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AN ADDITIONAL TERM BETWEEN ARITHMETIC MEAN AND GEOMETRIC MEAN

E. G. Kwon, S. T. Lee and K. H. Shon

Abstract. An additional term between the arithmetic mean and the geometric mean is presented.

1. Introduction

Throughout we let $X = (X, \mathcal{S}, \mu)$ and $Y = (Y, \mathcal{T}, \nu)$ be σ-finite measure spaces with positive measures μ and ν. When we call f defined on $X \times Y$ measurable it refers to $(\mathcal{S}\times\mathcal{T})$-measurable. $\mu\times\nu$ denotes the product measure of μ and ν (see [5, Chapter 7]). $L^1(\mu)$ denotes the space of those Lebesgue integrable (with respect to μ) functions defined on X and $L^p(\mu)$, $0 < p < \infty$, denotes the space of those complex measurable f defined on X for which $|f|^p \in L^1(\mu)$.

In Section 2 we give preliminary results. In Section 3 we give our main results which are related to a result of H. Z. Chuan ([2]) on the arithmetic-geometric mean inequality. In Section 4 we give proofs of the results of Section 3.

2. Preliminaries

Concerning the arithmetic-geometric mean inequality, H. Z. Chuan ([2]) inserted a continuum of additional terms between the two sides of the inequality as follows.

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THEOREM A ([2]). *If n is a natural number, $s > 0$, $a_j > 0$, $q_j > 0$ $(j = 1, ..., n)$, and $q_1 + ... + q_n = 1$ then*

$$(2.1)\quad \prod_{j=1}^{n} a_j^{q_j} \leq \left(s \int_0^\infty \left[\prod_{j=1}^{n} (x + a_j)^{q_j} \right]^{-s-1} dx \right)^{-\frac{1}{s}} \leq \sum_{j=1}^{n} q_j a_j.$$

Theorem A was generalized as

THEOREM B ([4, Theorem 2]). *If $s > 0$, $\mu(X) = 1$, and f is a positive function of $L^1(\mu)$, then*
(2.2)

$$G_X f \leq \left\{ s \int_0^\infty \exp\left(\int_X \log(y + f(x))^{-s-1} \, d\mu(x) \right) dy \right\}^{-1/s} \leq A_X f.$$

Either of the equalities in (2.2) holds if and only if $f(x) =$ constant a. e. $[\mu]$.

Here

$$G_X f = \exp\left(\int_X \log f(x) \, d\mu(x) \right)$$

and

$$A_X f = \int_X f(x) \, d\mu(x)$$

are respectively the geometric mean and the arithmetic mean of f over X.

If $0 \leq x \leq 1$ then Hölder's inequality says that
(2.3)

$$\int_Y f_1(y)^x f_2(y)^{1-x} d\nu(y) \leq \left(\int_Y f_1(y) d\nu(y) \right)^x \left(\int_Y f_2(y) d\nu(y) \right)^{1-x}$$

for all positive functions f_1 and f_2 of $L^1(\nu)$. It is known that (2.3) can be extended to the case of a multiple product of functions (See, for example, [1], [3], and [6]), and even to a countable product of functions (provided the product converges). The following is a continuous form of (2.3).

THEOREM C ([4, Theorem 1]). *Let $\mu(X) = 1$. Let $f(x,y)$ be a positive measurable function defined on $X \times Y$. Then*

$$\int_Y \exp\left(\int_X \log f\,d\mu\right)d\nu \;\le\; \exp\left\{\int_X \log\left(\int_Y f\,d\nu\right)d\mu\right\}. \tag{2.4}$$

Equality holds in (2.4) as a nonzero finite value if and only if

$$f(x,y) = g(x)h(y) \quad a.e. \quad \mu \times \nu$$

for a positive μ-measurable function g with $-\infty < \int_X \log g\,d\mu < \infty$ and a positive ν-measurable h with $\int_Y h\,d\nu = 1$.

If we take

$$X = \{1,2\},\; f(x,y) \;=\; f_x(y), \;\; x \in X,$$

and

$$d\mu \;=\; \left(t\chi_{\{1\}} + (1-t)\chi_{\{2\}}\right)dm,$$

where dm is the counting measure and $\chi_{\{.\}}$ is the corresponding characteristic function, then (2.4) reduces to (2.3).

LEMMA ([5, p. 74]). *If $\mu(X) = 1$ and F is positive function of $L^1(\mu)$ then*

$$\lim_{p\to 0}\left(\int_X F^p\,d\mu\right)^{\frac{1}{p}} = \exp\left(\int_X \log F\,d\mu\right), \tag{2.5}$$

and the convergence is monotone decreasing.

3. Main Results

THEOREM 1. *If n is a natural number, $s > 0$, $0 < p \le 1$, $a_j > 0$, $q_j > 0$ $(j = 1,\dots,n)$, and $q_1 + \dots + q_n = 1$ then*

(3.1)

$$\prod_{j=1}^{n} a_j^{q_j} \le \left(s\int_0^\infty \left[\sum_{j=1}^{n}(x+a_j)^{q_j p}\right]^{-(s+1)/p} dx\right)^{-\frac{1}{s}} \le \sum_{j=1}^{n} q_j a_j.$$

In view of Lemma, Theorem 1 reduces to Theorem A by letting $p \to 0$. A continuous form of Theorem 1 is as follows.

THEOREM 2. *If $s > 0$, $0 < p \leq 1$, $\mu(X) = 1$, and f is a positive function of $L^1(\mu)$, then*
(3.2)

$$G_X f \leq \left\{ s \int_0^\infty \left(\int_X (y + f(x))^p \, d\mu(x) \right)^{-(s+1)/p} dy \right\}^{-1/s} \leq A_X f.$$

Either of the equalities in (3.2) *holds if and only if $f(x) =$ constant a. e. $[\mu]$.*

4. Proofs

Proof of Theorem 2. To prove the first inequality of (3.2), note that by Lemma

$$\left(\int_X (y + f(x))^p \, d\mu(x) \right)^{1/p} \geq \exp \left(\int_X \log(y + f(x)) \, d\mu(x) \right), \quad y > 0,$$

so that by Theorem C

$$\begin{aligned} &\int_0^\infty \left(\int_X (y + f(x))^p \, d\mu(x) \right)^{-(s+1)/p} dy \\ &\leq \int_0^\infty \left\{ \exp \left(\int_X \log (y + f(x)) \, d\mu(x) \right) \right\}^{-s-1} dy \\ &\leq \exp \left\{ \int_X \log \left(\int_0^\infty (y + f(x))^{-s-1} dy \right) d\mu(x) \right\} \qquad (4.1) \\ &= \exp \left(\int_X \log \frac{1}{s} f(x)^{-s} \, d\mu(x) \right) \\ &= \frac{1}{s} \exp \left(\int_X \log f(x) \, d\mu(x) \right)^{-s} = \frac{1}{s} (G_X f)^{-s}, \end{aligned}$$

which is the desired.

To prove the second inequality of (3.2), we note that if $0 < p \leq 1$ then by Jensen's inequality

$$\left(\int_X (y + f(x))^p \, d\mu(x) \right)^{1/p} \leq \int_X (y + f(x)) \, d\mu(x) = y + A_X f$$

for all $y > 0$. Thus for $s > 0$

$$\begin{aligned} & \int_0^\infty \left(\int_X (y + f(x))^p \, d\mu(x) \right)^{-(s+1)/p} dy \\ (4.2) \qquad & \geq \int_0^\infty (y + A_X f)^{-s-1} dy \\ & = \frac{1}{s} (A_X f)^{-s}. \end{aligned}$$

This gives the second inequality of (3.2).

Next, in order to have equality sign in the one of the inequalities of (3.2), we should have equality either in (4.1) or in (4.2). By considering the equality case of Jensen's inequality and that of (2.4) as stated in Theorem 1, we conclude that f = constant a.e. $[\mu]$. The proof is complete. □

Proof of Theorem 1. Theorem 1 follows from Theorem 2 by taking X and μ appropriately. □

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E. G. Kwon, Department of Mathematics Education, Andong National University, Andong 760-749, Korea
E-mail: egkwon@andong.ac.kr

K. H. Shon, Department of Mathematics, College of Natural Sciences, Pusan National University, Pusan 609-735, Korea
E-mail: khsohn@hyowon.pusan.ac.kr