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GENERALIZED SOLUTION OF TIME DEPENDENT IMPULSIVE CONTROL SYSTEM CORRESPONDING TO VECTOR-VALUED CONTROLS OF BOUNDED VARIATION

CHANG EON SHIN AND JI HYUN RYU

ABSTRACT. This paper is concerned with the impulsive Cauchy problem where the control function u is a possibly discontinuous vector-valued function with finite total variation. We assume that the vector fields $f, g_i(i = 1, \cdots, m)$ are dependent on the time variable. The impulsive Cauchy problem is of the form

$$\dot{x}(t) = f(t,x) + \sum_{i=1}^{m} g_i(t,x)\dot{u}_i(t), \quad t \in [0,T], \quad x(0) = \bar{x} \in \mathbb{R}^n,$$

where the vector fields $f, g_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are measurable in t and Lipschitz continuous in x. If $g_i's$ satisfy a condition that

$$\sum_{i=1}^{m} |g_i(t_2,x) - g_i(t_1,x)| \le \phi(t_2) - \phi(t_1), \quad \forall t_1 < t_2, x \in \mathbb{R}^n,$$

for some increasing function ϕ, then the input-output function can be continuously extended to measurable functions of bounded variation.

1. Introduction

Given vector fields f, $g_i(i = 1, \cdots, m)$ defined on $\mathbb{R} \times \mathbb{R}^n$, consider

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the time dependent impulsive control system with initial value

$$(1.1_1) \qquad \dot{x}(t) = f(t,x) + \sum_{i=1}^{m} g_i(t,x)\dot{u}_i(t), \quad t \in [0,T],$$

$$(1.1_2) \qquad x(0) = \bar{x} \in \mathbb{R}^n.$$

where $\cdot = \frac{d}{dt}$ and $u(\cdot) = (u_1, \cdots, u_m)(\cdot)$ is a vector-valued control function. Assume that vector fields f and $g_i{}'s$ satisfy the following assumptions:

A1. There exists $M > 0$ such that for any $(t,x) \in \mathbb{R} \times \mathbb{R}^n$,

$$|f(t,x)| \le M, \qquad |g_i(t,x)| \le M.$$

A2. For each $x \in \mathbb{R}^n$, the map $t \mapsto f(t,x)$ and $t \mapsto g(t,x)$ measurable, g_i are C^1 w.r.t x and there exists $L > 0$ such that for any $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$,

$$|f(t,x) - f(t,y)| \le L|x-y|, \quad |g_i(t,x) - g_i(t,y)| \le L|x-y|.$$

A3. There is an increasing function ϕ defined on $[0,T]$ such that

$$\sum_{i=1}^{m} |g_i(t_2,x) - g_i(t_1,x)| \le \phi(t_2) - \phi(t_1), \quad \forall t_1 < t_2, x \in \mathbb{R}^n.$$

By possibly replacing ϕ with

$$\tilde{\phi}(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 + t + \phi(t) & \text{if } 0 \le t < T, \\ 2 + T + \phi(T) & \text{if } t \ge T, \end{cases}$$

it is not restrictive to assume that

$$\phi(0+) - \phi(0-) \ge 1, \quad \phi(T+) - \phi(T-) \ge 1, \quad \dot{\phi}(t) \ge 1 \text{ a.e.}$$

For $K > 0$, let

$S_K = \{u = (u_1, \cdots, u_m) : [0,T] \to \mathbb{R}^m|$ each u_i is a piecewise constant function such that u is right continuous, the discontinuities of do not happen at the discontinuities of ϕ and the total variation of u is less than or equal to $K\}$.

For a C^1- control function u, we write by $x_u(\cdot)$ the Carathéodory solution of (1.1), that is, x_u is an absolutely continuous function and satisfies (1.1) a.e. In §2, depending on the canonical graph completion of u ([4]) which provides a definition of generalized solutions for an autonomous impulsive control systems corresponding to vector-valued control functions of bounded variation, we define a generalized solution $x(u,t)$ of (1.1) corresponding to u in S_K. In Theorem 1.1, we show that $\psi : u \to x_u(t)$ is Lipschitz continuous on the set of C^1- functions whose total variations are uniformly bounded and by the continuity of ψ, we can define the generalized solution $x(u,t)$ of (1.1) corresponding to a measurable function u having total variation less than or equal to K.

THEOREM 1.1. *We assume* A1, A2, *and* A3. *There exists* $\bar{M} > 0$ *such that for any vector valued* C^1- *control functions* u, v *whose total variations are less than or equal to* K

$$|x_u(T) - x_v(T)| \leq \bar{M} \int_0^T |u(s) - v(s)| d\phi(s). \tag{1.2}$$

Inequality (1.2) makes it possible to extend the map $u \to x(u,T)$ continuously to discontinuous controls of bounded variation and define the corresponding generalized solution of (1.1).

DEFINITION 1.2. Let f and g_i's satisfy A1, A2, and A3. For a measurable control function u having a finite total variation, a generalized solution $x(u,t)$ is defined as the limit of the Carathéodory solutions of (1.1) corresponding to C^1- control functions u_n which converge to u in $L^1(d\phi)$, that is, if $\{u_n\}$ is a sequence of C^1-functions tending to u in $L^1(d\phi)$,

$$x(u,t) = \lim_{n \to \infty} x_{u_n}(t).$$

2. Definition of Generalized Solution Corresponding to Step Functions

The aim of this chapter is to define the generalized solution $x(u,t)$ corresponding to $u \in S_K$ of (1.1) depending on the definition of the generalized solution for the autonomous impulsive control system in [4].

Denote by $e^{s\sum_{i=1}^m \varepsilon_i g_i(t)}z$ the value at time s of the Cauchy problem

$$\frac{dw}{ds} = \sum_{i=1}^m \varepsilon_i g_i(t, w(s)), \quad w(0) = z,$$

and in particular, at $s = 1$, we write

$$\hat{e}^{\sum_{i=1}^m \varepsilon_i g_i(t)}z = e^{1\sum_{i=1}^m \varepsilon_i g_i(t)}z.$$

Let $s \to \Phi(s, \sum_{i=1}^m \varepsilon_i g_i(t), z)$ be the fundamental matrix solution of the linear system

$$\dot{v}(s) = D_x \sum_{i=1}^m \varepsilon_i g_i(t, e^{s\sum_{i=1}^m \varepsilon_i g_i(t) z}) \cdot v(s),$$

where $D_x(\sum_{i=1}^m \varepsilon_i g_i(t,\cdot))$ is the Jacobian matrix of the first derivatives of the map $x \to \sum_{i=1}^m \varepsilon_i g_i(t,x)$ and $\Phi(0, \sum_{i=1}^m \varepsilon_i g_i(t), z)$ is the identity matrix.

Let $u \in S_K$. For $t \in [0,T]$, let $V(t)$ be the total variation of u on $[0,t]$. Since the function $t \to t + V(t)$ is right continuous and strictly increasing, for every $s \in [0, T + V(T)]$, there is a unique $t = t(s)$ such that

$$s \le t + V(t), \ \bar{t} + V(\bar{t}) < s, \ \forall \bar{t} < t. \tag{2.1}$$

Write

$$u^-(t) = \lim_{\bar{t}\to t^-} u(\bar{t}), \quad V^-(t) = \lim_{\bar{t}\to t^-} V(\bar{t}),$$

$$W(t) = t + V(t), \quad W^-(t) = t + V^-(t).$$

We recall the canonical graph completion defined in [4]. The canonical graph completion is the map $\varphi = (\varphi_0, \cdots, \varphi_m) : [0, W(T)] \to [0,T] \times \mathbb{R}^m$ defined by $\varphi(s) = (t(s), v(s))$, where $t(s)$ is defined in (2.1) and $v(s)$ is $u(t(s))$ if u is continuous at $t(s)$, while $v(s) = \frac{s-W^-(t)}{W(t)-W^-(t)}u(t) + \frac{W(t)-s}{W(t)-W^-(t)}u^-(t)$ if the jump of u occurs at $t = t(s)$. The above map φ is Lipschitz continuous with Lipschitz constant 1.

We write the characteristic function on I by χ_I. For $u \in S_K$, if $u = \sum_{i=1}^k d_i \chi_{I_i}$, where $I_i = [t_{i-1}, t_i)$ for $i = 1, \cdots, k-1$, $I_k = [t_{k-1}, t_k]$ and $0 = t_0 < \cdots < t_k = T$, we say that $\sum_{i=1}^k d_i \chi_{I_i}$ is a *standard representation* of u. Choose c_i in (t_{i-1}, t_i) for $i = 2, \cdots, k-1$ and put $c_1 = 0, c_k = T$. Write $J_i = [c_i, c_{i+1})$ for $i = 1, \cdots, k-2$ and $J_{k-1} = [c_{k-1}, T]$. Let $\varphi = (\varphi_0, \cdots, \varphi_m)$ be the canonical graph completion of u and for $i = 1, \cdots, k$, let each τ_k be a point in $[0, T+V(T)]$ with $\varphi_0(\tau_k) = c_k$. Write $J'_i = [\tau_i, \tau_{i+1}), (i = 1, \cdots, k-2)$ and $J'_{k-1} = [\tau_{k-1}, \tau_k]$.

Note that on J_i, the system

$$\dot{x} = f(t, x) + \sum_{j=1}^{m} g_j(t, x)\dot{u}_j \tag{2.2}$$

is equivalent to

$$\dot{x} = f(t, x) + \sum_{j=1}^{m} g_j(t_i, x)\dot{u}_j \tag{2.3}$$

since $\dot{u}_j = 0$, on $J_i \setminus \{t_i\}$.

If $x(c_i) = x_i$ is given, then we can define the generalized solution $x(u,t)$ of (2.3) corresponding to u on J_i as $x(u,t) = y(\varphi_0^+(s))$, where $\varphi_0^+(s) = \max\{s : t = \varphi_0(s)\}$ and $y(s)$ is the solution of the Cauchy problem on the interval J'_i,

$$\begin{aligned} \frac{d}{ds}y(s) =& f(\varphi_0(s), y(s))\frac{d}{ds}\varphi_0(s) + \sum_{j=1}^{m} g_j(t_i, y(s))\frac{d}{ds}\varphi_j(s), \\ & s \in J'_i, \quad y(\tau_i) = x_i. \end{aligned} \tag{2.4}$$

Hence we define the generalized solution $x(u,t)$ on [0,T] corresponding to $u \in S_K$.

DEFINITION 2.1. For $u \in S_K$. Let $y(s)$ be the Carathéodory solution of the Cauchy problem

$$\frac{d}{ds}y(s) = f(\varphi_0(s), y(s))\frac{d}{ds}\varphi_0(s) + \sum_{j=1}^{m} g_j(t_i, y(s))\frac{d}{ds}\varphi_j(s), \tag{2.5}$$
$$s \in J_i', \ (i = 1, \cdots, k-1), \quad y(0) = \bar{x},$$

where φ is the canonical graph completion of u. The map $t \mapsto x(u,t)$ defined by $x(u,t) = y(\varphi_0^+(s))$ is called the generalized solution corresponding to u of the Cauchy problem

$$\dot{x} = f(t,x) + \sum_{i=1}^{m} g_i(t,x)\dot{u}_i \qquad x(0) = \bar{x}, \tag{2.6}$$

where $\varphi_0^+(s) = \max\{s : t = \varphi_0(s)\}$.

Depending on the above definition, we claim that (1.2) holds for $u, v \in S_K$ and for $u, v \in C^1$ in the main theorem, and deduce Definition 1.2.

For $u \in S_K$, by the above definition, $x(u,\cdot)$ is right continuous and for any $t \in [0,T]$ write $x(u,t-) = \lim_{s\to t-} x(u,s)$. The left and the right limits of the generalized solutions are uniquely determined and satisfy the property (2.7).

PROPOSITION 2.2. *For $u \in S_K$, let $\sum_{i=1}^{k} d_i\chi_{I_i}$ be a standard representation of u for some $0 = t_0 < \cdots < t_k = T$. For each $j = 1, \cdots, k-1$,*

$$x(u,t_j) = \hat{e}^{\sum_{i=1}^{m}(u_i(t_j) - u_i(t_j-))g_i(t_j)}x(u,t_j-). \tag{2.7}$$

Proof. Let $\varphi = (\varphi_0, \cdots, \varphi_m)$ be the canonical graph completion of u and let $y(s)$ be the solution of (2.5) corresponding to φ. If $t_j + V^-(t_j) = s_1$ and $t_j + V(t_j) = s_2$, then

$$s_2 - s_1 = W(t_j) - W(t_j-), \ \ x(u,t_j) = y(s_2), \ \ x(u,t_j-) = y(s_1)$$

and

$$x(u,t_j) - x(u,t_j-) = y(s_2) - y(s_1). \tag{2.8}$$

Observing that on the interval $[s_1, s_2]$, $\varphi_0(s) = t_j$, $\frac{d}{ds}\varphi_0(s) = 0$ and $\frac{d}{ds}\varphi_i(s) = \frac{u_i(t_j)-u_i^-(t_j)}{W_i(t_j)-W_i^-(t_j)}$, for any $s \in [s_1, s_2]$

$$(2.9) \qquad y(s) - y(s_1) = \int_{s_1}^{s} \sum_{i=1}^{m} g_i(t_j, y(\tau)) \frac{d}{d\tau}\varphi_i(\tau) d\tau.$$

Define a map $z : [0,1] \to \mathbb{R}^n$ by $z(\sigma) = y(s_1 + \sigma(s_2 - s_1))$. By (2.9),

$$(2.10) \qquad \frac{d}{d\sigma}z(\sigma) = \sum_{i=1}^{m} g_i(t_j, z(\sigma))(u_i(t_j) - u_i^-(t_j)), \quad z(0) = y(s_1)$$

and

$$z(1) = \hat{e}^{\sum_{i=1}^{m}(u_i(t_j)-u_i^-(t_j))g_i(t_j)}z(0).$$

Since $z(1) = y(s_2)$ and $z(0) = x(u, t_j-)$, (2.7) holds. □

3. Some Lemmas

Denote by $\tilde{e}^{sf(\tau)}\bar{x}$ the value at time $\tau + s$ of the Cauchy problem

$$\dot{x}(t) = f(t, x(t)), \quad x(\tau) = \bar{x}.$$

To prove the main theorem, we need to prove inequality (3.1). To do this, we need:

LEMMA 3.1. *For any σ_1, $t_1 \in [0,T]$ and $0 \le s \le t \le T$, let* $v(t) = \Phi(t, \sum_{i=1}^{m} \varepsilon_i g_i(t_1), \tilde{e}^{\sigma_1 f(0)}\bar{x}) f(\sigma_1, \tilde{e}^{\sigma_1 f(0)})$.

(i) $|v(t)| \le Me^{\sum_{i=1}^{m}|\varepsilon_i| n^2 LT}$.
(ii) $|v(t) - v(s)| \le |t-s| \sum_{i=1}^{m} |\varepsilon_i| n^2 LMe^{\sum_{i=1}^{m}|\varepsilon_i| n^2 LT}$.

Proof. *(i)* Observing that

$$\frac{d}{dt}|v(t)| \le |D_x \sum_{i=1}^{m} \varepsilon_i g_i(t_1, e^{t\sum_{i=1}^{m}\varepsilon_i g_i(t_1)}\tilde{e}^{\sigma_1 f(0)}\bar{x}) \cdot v(t)|$$
$$\le \sum_{i=1}^{m} |\varepsilon_i| n^2 L |v(t)|,$$

by Gronwall's inequality, we have

$$\begin{aligned}|v(t)| &\le |f(\sigma_1, \tilde{e}^{\sigma_1 f(0)}\bar{x})| e^{\sum_{i=1}^m |\varepsilon_i| n^2 LT} \\ &\le M e^{\sum_{i=1}^m |\varepsilon_i| n^2 LT}.\end{aligned}$$

(ii)

$$\begin{aligned}|v(t)-v(s)| &= \left| \int_s^t D_x \sum_{i=1}^m \varepsilon_i g_i(t_1, e^{\sigma \sum_{i=1}^m \varepsilon_i g_i} \tilde{e}^{\sigma_1 f(0)}\bar{x}) \cdot v(\sigma) d\sigma \right| \\ &\le \int_s^t \sum_{i=1}^m |\varepsilon_i| n^2 LM e^{\sum_{i=1}^m |\varepsilon_i| nLT} d\sigma \\ &\le |t-s| \sum_{i=1}^m |\varepsilon_i| n^2 LM e^{\sum_{i=1}^m |\varepsilon_i| n^2 LT}.\end{aligned}$$

□

LEMMA 3.2. *There exists a positive constant C such that for any $0 < t_1 < t_2 < T$, $\bar{x} \in \mathbb{R}^n$, and $|\varepsilon_i| \le K$, $i = 1, \cdots, m$,*

$$\begin{aligned}&|\tilde{e}^{(t_2-t_1)f(t_1)} \hat{e}^{\sum_{i=1}^m \varepsilon_i g_i(t_1)} \bar{x} - \hat{e}^{\sum_{i=1}^m \varepsilon_i g_i(t_2)} \tilde{e}^{(t_2-t_1)f(t_1)} \bar{x}| \\ &\le C \sum_{i=1}^m |\varepsilon_i| |\phi(t_2) - \phi(t_1)|.\end{aligned} \tag{3.1}$$

Proof. Let $0 < t_1 < t_2 < T$, $\bar{x} \in \mathbb{R}^n$ and $|\varepsilon_i| \le K$, $i = 1, \cdots, m$. Note that

$$\begin{aligned}&|\tilde{e}^{(t_2-t_1)f(t_1)} \hat{e}^{\sum_{i=1}^m \varepsilon_i g_i(t_1)} \bar{x} - \hat{e}^{\sum_{i=1}^m \varepsilon_i g_i(t_2)} \tilde{e}^{(t_2-t_1)f(t_1)} \bar{x}| \\ &\le |\tilde{e}^{(t_2-t_1)f(t_1)} \hat{e}^{\sum_{i=1}^m \varepsilon_i g_i(t_1)} \bar{x} - \hat{e}^{\sum_{i=1}^m \varepsilon_i g_i(t_1)} \tilde{e}^{(t_2-t_1)f(t_1)} \bar{x}| \\ &\quad + |\hat{e}^{\sum_{i=1}^m \varepsilon_i g_i(t_1)} \tilde{e}^{(t_2-t_1)f(t_1)} \bar{x} - \hat{e}^{\sum_{i=1}^m \varepsilon_i g_i(t_2)} \tilde{e}^{(t_2-t_1)f(t_1)} \bar{x}| \\ &\doteq E_1 + E_2.\end{aligned} \tag{3.2}$$

We show that there exists $C_1, C_2 > 0$ such that

$$E_i \le C_i \sum_{i=1}^m |\varepsilon_i| |\phi(t_2) - \phi(t_1)|, \; i = 1, 2.$$

Define a map $h : [0,T] \times [0,1] \to \mathbb{R}^n$ by

$$h(\sigma_1,\sigma_2) = \tilde{e}^{\sigma_1 f(t_1)} e^{\sigma_2 \sum_{i=1}^m \varepsilon_i g_i(t_1)} \bar{x} - e^{\sigma_2 \sum_{i=1}^m \varepsilon_i g_i(t_1)} \tilde{e}^{\sigma_1 f(t_1)} \bar{x}.$$

We have

$$\begin{aligned}\frac{\partial h}{\partial \sigma_1} =& f(t_1+\sigma_1, \tilde{e}^{\sigma_1 f(t_1)} e^{\sigma_2 \sum_{i=1}^m \varepsilon_i g_i(t_1)} \bar{x}) \\ & - \Phi\left(\sigma_2, \sum_{i=1}^m \varepsilon_i g_i(t_1), \tilde{e}^{\sigma_1 f(t_1)} \bar{x}\right) f(\sigma_1, \tilde{e}^{\sigma_1 f(t_1)} \bar{x}),\end{aligned}$$

and by Lemma 3.1,

$$\begin{aligned}&\left|\frac{\partial h}{\partial \sigma_1}(\sigma_1,\sigma_2) - \frac{\partial h}{\partial \sigma_1}(\sigma_1,\sigma_2{}')\right| \\ &\leq |f(t_1+\sigma_1, \tilde{e}^{\sigma_1 f(t_1)} e^{\sigma_2 \sum_{i=1}^m \varepsilon_i g_i(t_1)} \bar{x}) \\ &\quad - f(t_1+\sigma_1, \tilde{e}^{\sigma_1 f(t_1)} e^{\sigma_2{}' \sum_{i=1}^m \varepsilon_i g_i(t_1)} \bar{x})| \\ &\quad + |\Phi(\sigma_2, \sum_{i=1}^m \varepsilon_i g_i(t_1), \tilde{e}^{\sigma_1 f(t_1)} \bar{x}) f(\sigma_1, \tilde{e}^{\sigma_1 f(t_1)} \bar{x}) \\ &\quad - \Phi(\sigma_2{}', \sum_{i=1}^m \varepsilon_i g_i(t_1), \tilde{e}^{\sigma_1 f(t_1)} \bar{x}) f(\sigma_1, \tilde{e}^{\sigma_1 f(t_1)} \bar{x})| \\ &\leq L|\tilde{e}^{\sigma_1 f(t_1)} e^{\sigma_2 \sum_{i=1}^m \varepsilon_i g_i(t_1)} \bar{x} - \tilde{e}^{\sigma_1 f(t_1)} e^{\sigma_2{}' \sum_{i=1}^m \varepsilon_i g_i(t_1)} \bar{x}| \\ &\quad + |\sigma_2 - \sigma_2{}'| \sum_{i=1}^m |\varepsilon_i| n^2 L M e^{\sum_{i=1}^m |\varepsilon_i| n^2 LT} \\ &\leq |\sigma_2 - \sigma_2{}'| \sum_{i=1}^m |\varepsilon_i| LM(e^{LT} + n^2 e^{\sum_{i=1}^m |\varepsilon_i| n^2 LT}).\end{aligned}$$

Since $h(0,1)=0$ and $\frac{\partial h}{\partial \sigma_1}(\tau,0)=0$ for any $\tau \in [0,T]$,

$$\begin{aligned} h(t_2-t_1,1) &= h(t_2-t_1,1)-h(0,1) \\ &= \left|\int_0^{t_2-t_1} \frac{\partial h}{\partial \sigma_1}(\tau,1)d\tau\right| \\ &\le \int_0^{t_2-t_1}\left|\frac{\partial h}{\partial \sigma_1}(\tau,1)-\frac{\partial h}{\partial \sigma_1}(\tau,0)\right|d\tau \\ &\le |t_2-t_1|\sum_{i=1}^{m}|\varepsilon_i|LM(e^{LT}+n^2e^{\sum_{i=1}^{m}|\varepsilon_i|n^2LT}). \end{aligned}$$

Hence

$$|E_1| \le C_1\sum_{i=1}^{m}|\varepsilon_i||\phi(t_2)-\phi(t_1)|, \tag{3.3}$$

where $C_1 = LM(e^{LT}+n^2e^{mKn^2L})$.

Next, we show that there exists $C_2>0$ such that

$$|E_2| \le C_2\sum_{i=1}^{m}|\varepsilon_i||\phi(t_2)-\phi(t_1)|.$$

Let $x_0 = \tilde{e}^{(t_2-t_1)f(t_1)}\bar{x}$ and let each $x_j, (j=1,2)$ is the solution of the Cauchy problem

$$\dot{x} = \sum_{i=1}^{m}\varepsilon_i g_i(t_j,x),\ x(0)=x_0.$$

Then $|E_2| = |x_1(1)-x_2(1)|$. Since

$$\begin{aligned} &|\dot{x_1}-\dot{x_2}| \\ &= \left|\sum_{i=1}^{m}\varepsilon_i g_i(t_1,x_1)-\sum_{i=1}^{m}\varepsilon_i g_i(t_2,x_2)\right| \\ &= \left|\sum_{i=1}^{m}\varepsilon_i g_i(t_1,x_1)-\sum_{i=1}^{m}\varepsilon_i g_i(t_2,x_1)\right|+\left|\sum_{i=1}^{m}\varepsilon_i g_i(t_2,x_1)-\sum_{i=1}^{m}\varepsilon_i g_i(t_2,x_2)\right| \\ &\le \sum_{i=1}^{m}|\varepsilon_i||\phi(t_2)-\phi(t_1)|+\sum_{i=1}^{m}|\varepsilon_i|L|x_2-x_1|, \end{aligned}$$

by Gronwall's inequality,

$$
(3.4)\qquad
\begin{aligned}
|x_1(1) - x_2(1)| &\le \int_0^1 \sum_{i=1}^{m} |\varepsilon_i||\phi(t_2) - \phi(t_1)| e^{\int_s^1 \sum_{i=1}^m |\varepsilon_i| L d\sigma} ds \\
&= C_2 \sum_{i=1}^{m} |\varepsilon_i||\phi(t_2) - \phi(t_1)|,
\end{aligned}
$$

where $C_2 = e^{mKL}$. By (3.2), (3.3) and (3.4), lemma holds with $C = C_1 + C_2$. □

4. Proof of Theorem 1.1

We show that inequality (1.2) holds for $u, v \in S_K$ and then show that it holds for C^1−functions u, v. Let

$$u(t) = (u_1, \cdots, u_m)(t) = \sum_{i=1}^{N} (\alpha_{i,1}, \cdots, \alpha_{i,m}) \chi_{I_i}(t)$$

and

$$v(t) = (v_1, \cdots, v_m)(t) = \sum_{i=1}^{N} (\beta_{i,1}, \cdots, \beta_{i,m}) \chi_{I_i}(t),$$

where $I_i = [t_{i-1}, t_i), i = 1, \cdots, N-1$, $I_N = [t_{N-1}, t_N]$, and $0 = t_0 < \cdots < t_N = T$. For $i = 1, \cdots, N-1$ and $j = 1, \cdots, m$, put

$$k_{i,j} = \alpha_{i+1,j} - \alpha_{i,j},\ \ s_{i,j} = \beta_{i+1,j} - \beta_{i,j},$$

and for $i = 1, \cdots, N$ and $j = 1, \cdots, m$, put

$$l_i = t_i - t_{i-1},\ \ d_{i,j} = \alpha_{i,j} - \beta_{i,j}.$$

For each $i = 1, \cdots, N-1, j = 1, \cdots, m$, $|k_{i,j}| \le K$, $|s_{i,j}| \le K$. By Proposition 2.2, the solution $x(u,t)$ of (1.1) corresponding to u at the points of discontinuities satisfies

$$x(u, t_i) = e^{\sum_{j=1}^m (u_j(t_i) - u_j^-(t_i)) g_j(t_i)} x(u, t_i-).$$

By Definition 2.1, $x(u,T)$, and $x(v,T)$ can be written as

$$x(u,T) = \tilde{e}^{l_N f(t_{N-1})} \hat{e}^{\sum_{i=1}^{m} k_{N-1,i} g_i(t_{N-1})} \cdot \\ \tilde{e}^{l_{N-1} f(t_{N-2})} \ldots \tilde{e}^{l_2 f(t_1)} \hat{e}^{\sum_{i=1}^{m} k_{1,i} g_i(t_1)} \tilde{e}^{l_1 f(t_0)} \bar{x},$$

and

$$x(v,T) = \tilde{e}^{l_N f(t_{N-1})} \hat{e}^{\sum_{i=1}^{m} s_{N-1,i} g_i(t_{N-1})} \cdot \\ \tilde{e}^{l_{N-1} f(t_{N-2})} \ldots \tilde{e}^{l_2 f(t_1)} \hat{e}^{\sum_{i=1}^{m} s_{1,i} g_i(t_1)} \tilde{e}^{l_1 f(t_0)} \bar{x}.$$

By Lemma 3.2 and following similar computation of the proof of Theorem 2.3 in [12], there exists $\bar{M} > 0$ so that

$$|x(u,T) - x(v,T)| \le \bar{M} \int_0^T |u(s) - v(s)| d\phi(s).$$

Next we claim that inequality (1.2) holds for C^1−functions u, v whose total variations are bounded by K. Suppose that for any $u \in C^1$, there exists a sequence $\{u_n\}$ in S_K such that $u_n \to u$ in $L^1(d\phi)$ and $x(u_n,T) \to x_u(T)$ as $n \to \infty$. For u and $v \in C^1$, we have sequences $\{u_n\}$ and $\{v_n\}$ so that $u_n \to u, v_n \to v$ in $L^1(d\phi)$, $x(u_n,T) \to x_u(T)$ and $x(v_n,T) \to x_v(T)$ as $n \to \infty$. We thus have that for any $n \in \mathbb{N}$

$$\begin{aligned} &|x_u(T) - x_v(T)| \\ &\le |x_u(T) - x(u_n,T)| + |x_v(T) - x(v_n,T)| + |x(u_n,T) - x(v_n,T)| \\ &\le |x_u(T) - x(u_n,T)| + |x_v(T) - x(v_n,T)| \\ &\quad + \bar{M} \int_0^T |u_n(s) - v_n(s)| d\phi(s) \end{aligned}$$

and take $n \to \infty$ to get

$$|x_u(T) - x_v(T)| \le \bar{M} \int_0^T |u(s) - v(s)| d\phi(s).$$

Hence we only have to show that for any $u \in C^1$, there exists a sequence $\{u_n\}$ in S_K such that $u_n \to u$ in $L^1(d\phi)$ and $x(u_n, T) \to x_u(T)$ as $n \to \infty$.

Let $u : [0,T] \to \mathbb{R}^m$ be a continuously differentiable map and let $\{u_n\}$ be a sequence of piecewise constant maps on $[0,T]$ such that u_n converges uniformly to u and the total variations of u_n are uniformly bounded by K. There exists $K_1 > 0$ so that for any $t \in [0,T]$, $|\dot{u}(t)| \leq K_1$. Write $u_n = \sum_{k=1}^{\delta(n)} d_{n,k}\chi_{I_{n,k}}$ where $I_{n,k} = [t_{n,k-1}, t_{n,k})$ for $k = 1, \cdots, \delta(n) - 1$ and $I_{n,\delta(n)} = [t_{n,\delta(n)-1}, T]$, and $0 = t_{n,0} < t_{n,1} < \cdots < t_{n,\delta(n)} = T$. Choose $c_{n,k}$ in $(t_{n,k-1}, t_{n,k})$ for $k = 2, \cdots, \delta(n) - 1$ and put $c_{n,1} = 0$ and $c_{n,\delta(n)} = T$. Write $J_{n,k} = [c_{n,k}, c_{n,k+1})$ for $k = 1, \cdots, \delta(n) - 2$ and $J_{n,\delta(n)-1} = [c_{n,\delta(n)-1}, c_{n,\delta(n)}]$. Let $\psi_n(s) = (\psi_n^0(s), \bar{\psi}_n(s)) \in \mathbb{R}^{m+1}$ be the canonical graph completion of u_n, where $\bar{\psi}_n(s) = (\psi_n^1(s), \cdots, \psi_n^m(s))$. Then ψ_n are Lipschitz continuous with constant 1. For $k = 1, \cdots, \delta(n)$, let $\tau_{n,k}$ be a point in $[0, S_n]$ with $\psi_n^0(\tau_{n,k}) = c_{n,k}$. If ψ_n is defined on $[0, S_n] \subset [0, T + K]$, by defining $\psi_n(s) = \psi_n(s_n)$ for $s \geq s_n$, we may assume that ψ_n are defined on [0, T+K]. Write $J'_{n,i} = [\tau_i, \tau_{i+1})$ $(i = 1, \cdots, \delta(n) - 2)$ and $J'_{n,\delta(n)-1} = [\tau_{\delta(n)-1}, T + K]$. If $x_n(u_n, t)$ is the generalized solution of the Cauchy problem

$$\dot{x} = f(t,x) + \sum_{i=1}^{m} g_i(t,x)\dot{u}_n, \quad x(0) = \bar{x}, \tag{4.1}$$

then it is the generalized solution of the Cauchy problem

$$\begin{gathered} \dot{x} = f(t,x) + \sum_{i=1}^{m} g_i(t_k,x)\dot{u}_n, \\ t \in J_{n,k},\ k = 1, \cdots, \delta(n) - 1, \quad x(0) = \bar{x}. \end{gathered} \tag{4.2}$$

Let y_n be the Carathéodory solution of the Cauchy problem.

$$\begin{gathered} \frac{d}{ds} y_n(s) = f(\psi_n^0(s), y_n(s)) \frac{d}{ds}\psi_n^0(s) + \sum_{i=1}^{m} g_i(t_k, y_n(s)) \frac{d}{ds}\psi_n^i(s), \\ s \in J'_{n,k},\ k = 1, \cdots, \delta(n) - 1, \quad y_n(0) = \bar{x}. \end{gathered} \tag{4.3}$$

Due to the Lipschitz continuity of ψ_n with Lipschitz constant 1,

$$\left|\frac{d}{ds}y_n\right| \leq M(1+mM). \tag{4.4}$$

Let $\tilde{x}_n$ be the Carathéodory solution of the Cauchy problem

$$\begin{gathered}\dot{\tilde{x}}_n = f(t,\tilde{x}_n) + \sum_{i=1}^{m} g_i(t_k,\tilde{x}_n)\dot{u}(t), \\ t \in J_{n,k},\ k = 1,\cdots,\delta(n)-1, \quad \tilde{x}(0) = \bar{x}\end{gathered} \tag{4.5}$$

and let $\tilde{x}$ be the Carathéodory solution of the Cauchy problem.

$$\dot{\tilde{x}} = f(t,\tilde{x}) + \sum_{i=1}^{m} g_i(t,\tilde{x})\dot{u}(t), \quad x(0) = \bar{x}. \tag{4.6}$$

Put $\tilde{y}_n(s) = \tilde{x}_n(\psi_n^0(s))$ and $\tilde{\varphi}_n(s) = (\tilde{\varphi}_n^1(s),\cdots,\tilde{\varphi}_n^m(s)) = u(\psi_n^0(s))$. Then $\tilde{\varphi}_n$ is Lipschitz continuous with constant K_1 and $\tilde{y}_n(s)$ satisfies the ordinary differential equation

$$\frac{d}{ds}\tilde{y}_n(s) = f(\psi_n^0(s),\tilde{y}_n(s))\frac{d}{ds}\psi_n^0(s) + \sum_{i=1}^{m} g_i(t_k,\tilde{y}_n(s))\frac{d}{ds}\tilde{\varphi}_n^i(s),$$

$$s \in J'_{n,k} \quad k = 1,\cdots,\delta(n)-1, \quad \tilde{y}_n(0) = \bar{x}.$$

The theorem is proved by showing that

$$\lim_{n\to\infty} |\tilde{x}_n(t) - \tilde{x}(t)| = 0 \quad \text{for any } t \in [0,T] \tag{4.7}$$

and

$$\lim_{n->\infty} |\tilde{y}_n(s) - y_n(s)| = 0 \quad \text{for any } s \in [0,T+K]. \tag{4.8}$$

Since u is uniformly continuous and u_n converges uniformly to u, we may assume that for any $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that $|u_n(s)-$

$u(t)| < \varepsilon$ whenever $n \geq n(\varepsilon)$, and $s, t \in J_{n,k}, k \in \{1, \cdots, \delta(n)-1\}$. Thus for any $\varepsilon > 0$, $n \geq n(\varepsilon)$ and $s \in [0, T+K]$,

$$|\bar{\psi}_n(s) - \tilde{\varphi}_n(s)| < \varepsilon. \tag{4.9}$$

Moreover since ϕ is an integrable function by choosing u_n so that $\max\{t_{n,k} - t_{n,k-1} : k = 1, \cdots, \delta(n)\}$ are sufficiently small, we may assume that $\phi_n(t) \to \phi(t)$ in $L^1(dm)$, where $\phi_n(t) = \sum_{k=1}^{\delta(n)-1} \phi(t_k)\chi_{J_{n,k}}$, and dm is the Lebesque measure. We claim that (4.7) holds. Let $n \geq n(\varepsilon)$ and $t' \in J_{n,k}$. For simplicity, we may assume that $t' = c_{n,k}$. We compute a bound for $|\tilde{x}_n(t') - \tilde{x}(t')|$ to get

$$\begin{aligned}|\tilde{x}_n(t') - \tilde{x}(t')| \leq &\int_0^{t'} |f(t, \tilde{x}_n(t)) - f(t, \tilde{x}(t))| dt \\ &+ \sum_{j=1}^{k} \int_{J_{n,j}} \sum_{i=1}^{m} |g_i(t_j, \tilde{x}_n(t)) - g_i(t, \tilde{x}(t))||\dot{u}(t)| dt \\ \doteq &E_1 + E_2,\end{aligned}$$

where $E_1 \leq \int_0^{t'} L|\tilde{x}_n(t) - \tilde{x}(t)| dt$ and

$$\begin{aligned}E_2 \leq &\sum_{j=1}^{k} \int_{J_{n,j}} \sum_{i=1}^{m} |g_i(t_j, \tilde{x}_n(t)) - g_i(t, \tilde{x}_n(t))||\dot{u}(t)| dt \\ &+ \sum_{j=1}^{k} \int_{J_{n,j}} \sum_{i=1}^{m} |g_i(t, \tilde{x}_n(t)) - g_i(t, \tilde{x}(t))||\dot{u}(t)| dt \\ \leq &\sum_{j=1}^{k} \int_{J_{n,j}} \left[K_1|\phi(t_j) - \phi(t)| + K_1 L|\tilde{x}_n(t) - \tilde{x}(t)|\right] dt \\ \leq &K_1 \int_0^T |\phi_n(t) - \phi(t)| dt + \int_0^t K_1 L|\tilde{x}_n(t) - \tilde{x}(t)| dt.\end{aligned}$$

Thus by Gronwall's inequality,

$$|\tilde{x}_n(t') - \tilde{x}(t')| \leq K_1 e^{L(K_1+1)T} \int_0^T |\phi_n(t) - \phi(t)| dt.$$

Hence $\tilde{x}_n$ converges uniformly to $\tilde{x}$ and (4.7) holds. Next, we show that (4.8) holds. Let $\varepsilon > 0$, $n > n(\varepsilon)$ and let $\tau \in J'_{n,k}$ for some k. For simplicity, we assume that $\tau = \tau_{n,k}$. Then

(4.10)
$$\begin{aligned} &|y_n(\tau) - \tilde{y}_n(\tau)| \\ &\le \left|\sum_{j=1}^{k} \int_{J'_{n,j}} [f(\psi_n^0(s), y_n(s)) - f(\psi_n^0(s), \tilde{y}_n(s))] \frac{d}{ds}\psi_n^0(s) ds\right| \\ &\quad + \left|\sum_{j=1}^{k} \int_{J'_{n,j}} \left[\sum_{i=1}^{m} g_i(t_j, y_n(s)) \frac{d}{ds}\psi_n^i(s) - \sum_{i=1}^{m} g_i(t_j, \tilde{y}_n(s)) \frac{d}{ds}\tilde{\varphi}_n^i(s)\right] ds\right| \\ &\doteq D_1 + D_2. \end{aligned}$$

By Lipschitz continuity of ψ_n^0 with constant 1,

$$\begin{aligned} D_1 &\le \sum_{j=1}^{k} \int_{J'_{n,j}} L|y_n(s) - \tilde{y}_n(s)| ds \\ &= \int_0^\tau L|y_n(s) - \tilde{y}_n(s)| ds \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} D_2 &\le \left|\sum_{j=1}^{k} \int_{J'_{n,j}} \sum_{i=1}^{m} g_i(t_j, y_n(s)) \left(\frac{d}{ds}\psi_n^i(s) - \frac{d}{ds}\tilde{\varphi}_n^i(s)\right) ds\right| \\ &\quad + \left|\sum_{j=1}^{k} \int_{J'_{n,j}} \sum_{i=1}^{m} (g_i(t_j, y_n(s)) - g_i(t_j, \tilde{y}_n(s))) \frac{d}{ds}\tilde{\varphi}_n^i(s) ds\right| \\ &\doteq D_{2,1} + D_{2,2}. \end{aligned} \tag{4.12}$$

In order to determine a bound of D_2, we have to get bounds of $D_{2,1}$

and $D_{2,2}$. By integration by part,

$$
\begin{aligned}
D_{2,1} \le & \left| \sum_{j=1}^{k} \left[\sum_{i=1}^{m} g_i(t_j, y_n(\tau_j))(\psi_n^i(\tau_j) - \tilde{\varphi}_n^i(\tau_j)) \right.\right. \\
& \left.\left. - \sum_{i=1}^{m} g_i(t_j, y_n(\tau_{j-1}))(\psi_n^i(\tau_{j-1}) - \tilde{\varphi}_n^i(\tau_{j-1})) \right] \right| \\
& + \left| \sum_{j=1}^{k} \int_{J'_{n,j}} \left[\frac{d}{ds} \sum_{i=1}^{m} g_i(t_j, y_n(s)) \right] (\psi_n^i(s) - \tilde{\varphi}_n^i(s)) ds \right| \\
\doteq & D_{2,1,1} + D_{2,1,2},
\end{aligned} \tag{4.13}
$$

by (4.9)

$$
\begin{aligned}
D_{2,1,1} \le & \sum_{i=1}^{m} |g_i(t_k, y_n(\tau_k))(\psi_n^i(\tau_k) - \tilde{\varphi}_n^i(\tau_k))| \\
& + \sum_{j=2}^{k} \sum_{i=1}^{m} \big| \big(g_i(t_{j-1}, y_n(\tau_{j-1})) \\
& \qquad - g_i(t_j, y_n(\tau_{j-1})) \big) \big(\psi_n^i(\tau_{j-1}) - \tilde{\varphi}_n^i(\tau_{j-1}) \big) \big| \\
& + \sum_{i=1}^{m} |g_i(t_1, y_n(\tau_0))(\psi_n^i(\tau_0) - \tilde{\varphi}_n^i(\tau_0))| \\
\le & mM\varepsilon + (\phi(t_k) - \phi(t_1))\varepsilon + mM\varepsilon
\end{aligned} \tag{4.14}
$$

and by (4.12) and $|\frac{d}{ds} g_i(t_j, y_n(s))| \le n^2 LM(1 + mM)$,

$$
\begin{aligned}
D_{2,1,2} \le & \sum_{j=1}^{k} \int_{J'_{n,j}} mn^2 LM(1 + mM)\varepsilon ds \\
= & Tmn^2 LM(1 + mM)\varepsilon.
\end{aligned} \tag{4.15}
$$

By (4.13), (4.14) and (4.15),

$$
D_{2,1} \le B_1 \varepsilon, \tag{4.16}
$$

where $B_1 = 2mM + \phi(T) - \phi(0) + Tmn^2LM(1+mM)$. From (4.11),

$$
\begin{aligned}
D_{2,2} &\le \sum_{j=1}^{k} \int_{J'_{n,j}} \sum_{i=1}^{m} L|y_n(s) - \tilde{y}_n(s)|K_1 ds \\
&= \int_0^T mL|y_n(s) - \tilde{y}_n(s)|K_1 ds.
\end{aligned} \tag{4.17}
$$

By (4.10)-(4.13) and (4.17),

$$|y_n(\tau) - \tilde{y}_n(\tau)| \le B_1\varepsilon + \int_0^T (L + mLK_1)|y_n(s) - \tilde{y}_n(s)|ds.$$

Due to Gronwall's inequality,

$$|y_n(\tau) - \tilde{y}_n(\tau)| \le B_1\varepsilon e^{(L+mLK_1)T}.$$

As a consequence, the theorem is proved.

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DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, KOREA
E-mail: shinc@ccs.sogang.ac.kr