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CHARACTERIZATIONS OF BOUNDED VECTOR MEASURES

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ABSTRACT. Let X be a locally convex space. A series of clearcut characterizations for the boundedness of vector measure $\mu : \Sigma \to X$ is obtained, e.g., μ is bounded if and only if $\mu(A_j) \to 0$ weakly for every disjoint $\{A_j\} \subseteq \Sigma$ and if and only if $\{\frac{1}{j^j}\mu(A_j)\}_{j=1}^{\infty}$ is bounded for every disjoint $\{A_j\} \subseteq \Sigma$.

Let Σ be an algebra of subsets of a set Ω and X a locally convex space with the dual X'. A function $\mu : \Sigma \to X$ is said to be a *measure* if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \Sigma$ and $A \cap B = \emptyset$. A measure $\mu : \Sigma \to X$ is bounded if $\{\mu(A) : A \in \Sigma\}$ is a bounded subset of X. If X is a Banach space, then $\mu : \Sigma \to X$ is bounded if and only if μ is of bounded semivariation, i.e., $\|\mu\|(\Omega) = \sup\{\| \sum_{A_j \in \Pi} \epsilon_j \mu(A_j)\| :$ Π is a finite Σ-partition of $\Omega, |\epsilon_j| \le 1\} < +\infty$ ([3], p. 4).

A measure $\mu : \Sigma \to X$ is said to be *strongly bounded* if $\mu(A_j) \to 0$ for every pairwise disjoint $\{A_j\} \subseteq \Sigma$ ([3], p. 9). Strongly bounded measures are bounded but the converse is not true.

EXAMPLE 1. Let $\Sigma = \{A \subseteq \mathbb{N} : A$ is finite or $\mathbb{N} \backslash A$ is finite$\}$ and $\mu : \Sigma \to c_0$, $\mu(A) = \chi_A$ if A is finite and $\mu(A) = -\chi_{\mathbb{N}\backslash A}$ if $\mathbb{N}\backslash A$ is finite, where χ_B is the characteristic function of $B \subseteq \mathbb{N}$. Then μ is a bounded measure but μ is not strongly bounded: $\mu(\{j\}) = \chi_{\{j\}} \not\to 0$ as $j \to +\infty$.

If X is a Banach space, then a measure $\mu : \Sigma \to X$ is of bounded variation if $|\mu|(\Omega) = \{\Sigma_{A \in \Pi} \|\mu(A)\| : \Pi$ is a finite Σ–partition of $\Omega\} < +\infty$ ([3], p. 4).

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For the variation boundedness and the strong boundedness, we have the following characterizations.

THEOREM 2. *Let Σ be an algebra of subsets of a set Ω and X a Banach space. Then the following* (A) *and* (B) *hold.*

(A) *A measure $\mu : \Sigma \to X$ is of bounded variation if and only if $\sum_{j=1}^{\infty} \|\mu(A_j)\| < +\infty$ for every pairwise disjoint sequence $\{A_j\} \subseteq \Sigma$* ([8], Lemma 4.1).

(B) *A measure $\mu : \Sigma \to X$ is strongly bounded if and only if μ is strongly additive, i.e., for every pairwise disjoint sequence $\{A_j\} \subseteq \Sigma$, the series $\sum_{j=1}^{\infty} \mu(A_j)$ converges* ([3], p. 9).

However, there is no any remarkable description for the boundedness of vector measures, though there is many uniform boundedness results for families of bounded vector measures ([1]-[3], [6], [7]).

In this note, we will show a series of clearcut characterizations for the boundedness of vector measures.

Throughout this note, Σ will denote an algebra of subsets of a set Ω.

THEOREM 3. *Let X be a locally convex space with the dual X'. For a measure $\mu : \Sigma \to X$, the following conditions are equivalent.*

(1) *μ is bounded, i.e., $\{\mu(A) : A \in \Sigma\}$ is bounded.*

(2) *For every pairwise disjoint $\{A_j\} \subseteq \Sigma$ and $\{t_j\} \in c_0$, i.e., $t_j \to 0$ in $\mathbb{C}$, the sequence $\{\sum_{j=1}^{n} t_j\mu(A_j)\}_{n=1}^{\infty}$ is Cauchy.*

(3) *For every pairwise disjoint $\{A_j\} \subseteq \Sigma$, the series $\sum_{j=1}^{\infty} \mu(A_j)$ is weakly unconditionally Cauchy, i.e., $\sum_{j=1}^{\infty} |f(\mu(A_j))| < +\infty$ for all $f \in X'$.*

(4) *For every pairwise disjoint $\{A_j\} \subseteq \Sigma$, $\mu(A_j) \to 0$ weakly.*

(5) *For every pairwise disjoint $\{A_j\} \subseteq \Sigma$, $\frac{1}{j^j}\mu(A_j) \to 0$ weakly.*

(6) *For every pairwise disjoint $\{A_j\} \subseteq \Sigma$, $\left\{\frac{1}{j^j}\mu(A_j)\right\}_{j=1}^{\infty}$ is bounded.*

Proof. (1)$\Rightarrow$(2). Suppose that (1) holds. Then for every $f \in X'$, we have that $\sup_{A\in\Sigma} |f(\mu(A))| < +\infty$. Let $A_j \in \Sigma$, $A_i \cap A_j = \emptyset$ $(i \neq j)$ and $t_j \to 0$ in $\mathbb{C}$. We claim that $K = \{\sum_{j\in\Delta} b_j\mu(A_j) : \Delta \subseteq \mathbb{N}$ finite, $|b_j| \leq 1$ for all $j\}$ is bounded. In fact, letting $\Delta_1 = \{j \in \mathbb{N} : \mathrm{Re} f(\mu(A_j)) > 0\}$, $\Delta_2 = \{j \in \mathbb{N} : \mathrm{Re} f(\mu(A_j)) < 0\}$, $\Delta_3 = \{j \in \mathbb{N} : \mathrm{Im} f(\mu(A_j)) > 0\}$ and

$\Delta_4 = \{j \in \mathbb{N} : \mathrm{Im} f(\mu(A_j)) < 0\}$ for every $f \in X'$, we have that

$$\begin{aligned}\left| f\Big(\sum_{j\in\Delta} b_j\mu(A_j)\Big)\right| &\le \sum_{j\in\Delta} |f(\mu(A_j))| \\ &\le \mathrm{Re} f\Big(\mu\Big(\bigcup_{j\in\Delta\cap\Delta_1} A_j\Big)\Big) - \mathrm{Re} f\Big(\mu\Big(\bigcup_{j\in\Delta\cap\Delta_2} A_j\Big)\Big) \\ &\quad + \mathrm{Im} f\Big(\mu\Big(\bigcup_{j\in\Delta\cap\Delta_3} A_j\Big)\Big) - \mathrm{Im} f\Big(\mu\Big(\bigcup_{j\in\Delta\cap\Delta_4} A_j\Big)\Big) \\ &\le 4 \sup_{A\in\Sigma} |f(\mu(A))|, \ \forall \Delta \subseteq \mathbb{N} \text{ finite}, \ |b_j| \le 1, \ \forall j.\end{aligned}$$

This shows that K is weakly bounded and hence bounded in X by the Mackey theorem.

Now let $\alpha_k = \sup\{|t_j| : j \ge k\}$. Then $\alpha_k \to 0$. Without loss of generality, assume that $t_j \ne 0$ for infinitely many j and hence $\alpha_k > 0$ for all k. Let U be a neighborhood of $0 \in X$. Then there is a $\delta > 0$ such that

$$\alpha \sum_{j\in\Delta} b_j\mu(A_j) \in U \text{ for all } |\alpha| \le \delta, \quad \sum_{j\in\Delta} b_j\mu(A_j) \in K.$$

Thus, there is a $k_0 \in \mathbb{N}$ such that

$$\sum_{j=k}^{m} t_j\mu(A_j) = \alpha_k \sum_{j=k}^{m} (t_j/\alpha_k)\mu(A_j) \in U$$

for all $m \ge k > k_0$, i.e., $\{\sum_{j=1}^{n} t_j\mu(A_j)\}_{n=1}^{\infty}$ is Cauchy in X.

(2)$\Rightarrow$(3). If $A_j \in \Sigma$, $A_i \cap A_j = \emptyset$ $(i \ne j)$, then (2) implies that the series $\sum_{j=1}^{\infty} t_j f(\mu(A_j))$ converges for every $f \in X'$ and $\{t_j\} \in c_0$. Therefore, $\sum_{j=1}^{\infty} |f(\mu(A_j))| < +\infty$ for all $f \in X'$.

(3)$\Rightarrow$(4)$\Rightarrow$(5)$\Rightarrow$(6) is trivial.

(6)$\Rightarrow$(1). Suppose that (6) holds but $\{\mu(A) : A \in \Sigma\}$ is not bounded. Then there is an $f \in X'$ such that $\sup_{A\in\Sigma} |f(\mu(A))| = +\infty$. Pick an $A_1 \in \Sigma$. Then

$$\sup_{A\in\Sigma, A\subseteq A_1} |f(\mu(A))| = +\infty \quad \text{or} \quad \sup_{A\in\Sigma, A\subseteq\Omega\setminus A_1} |f(\mu(A))| = +\infty.$$

In fact, if $\sup_{A\in\Sigma,A\subseteq A_1}|f(\mu(A))| = M < +\infty$ and $\sup_{A\in\Sigma,A\subseteq\Omega\backslash A_1}$ $|f(\mu(A))| = N < +\infty$, then

$$\begin{aligned}|f(\mu(A))| &\le |f(\mu(A\cap A_1))| + |f(\mu(A\cap(\Omega\backslash A_1)))|\\ &\le M+N\end{aligned}$$

for all $A\in\Sigma$ and hence $\sup_{A\in\Sigma}|f(\mu(A))| \le M+N < +\infty$, but this is impossible.

Now let $B_1 = A_1$ if $\sup_{A\in\Sigma,A\subseteq A_1}|f(\mu(A))| = +\infty$ and, otherwise, let $B_1 = \Omega\backslash A_1$. Then $\sup_{A\in\Sigma,A\subseteq B_1}|f(\mu(A))| = +\infty$ and hence there is an $A_2\subseteq B_1$ ($A_2\in\Sigma$) such that

$$|f(\mu(A_2))| > 2^2 + |f(\mu(A_1))| + |f(\mu(\Omega\backslash A_1))|.$$

Let $B_2 = A_2$ if $\sup_{A\in\Sigma,A\subseteq A_2}|f(\mu(A))| = +\infty$ and, otherwise, let $B_2 = B_1\backslash A_2$. Then $B_2\subseteq B_1$ and $\sup_{A\in\Sigma,A\subseteq B_2}|f(\mu(A))| = +\infty$. We claim that $|f(\mu(B_1\backslash B_2))| > 2^2$. In fact, if $B_1 = A_1$ and $B_2 = A_2$, then $A_2\subseteq A_1$ and hence

$$\begin{aligned}|f(\mu(B_1\backslash B_2))| &= |f(\mu(A_1)) - f(\mu(A_2))|\\ &\ge |f(\mu(A_2))| - |f(\mu(A_1)) > 2^2;\end{aligned}$$

if $B_1 = A_1$ and $B_2 = B_1\backslash A_2$, then $B_2 = A_1\backslash A_2$, $A_2\subseteq A_1$ and hence

$$|f(\mu(B_1\backslash B_2))| = |f(\mu(A_2))| > 2^2;$$

if $B_1 = \Omega\backslash A_1$ and $B_2 = A_2$, then $A_2\subseteq\Omega\backslash A_1$ and hence

$$\begin{aligned}|f(\mu(B_1\backslash B_2))| &= |f(\mu(\Omega\backslash A_1)) - f(\mu(A_2))|\\ &\ge |f(\mu(A_2))| - |f(\mu(\Omega\backslash A_1)) > 2^2;\end{aligned}$$

if $B_1 = \Omega\backslash A_1$ and $B_2 = B_1\backslash A_2$, then $B_2\subseteq B_1$, $A_2\subseteq B_1$ and hence

$$|f(\mu(B_1\backslash B_2))| = |f(\mu(A_2))| > 2^2.$$

Now observing $\sup_{A\in\Sigma,A\subseteq B_2}|f(\mu(A))| = +\infty$, we have an $A_3\subseteq B_2$ ($A_3\in\Sigma$) such that

$$|f(\mu(A_3))| > 3^3 + |f(\mu(A_2))| + |f(\mu(B_1\backslash A_2))|.$$

Let $B_3 = A_3$ if $\sup_{A\in\Sigma, A\subseteq A_3} |f(\mu(A))| = +\infty$ and, otherwise, let $B_3 = B_2\backslash A_3$. Then $B_3 \subsetneq B_2$, $\sup_{A\in\Sigma, A\subseteq B_3} |f(\mu(A))| = +\infty$ and $f(\mu(B_2\backslash B_3))| > 3^3$.

Continuing this construction inductively, we obtain a sequence $\{B_j\}$ in Σ such that

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots \quad \text{and} \quad |f(\mu(B_j\backslash B_{j+1}))| > (j+1)^{j+1}$$

for all j. However, $\{B_j\backslash B_{j+1}\}_{j=1}^{\infty}$ is a pairwise disjoint sequence in Σ and

$$\left|\frac{1}{j^j} f(\mu(B_j\backslash B_{j+1}))\right| > \frac{(j+1)^{j+1}}{j^j} > j+1$$

for all j, i.e., $\left\{\frac{1}{j^j}\mu(B_j\backslash B_{j+1})\right\}_{j=1}^{\infty}$ is not bounded. This contradicts (6). □

If a locally convex space X contains no copy of $(c_0, \|\cdot\|_\infty)$, then X has a series of very nice properties, e.g., every continuous linear operator $T: c_0 \to X$ is both compact and sequentially compact, i.e., for every bounded $B \subseteq c_0$, $\overline{T(B)}$ is both compact and sequentially compact ([5], Theorem 4). Theorem 3 implies the following characterization of the c_0-absence.

COROLLARY 4. *A sequentially complete locally convex space X contains no copy of $(c_0, \|\cdot\|_\infty)$ if and only if every bounded X-valued measure is strongly additive.*

Proof. As was stated in Example 1, if X contains a copy of $(c_0, \|\cdot\|_\infty)$, then there exists a bounded measure which is not strongly additive.

Suppose that X contains no copy of $(c_0, \|\cdot\|_\infty)$ and $\mu : \Sigma \to X$ is a bounded measure. Let $\{A_j\}$ be a pairwise disjoint sequence in Σ. By Theorem 3, the series $\sum_{j=1}^{\infty} \mu(A_j)$ is weakly unconditionally Cauchy and hence the series $\sum_{j=1}^{\infty} \mu(A_j)$ converges because X contains no copy of $(c_0, \|\cdot\|_\infty)$ ([5], Theorem 4). This shows that μ is strongly additive. □

Let X and Y be Banach spaces. For a vector measure $\mu : \Sigma \to X$ and $A \in \Sigma$, the semivariation $\|\mu\|(A)$ is defined by

$$\text{(I)} \quad \|\mu\|(A) = \sup\left\{\left\| \sum_{B\in\Pi} \epsilon_B \mu(B) \right\| : \Pi \text{ is a finite } \Sigma\text{-partition of } A, \epsilon_B \in \mathbb{C},\ |\epsilon_B| \le 1\right\} \quad ([3], \text{p. } 4).$$

This definition is reasonable because $\|\mu\|(\Omega) < +\infty$ if and only if μ is bounded, i.e., $\{\mu(A) : A \in \Sigma\}$ is bounded. However, there is another definition of semivariation for operator-valued measure $\mu : \Sigma \to L(X, Y)$ and $A \in \Sigma$ as follows:

$$\text{(II)} \quad \|\mu\|(A) = \sup\left\{\left\| \sum_{B\in\Pi} \mu(B)(f(B)) \right\| : \Pi \text{ is a finite } \Sigma\text{-partition of } A, f \in X^{\Sigma}, \|f(\cdot)\| \le 1\right\} \quad ([8], \S 4).$$

According to the semivariation (II), $\mu : \Sigma \to L(X, Y)$ is of bounded semivariation if and only if for every pairwise disjoint $\{A_j\} \subseteq \Sigma$ the series $\sum_{j=1}^{\infty} \mu(A_j)\ (x_j)$ converges for every $\{x_j\} \in c_0(X)$, i.e., $x_j \to 0$ in X ([8], Theorem 4.2).

We would like to show that semivariation (I) and (II) are different for operator-valued measures, in general.

THEOREM 5. *Let X and Y be Banach spaces. If X is infinite-dimensional, then there exists a measure $\mu : 2^{\mathbb{N}} \to L(X, Y)$ such that μ is strongly additive and hence μ is bounded, i.e., μ is of bounded semivariation in the sense of* (I) *but μ is not of bounded semivariation in the sense of* (II).

Proof. By Theorem 4 of [4], there exists a sequence $\{T_j\} \subseteq L(X, Y)$ such that the series $\sum_{j=1}^{\infty} T_j$ is subseries convergent in the operator norm but $\sup_m \|\sum_{j=1}^{m} T_j(x_j)\| = +\infty$ for some $\{x_j\} \in c_0(X)$. Define $\mu : 2^{\mathbb{N}} \to L(X, Y)$ by $\mu(A) = \sum_{j\in A} T_j$ for $A \subseteq \mathbb{N}$. Then μ is strongly additive because the series $\sum_{j=1}^{\infty} T_j$ is also unconditionally convergent ([5]). However, μ is not of bounded semivariation in the sense of (II) because the series $\sum_{j=1}^{\infty} \mu(\{j\})(x_j) = \sum_{j=1}^{\infty} T_j(x_j)$ diverges for some $\{x_j\} \in c_0(X)$. □

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References

[1] P. Antosik and C. Swartz, *The Nikodym Boundedness Theorem and the Uniform Boundedness Principle*, Lecture Notes in Math., 1033, Springer-Verlag, Heidelberg, 1983.
[2] ———, *Matrix Methods in Analysis*, Lecture Notes in Math., 1113, Springer-Verlag, Heidelberg, 1985.
[3] J. Diestel and J. Uhl, *Vector Measures*, Amer. Math. Soc., Providence, 1977.
[4] Li Ronglu, Cui Chengri and Min-Hyung Cho, *Improvements of Thorp-Rolewicz theorems on operator series*, Bull. Korean Math. Soc. **35** (1998), no. 1, 75–82.
[5] Li Ronglu and Bu Qingying, *Locally convex spaces containing no copy of* c_0, J. Math. Anal. Appl. **172** (1993), no. 1, 205–211.
[6] C. Swartz, *The evolution of the uniform boundedness principle*, Math. Chron. **19** (1990), 1–18.
[7] ———, *Infinite Matrices and the Gliding Hump*, World Sci., Singapore-New Jersey-London-HongKong, 1996.
[8] B. L. D. Thorp, *Sequential-evolution convergence*, J. London Math. Soc. **44** (1969), 201–209.

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