ is called a *sink*.

Given a row finite directed graph E, a *Cuntz-Krieger E-family* consists of mutually orthogonal projections $\{P_v \mid v \in E^0\}$ and partial isometries $\{S_e \mid e \in E^1\}$ satisfying the Cuntz-Krieger relations

$$S_e^* S_e = P_{r(e)},\ e \in E^1, \text{ and } P_v = \sum_{s(e)=v} S_e S_e^*,\ v \in s(E^1).$$

From these relations, one can show that every non-zero word in S_e, P_v and S_f^* reduces to a partial isometry of the form $S_\alpha S_\beta^*$ for some $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$ ([8], Lemma 1.1).

THEOREM 2.1. ([8], Theorem 1.2) *For a row finite directed graph $E = (E^0, E^1)$, there exists a C^*-algebra $C^*(E)$ generated by a* Cuntz-Krieger *E-family $\{s_e, p_v \mid v \in E^0, e \in E^1\}$ of non-zero elements such that for any* Cuntz-Krieger *E-family $\{S_e, P_v \mid v \in E^0, e \in E^1\}$ of partial isometries acting on a* Hilbert *space $\mathcal{H}$, there is a representation $\pi : C^*(E) \to B(\mathcal{H})$ such that*

$$\pi(s_e) = S_e, \text{ and } \pi(p_v) = P_v$$

for all $e \in E^1, v \in E^0$.

Let $\{s_e, p_v \mid e \in E^1, v \in E^0\}$ be a Cuntz-Krieger E-family generating the C^*-algebra $C^*(E)$. Then for each $z \in \mathbb{T}$ we have another Cuntz-Krieger E-family $\{zs_e, p_v \mid e \in E^1,\ v \in E^0\}$ in $C^*(E)$, and by the universal property of $C^*(E)$ there exists an isomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$. In fact, $\gamma : z \mapsto \gamma_z \in \mathrm{Aut}(C^*(E))$ is a strongly continuous action of $\mathbb{T}$ on $C^*(E)$ and is called the *gauge action* ([4]).

We call a finite path α with $|\alpha| > 0$ a *loop* at v if $s(\alpha) = r(\alpha) = v$. It turns out that the distribution of loops in a graph E is very important to understand the structure of a graph C^*-algebra $C^*(E)$.

A graph E is said to satisfy *condition* (L) if every loop in E has an exit, and *condition* (K) if for any vertex v on a loop there exist at least two distinct loops based at v. Note that condition (K) is stronger than (L) and if E has no loops then the two conditions are trivially satisfied.

For two vertices v, w we simply write $v \geq w$ if there is a path $\alpha \in E^*$ from v to w. A subset H of E^0 is said to be *hereditary* if $v \geq w$ and

$v \in H$ imply $w \in H$, and a hereditary set H is *saturated* if $s^{-1}(v) \neq \emptyset$ and $\{r(e) \mid s(e) = v\} \subset H$ imply $v \in H$. The *saturation* of a hereditary set H is the smallest saturated subset of E^0 containing H.

Let H be a saturated hereditary subset of E^0. Then the ideal $I(H) = \overline{span}\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = r(\beta) \in H\}$ is clearly gauge-invariant and $I(H)$ is generated by $\{p_v \mid v \in H\}$.

In case E has no sinks, in [9], an isomorphism of the lattice of saturated hereditary subsets V of E^0 into the lattice of ideals $I(V)$ in $C^*(E)$ was established and it is shown that the quotient algebra $C^*(E)/I(V)$ is isomorphic to a graph algebra $C^*(G)$ for a certain subgraph G of E. More generally, the following was proved in [4].

THEOREM 2.2. ([4, Theorem 4.1]) *Let $E = (E^0, E^1, r, s)$ be a row finite directed graph. For each subset H of E^0, let $I(H)$ be the ideal in $C^*(E)$ generated by $\{p_v \mid v \in H\}$.*

(a) *The map $H \mapsto I(H)$ is an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of closed gauge-invariant ideals of $C^*(E)$.*

(b) *Suppose H is saturated and hereditary. If $F^0 := E^0 \backslash H$, $F^1 := \{e \in E^1 \mid r(e) \notin H\}$, and $F := (F^0, F^1, r, s)$, then $C^*(E)/I(H)$ is canonically isomorphic to $C^*(F)$.*

Note that if a graph E satisfies condition (K) then the isomorphism of Theorem 2.2.(a) maps onto the lattice of all closed ideals in $C^*(E)$, that is, every ideal is gauge-invariant. It is known ([4], [7]) that for a row-finite graph E, the graph C^*-algebra $C^*(E)$ is simple if and only if E is a cofinal graph satisfying condition (L), here we say that E is *cofinal* if every vertex connects to every infinite path. Before we review the following interesting dichotomy for simple graph C^*-algebras, let us recall that a C^*-algebra A is said to be *purely infinite* if every non-zero hereditary C^*-subalgebra of A has an infinite projection.

PROPOSITION 2.3. ([8], Corollary 3.11) *Let E be a locally finite graph which has no sinks, is cofinal, and satisfies condition* (L). *Then $C^*(E)$ is simple, and*

(i) *if E has no loops, then $C^*(E)$ is AF;*

(ii) *if E has a loop, then $C^*(E)$ is purely infinite.*

3. Extremally rich graph C^*-algebra

Let A be a unital C^*-algebra and $\mathcal{E}(A)$ (or simply $\mathcal{E}$) be the set of all extreme points in the (convex) closed unit ball A_1 of A. Then it is well known that an extreme point v in A_1 is characterized as a partial isometry satisfying $(1-vv^*)A(1-v^*v)=0$ ([11], Proposition 1.4.7.). Let A_+^{-1} be the set of all positive invertible elements of A. We call elements $x \in \mathcal{E}A_+^{-1}(= A^{-1}\mathcal{E}A^{-1})$ *quasi-invertible* ([3]) and denote the set of all quasi-invertible elements in A by A_q^{-1}. If A_q^{-1} is dense in A we say that A is *extremally rich*. For a non-unital C^*-algebra A, A is said to be extremally rich when its unitization $\tilde{A}$ is so. Obviously a C^*-algebra A with $sr(A)=1$ is always extremally rich since $A^{-1} \subset A_q^{-1}$. In particular, all AF-algebras are extremally rich. Other examples are purely infinite simple C^*-algebras ([11], Theorem 10.1), the Toeplitz algebra ([11], Corollary 9.2), commutative C^*-algebras $C(X)$ with $\dim(X) \leq 1$ (see [3], section 3), and all von Neumann algebras ([11], Theorem 4.2). Thus from Proposition 2.3 it follows that every simple graph C^*-algebra $C^*(E)$ is extremally rich for a row-finite directed graph E. Also if A is an extremally rich simple C^*-algebra then it is known that either it is purely infinite or it has stable rank one ([2], Corollary 10.5).

We will see some other nonsimple graph C^*-algebras which are extremally rich.

THEOREM 3.1. ([7, Theorem 3.3]) *Let $E=(E^0,E^1)$ be a row finite directed graph. Then E has no loop with an exit if and only if $sr(C^*(E))=1$.*

Besides the algebras with stable rank one we will see that there are extremally rich graph C^*-algebras including the Toeplitz algebra $\mathcal{T}$ with higher stable rank ($sr(\mathcal{T})=2$). These graph algebras will arise from directed graphs containing some loops with exits, so that they should have many infinite projections (generated by the exits of loops) and hence their stable rank is not one any more.

To this end note the following corollary of Theorem 2.2.

COROLLARY 3.2. ([7, Theorem 3.5]) *Let $E=(E^0,E^1,r,s)$ be a row-finite directed graph with the set V of sinks. Then there is a subgraph $G=(E^0 \setminus H, \{e \in E^1 \mid r(e) \notin H\})$ of E with no sinks such that $C^*(E)/I(V)$ is isomorphic to $C^*(G)$, where H is the saturation of V and $I(V)=\overline{span}\{s_\alpha s_\beta^* \mid \alpha,\beta \in E^*,\ r(\alpha)=r(\beta) \in V\}$.*

PROPOSITION 3.3. ([7, Proposition 3.7]) *Let G be a locally finite directed graph. If G is cofinal then either $sr(C^*(G)) = 1$ or it is purely infinite and simple.*

COROLLARY 3.4. *If the subgraph G in* Corollary 3.2 *is cofinal then the quotient algebra $C^*(E)/I$ is either of stable rank one or purely infinite simple.*

We also need to review briefly the following useful results on extremally rich C^*-algebras.

THEOREM 3.5. ([2],[3]) (a) *Every quotient, every direct sum or direct product and every hereditary C^*-subalgebra of an extremally rich C^*-algebra is again extremally rich.*
(b) *If A is strong* Morita *equivalent (or stably isomorphic) to an extremally rich C^*-algebra B then A is also extremally rich.*

Let A be a unital C^*-algebra and I be a closed two-sided ideal.
(c) *Suppose $sr(I) = 1$. Then A is extremally rich if and only if A/I is extremally rich and extreme partial isometries lift.*
(d) *$sr(A) = 1$ if and only if $sr(I) = sr(A/I) = 1$ and every invertible elements lifts, that is, $(\tilde{A}/I)^{-1} = \tilde{A}^{-1}/I$.*

For a C^*-algebra A and projections P, Q in A, the extreme points $\mathcal{E}(PAQ)$ of the closed convex set PA_1Q consist of elements $u \in PA_1Q$ which is a partial isometry such that $(P-uu^*)A(Q-u^*u) = \{0\}$. We say that the space PAQ is *extremally rich* if either $\mathcal{E}(PAQ) = \emptyset$ or $\mathcal{E}(PAQ) \neq \emptyset$ and $(PAP)^{-1}\mathcal{E}(PAQ)(QAQ)^{-1}$ is dense in PAQ. If $\mathcal{E}(PAQ) \neq \emptyset$ then PAQ is extremally rich if and only if $PA_1Q = conv(\mathcal{E}(PAQ))$ (see [2]).

For any non-zero projections P, Q acting on a Hilbert space $\mathcal{H}$, one can show that $\mathcal{E}(PB(\mathcal{H})Q) \neq \emptyset$ and the space $PB(\mathcal{H})Q$ is extremally rich by Proposition 11.4 of [2]; if A is a C^*-algebra with real rank zero and $\mathcal{E}(PAQ) \neq \emptyset$ for every pair of projections P, Q in A, then every such a space PAQ is, in fact, extremally rich.

PROPOSITION 3.6. ([2], Proposition 11.7) *Let I be a closed ideal with real rank zero in a unital C^*-algebra A, such that PIQ is extremally rich for any pair of projections such that $P \in A$ and $Q \in I$. If A/I is extremally rich and $\mathcal{E}(A/I)$ consists only of isometries and co-isometries then A is extremally rich.*

Now we prove the main theorem which generalizes Proposition 3.3. Note that the C^*-algebra $B(\mathcal{H})$ and its closed ideal $\mathcal{K}(\mathcal{H})$ of compact operators are known to have real rank zero.

THEOREM 3.7. *Let $E = (E^0, E^1)$ be a locally finite directed graph and V the set of sinks. If the subgraph G in* Corollary 3.2 *is cofinal then $C^*(E)$ is extremally rich.*

PROOF. Note first that $RR(I) = 0$, where $I = I(V)$ is the ideal in Corollary 3.2, since I is the direct sum of orthogonal ideals I_i which is isomorphic to $\mathcal{K}(\ell^2(E^*(v_i)))$ (see [8, Corollary 2.2]). Here we set $V = \{v_i \mid i \in \Lambda\}$ and $E^*(v_i) = \{\alpha \in E^* \mid r(\alpha) = v_i\}$. Let $P \in C^*(E)$, $Q \in I$ be any two projections. Then we have $PIQ \cong \oplus_i PI_iQ$ and $PI_iQ \subset I_i$. Furthermore it is easy to see that $PI_iQ \cong \bar{P}B(\mathcal{H}_i)\bar{Q}$ for some projections $\bar{P} \in B(\mathcal{H}_i)$ and $\bar{Q} \in \mathcal{K}(\mathcal{H}_i)$, where $\mathcal{H}_i$ denotes the Hilbert space $\ell^2(E^*(v_i))$. Thus the spaces PI_iQ are all extremally rich and so their direct sum PIQ is also extremally rich.

Now we prove the assertion case by case.

Suppose G has no loops. Then $C^*(E)/I$ is an AF algebra since G has no loops if and only if $C^*(G)$ is an AF algebra by ([8, Theorem 2.4]) and hence $sr(C^*(E)/I) = 1$. Since every invertible element in an AF algebra $\widetilde{C^*(E)/I}$ lifts to an invertible element in $\widetilde{C^*(E)}$, $sr(C^*(E)) = 1$ by Theorem 3.5.(d).

If G has precisely one loop and the loop has no exit then $sr(C^*(G)) = 1$ by Theorem 3.1, so that $sr(C^*(E)/I) = 1$, and hence every extreme partial isometry of $\widetilde{C^*(E)/I}$ is unitary. Therefore by Proposition 3.6 we conclude that $C^*(E)$ is extremally rich.

If the (cofinal) graph G has precisely one loop and it has an exit then G satisfies condition (L). Thus the algebra $C^*(G)$ is simple and its stable rank is not one since there are infinite projections in $C^*(G)$. Thus by Proposition 3.3 $C^*(G)$ is purely infinite simple and so extremally rich. Therefore $\widetilde{C^*(E)/I}$ is prime, and hence $\mathcal{E}(\widetilde{C^*(E)/I})$ consists only of isometries and coisometries. By Proposition 3.6, $C^*(E)$ is extremally rich.

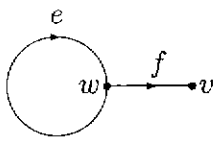
Finally let G have at least two loops α, β. If γ is a loop in G then we may assume that $\gamma \neq \alpha$ and consider an infinite path $x = \alpha\alpha\cdots = (x_1, x_2, \cdots)$. Since G is cofinal there exists a finite path $\delta \in G^*$ connecting the vertex $v = s(\gamma)$ to $x_n = r(\delta)$ for some n, which means that γ has an exit. Therefore G is a cofinal graph satisfying condition (L), so that $C^*(E)/I$ is

purely infinite and simple and thus it is extremally rich. Then we conclude that $C^*(E)$ is extremally rich by the same reason as in the preceding case. □

PROPOSITION 3.8. *Suppose a graph C^*-algebra $C^*(E)$ contains an ideal $I(H)$ with stable rank one such that the subgraph G generating the quotient algebra $C^*(G)$ ($\cong C^*(E)/I(H)$) is cofinal. If G satisfies condition (L) then $C^*(E)$ is extremally rich.*

PROOF. Since G satisfies condition (L), the third case as in the proof of Theorem 3.7 will not happen. Then it suffices to note that we applied Proposition 3.6 to prove only this case and for other three cases Theorem 3.5 (c) was useful and is still applicable under the assumptions of the proposition. □

EXAMPLE 3.9. Consider the following graph.



The sink v generates an ideal I which is isomorphic to $\mathcal{K}$, the compact operators acting on an infinite dimensional separable Hilbert space. Set $S = s_e + s_f$. Then $S^*S = 1$ and $SS^* = p_w < 1 = p_w + p_v$. Thus S is a proper isometry. Let $\mathcal{T}$ be the C^*-subalgebra of $C^*(E)$ generated by S. Since $S(1 - SS^*) = s_f$, it follows that $s_f \in \mathcal{T}$, hence $C^*(E) = \mathcal{T}$ and $\mathcal{T}$ is the Toeplitz algebra which is extremally rich. In this example, the subgraph G in Theorem 3.5 is the simple loop e. More generally, if a graph E consists of a simple loop with n vertices and each of the vertices emits an edge then we can conclude that the resulting graph algebra $C^*(E)$ is extremally rich.

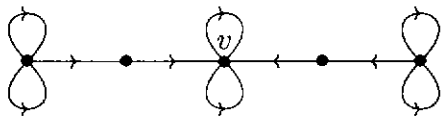
4. Examples of non-extremally rich prime graph C^*-algebras

In this section we show that there are many graph C^*-algebras with nice properties but are not extremally rich by constructing some examples.

Note from Corollary 3.7 in [10] that if I is a purely infinite simple ideal of a unital C^*-algebra A such that the quotient algebra A/I is also purely

infinite and simple then A should be extremally rich. On the other hand, it is known in [10, Remark 4.10] that there exists a non-extremally rich unital C^*-algebra B (non-separable) with two proper ideals $I_1 \subset I_2$ such that $I_1, I_2/I_1$ and B/I_2 are all purely infinite and simple (hence extremally rich). In the following we construct a separable unital C^*-algebra B with $RR(B) = 0$ which has exactly three proper ideals and every possible quotient is purely infinite and extremally rich, but B is not.

EXAMPLE 4.1. Consider the following locally finite directed graph $E = (E^0, E^1)$.



Let H be the smallest hereditary saturated vertex subset containing v. Then $C^*(E)$ has real rank zero by [7] since E satisfies condition (K) and $C^*(E)$ has exactly three proper ideals; recall that for a graph algebra $C^*(E)$ with finitely many ideals, $RR(C^*(E)) = 0$ if and only if E satisfies condition (K). It is easy to see that the ideal $I(H)$ corresponding to H is stably isomorphic to the graph algebra $C^*(G)$, where G is a subgraph of E with three vertices in the middle of E and four edges connecting them ([9]). Since G is cofinal and satisfies (K) (hence (L)) $C^*(G)$ is purely infinite and simple by Proposition 3.3. Thus $I(H)$ should be purely infinite and simple since it is well known that being purely infinite and simple is stable property under a stable isomorphism.

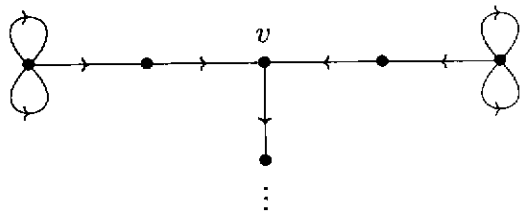
Moreover note that $I(H)$ is essential in $C^*(E)$, that is, it has nonzero intersection with every other nonzero closed ideal. Thus the graph algebra $C^*(E)$ is prime and hence its extreme point set of the unit ball consists of isometries or coisometries. Now consider the quotient algebra $C^*(E)/I(H)$, then it is isomorphic to the graph C^*-algebra $C^*(F)$, where $F = (E^0 \setminus H, \{e \mid r(e) \notin H\})$. Since $C^*(F)$ is isomorphic to the direct sum $\mathcal{O}_2 \oplus \mathcal{O}_2$ of the Cuntz algebra $\mathcal{O}_2$ which is purely infinite and simple, the quotient algebra is extremally rich.

Let s_1, s_2 be two isometries generating the Cuntz algebra $\mathcal{O}_2$. If $C^*(E)$ were extremally rich then by [2, Corollary 9.3] every extreme partial isometry of $C^*(E)/I(H)$ should lift. But the partial isometry $u = s_i \oplus s_i^*$ $(i = 1, 2)$ is extremal in the quotient algebra $C^*(E)/I(H)$ and cannot lift to an isometry or a coisometry. This proves the assertion. Note that $C^*(E)$

(and so every ideal) is purely infinite since E satisfies (L) and every vertex connects to a loop ([4, Proposition 5.3]).

In the following example, the essential ideal will be an AF (so finite) algebra.

EXAMPLE 4.2. Consider the following directed graph $E = (E^0, E^1)$.



Let H be the smallest hereditary saturated vertex subset containing v as above. Then E obviously satisfies condition (K) and hence $C^*(E)$ has real rank zero by [7]. As in the preceding example $C^*(E)$ has exactly three proper ideals and the ideal $I(H)$ corresponding to H is essential in $C^*(E)$. One different thing about $I(H)$ from Example 4.1 is that $I(H)$ is stably isomorphic to the simple C^*-algebra $\mathcal{K}$ of the compact operators acting on a separable infinite dimensional C^*-algebra. Now consider the quotient algebra $C^*(E)/I(H)$. Then it is isomorphic to the graph C^*-algebra $C^*(F)$, where $F = (E^0 \setminus H, \{e \mid r(e) \notin H\})$ by Theorem 2.2.(b). Since $C^*(F) \cong \mathcal{O}_2 \oplus \mathcal{O}_2$ as before, the quotient algebra is extremally rich and contains extremal partial isometries which cannot lift, and this proves the assertion.

REMARK 4.3. We have seen that a cofinal graph generates an extremally rich graph C^*-algebras in section 3 and the two graphs examined above generate nonextremally rich C^*-algebras. In fact these graphs are not cofinal, and it would be interesting to find conditions of graphs which are not cofinal but generate extremally rich C^*-algebras.

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