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NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS ON SEMI-RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, when N is a compact Riemannian manifold, we discuss the method of using warped products to construct timelike or null future(or past) complete Lorentzian metrics on $M = (-\infty, \infty) \times_f N$ with specific scalar curvatures.

1. Introduction

In a recent study [13, 14], M. C. Leung have studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of Riemannian warped metric with some prescribed scalar curvature function. In this paper, we study also the existence and nonexistence of Lorentzian warped metric with prescribed scalar curvature functions on some Lorentzian warped product manifolds. The methods of our proofs are similar to those of [13, 14], but the obtained results are in a sense very different.

By the results of Kazdan and Warner ([10, 11, 12]), if N is a compact Riemannian $n-$manifold without boundary, $n \geq 3$, then N belongs to one of the following three catagories:

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

(B) A Smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

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(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [10, 11, 12], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

For noncompact Riemannian manifolds, many important works have been done on the question of how to determine which smooth functions are scalar curvatures of complete Riemannian metrics on an open manifold. Results of Gromov and Lawson ([8]) show that some open manifolds cannot carry complete Riemannian metrics of positive scalar curvature, for example, weakly enlargeable manifolds. Furthermore, they show that some open manifolds cannot even admit complete Riemannian metrics with scalar curvatures uniformly positive outside a compact set and with Ricci curvatures bounded ([8], [15] p. 322).

On the other hand, it is well known that each open manifold of dimension bigger than 2 admits a complete Riemannian metric of constant negative scalar curvature ([5]). It follows from the results of Aviles and McOwne ([1]) that any bounded negative function on an open manifold of dimension bigger than 2 is the scalar curvature of a complete Riemannin metric.

In [13, 14], the author considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric. And also in [7], authors considered the existence of a nonconstant warping function on a Lorentzian warped product manifold such that the resulting warped product metric produces the constant scalar curvature when the fiber manifold has the constant scalar curvature.

Ironically, even though there exists some obstruction of positive or zero scalar curvature on a Riemannian manifold, results of [7], say, Theorem 3.1, Theorem 3.5 and Theorem 3.7 of [7] show that there exists no obstruction of positive scalar curvature on a Lorentzian warped product manifold, but there may exist some obstruction of negative or zero scalar curvature.

In this paper, when N is a compact Riemannian manifold, we discuss the method of using warped products to construct timelike or null future(or past) complete Lorentzian metrics on $M = (-\infty, \infty) \times_f N$ with specific scalar curvatures. By making use of the boundary, we can construct warped products at the ends of M. It is shown that if the fiber manifold N belongs to class (A) or (B), then M admits a Lorentzian metric with negative scalar curvature approaching zero near the end outside a compact set.

2. Fiber manifold in class (A) or (B)

Let (N, g) be a Riemannian manifold of dimension n and let $f : (-\infty, \infty) \to R^+$ be a smooth function. The Lorentzian warped product of N and $(-\infty, \infty)$ with warping function f is defined to be the product manifold $((-\infty, \infty) \times_f N, g')$ with

$$g' = -dt^2 + f^2(t)g \,. \tag{2.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t, x)$ of g' is given by the equation

$$R(t,x) = \frac{1}{f^2(t)}\{R(g)(x) + 2nf(t)f''(t) + n(n-1)|f'(t)|^2\} \tag{2.2}$$

for $t \in (-\infty, \infty)$ and $x \in N$. (For details, cf. [2], [6] or [7]) If we denote

$$u(t) = f^{\frac{n+1}{2}}(t),$$

then equation (2.2) can be changed into

$$\frac{4n}{n+1}u''(t) - R(t,x)u(t) + R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian $n-$manifold without boundary. Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [7], we have the following proposition.

PROPOSITION 2.1. *If the scalar curvature of the fiber manifold N is arbitrary constant, then there exists a nonconstant warping function $f(t)$ on $(-\infty, \infty)$ such that the resulting Lorentzian warped product metric on $(-\infty, \infty) \times_f N$ produces positive constant scalar curvature.*

Proposition 2.1 implies that in Lorentzian warped product there is no obstruction of metric with positive scalar curvature. However, the results of [7] show that there may exist some obstruction about the Lorentzian warped product metric with negative or zero scalar curvature when the fiber manifold has constant scalar curvature.

REMARK 2.2. Theorem 5.5 in [16] implies that all timelike geodesics are future (resp. past) complete on $(-\infty, +\infty) \times_{v(t)} N$ if and only if $\int_{t_0}^{+\infty} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$ (resp. $\int_{-\infty}^{t_0} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$) for some $t_0 \in (-\infty, \infty)$ and Remark 2.58 in [3] implies that all null geodesics are future (resp. past) complete if and only if $\int_{t_0}^{+\infty} v^{\frac{1}{2}} dt = +\infty$ (resp. $\int_{-\infty}^{t_0} v^{\frac{1}{2}} dt = +\infty$) for some $t_0 \in (-\infty, \infty)$ (cf. Theorem 4.1 and Remark 4.2 in [4]).

If N admits a Riemannian metric of negative or zero scalar curvature, then we let $u(t) = (c_0 + t^2)^\alpha$ in (2.3), where c_0 is a positive number and $\alpha \in (0, 1)$ is a constant. So we have

$$R(t, x) \leq -\frac{4n}{n+1} \frac{-2\alpha c_0 + 2\alpha(1-2\alpha)t^2}{(c_0 + t^2)^2}.$$

Therefore, from the above fact, Remark 2.2 implies the following:

THEOREM 2.3. *For $n \geq 3$, let $M = (-\infty, \infty) \times_f N$ be the Lorentzian warped product $(n+1)$-manifold with N compact n-manifold. Suppose that N is in class (A) or (B), then on M there is a nonspacelike future geodesically complete Lorentzian metric of negative scalar curvature outside a compact set.*

We note that $2\alpha(1-2\alpha)$ achieves its maximum value $\frac{1}{4}$ when $\alpha = \frac{1}{4}$. If $R(t, x)$ is the function of only t-variable, then we have the following proposition whose proof is similar to that of Lemma 1.8 in [14].

PROPOSITION 2.4. *If $R(g) = 0$, then there is no positive solution to equation (2.3) with*

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \textit{for} \quad t > t_0 \quad \textit{or} \quad t < -t_0,$$

where $c > 1$ and t_0 are positive constants.

Proof. In case that

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t \geq t_0,$$

we have the same proof as in [J.].

Assume that

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t \leq -t_0,$$

with $c > 1$. Equations (2.3) gives

$$t^2 u'' + \frac{c}{4}u \leq 0.$$

Let

$$u(t) = (-t)^\alpha v(t), \quad t \leq -t_0,$$

where $\alpha > 0$ is a constant and $v(t) > 0$ is a smooth function. Then we have

$$u'' = \alpha(\alpha-1)(-t)^{\alpha-2}v(t) - 2\alpha(-t)^{\alpha-1}v'(t) + (-t)^\alpha v''(t).$$

And we obtain

$$(2.4) \quad (-t)^\alpha v(t)[\alpha(\alpha-1) + \frac{c}{4}] - 2\alpha(-t)^{\alpha+1}v'(t) + (-t)^{\alpha+2}v''(t) \leq 0.$$

Let δ be a positive constant such that $\delta^2 = \frac{c-1}{4}$. Then we have

$$\alpha(\alpha-1) + \frac{c}{4} = (\alpha - \frac{1}{2})^2 + \frac{c-1}{4} \geq \delta^2.$$

Then δ is a constant independent on α. Equation (2.4) gives

$$2\alpha t v'(t) + (-t)^2 v''(t) \leq -\delta^2 v(t). \tag{2.5}$$

Let $\beta = 2\alpha$ and we choose $\alpha > 0$ such that $\beta < 1$, that is, $\alpha < \frac{1}{2}$. Then (2.5) becomes

$$((-t)^\beta v'(t))' \leq -\frac{\delta^2 v(t)}{(-t)^{2-\beta}}.$$

Upon integration we have

$$(-\tau)^\beta v'(\tau) - (-t)^\beta v'(t) \leq -\int_t^\tau \frac{\delta^2 v(s)}{(-s)^{2-\beta}} ds, \quad t < \tau < -t_0. \tag{2.6}$$

Here we have two following cases:

i) If $v'(\tau) \geq 0$ for some $\tau < -t_0$, then (2.6) implies that

$$-(-t)^\beta v'(t) \leq -C$$

for some positive constant C. We have

$$v(t) \leq v(\tau) - \int_t^\tau \frac{C}{(-s)^\beta} ds = v(\tau) + \frac{C(-s)^{1-\beta}}{1-\beta}|_t^\tau \to -\infty,$$

as $\beta < 1$. Hence $v(t) < 0$ for some t, contradicting that $v(t) > 0$ for all $t \leq -t_0$.

ii) We have $v'(t) < 0$ for all $t < -t_0$. Equation (2.6) implies that

$$-(-\tau)^\beta v'(\tau) - \int_t^\tau \frac{\delta^2 v(s)}{(-s)^{2-\beta}} ds \geq 0$$

for all $t < \tau < -t_0$. As $v'(t) < 0$ for all $t < -t_0$, we have

$$-(-\tau)^\beta v'(\tau) \geq v(\tau) \int_t^\tau \frac{\delta^2}{(-s)^{2-\beta}} ds = v(\tau)[\frac{1}{(-s)^{1-\beta}} \frac{\delta^2}{1-\beta}]|_t^\tau$$

Let $t \to \infty$ we have

$$-(-\tau)^\beta v'(\tau) \geq \frac{v(\tau)}{(-\tau)^{1-\beta}} \frac{\delta^2}{1-\beta}.$$

Or after changing the parameter we have

$$\frac{v'(t)}{v(t)} \leq \frac{1}{t} \frac{\delta^2}{1-\beta}, \quad -\infty < t < -t_0.$$

Choosing $\alpha < \frac{1}{2}$ close to $\frac{1}{2}$ so that $\beta < 1$ is close to 1 and using the fact that δ is independent on α or β, we have

$$\frac{v'(t)}{v(t)} \leq \frac{N}{t}$$

for a big integer $N > 2$. This gives

$$v(t) \geq C(-t)^N, \quad t < -t_0,$$

where C is a positive constant. (2.6) implies that

$$(-t)^\beta v'(t) \geq (-\tau)^\beta v'(\tau) + \int_t^\tau \frac{C\delta^2(-s)^N}{(-s)^{2-\beta}} ds \to \infty \text{ as } t \to -\infty.$$

Thus $v'(t) > 0$ for t small, which is also a contradiction. Hence there is no solution to equation (2.3). □

In particular, if $R(g) \leq 0$, then using Lorentzian warped product it is impossible to obtain a Lorentzian metric of uniformly negative scalar curvature outside a compact subset. The best we can do is when $u(t) = (c_0 + t^2)^{\frac{1}{4}}$, or $f(t) = (c_0 + t^2)^{\frac{1}{2(n+1)}}$, for some positive number c_0, where the scalar curvature is negative but goes to zero at infinity.

THEOREM 2.5. *Suppose that $R(g) = 0$ and $R(t, x) = R(t) \in C^\infty((-\infty, \infty))$. Assume that for $|t| > t_0$ there exist an upper solution u_+ and a lower solution u_- such that $0 < u_- < u_+$. Then there exists a solution u of equation (2.3) such that $0 < u_- < u < u_+$ for $|t| > t_0$.*

Proof. We have only to show that there exist an upper solution $\tilde{u}_+$, and a lower solution $\tilde{u}_-$ such that for all $t \in (-\infty,\infty)$ $\tilde{u}_- < \tilde{u}_+$. Since $R(t) \in C^\infty((-\infty,\infty))$, there exists a positive constant b such that $|R(t)| \le b^2$ for $|t| \le t_0$.

Since $\frac{4n}{n+1}u_+''(t) + R(t)u_+(t) \le \frac{4n}{n+1}(u_+''(t) + b^2u_+(t))$, if we divide the given interval $[-t_0,t_0]$ into small intervals $\{I_i\}_{i=1}^n$, then for each interval I_i we have an upper solution $u_+^i(t)$ by parallel transporting *cosbt* such that $0 < c_0 \le u_+^i(t) \le 1$. That is to say, for interval I_i, $\frac{4n}{n+1}u_+^i(t)'' + R(t)u_+^i(t) \le \frac{4n}{n+1}(u_+^i(t)'' + b^2u_+^i(t)) = 0$, which means that $u_+^i(t)$ is an upper solution for each interval I_i. Then put $\tilde{u}_+(t) = u_+^i(t)$ for $t \in I_i$ and $\tilde{u}_+(t) = u_+(t)$ for $|t| > t_0$, which is our desired(weak) upper solution such that $c_0 \le \tilde{u}_+(t) \le 1$ for all $|t| > t_0$.

Put $\tilde{u}_-(t) = c_0e^{-\alpha|t|}$ for $|t| \le t_0$ and some large positive α, which will be determined later, and $\tilde{u}_-(t) = u_-(t)$ for $|t| > t_0$. Then, for $|t| \le t_0$, $\frac{4n}{n+1}u_-''(t)+R(t)u_-(t) \ge \frac{4n}{n+1}(u_-''(t)-b^2u_-(t)) = \frac{4n}{n+1}c_0e^{-\alpha t^2}(\alpha^2-b^2) \ge 0$ for large α. Thus $\tilde{u}_-(t)$ is our desired (weak) lower solution such that for all $t \in (-\infty,\infty)$ $0 < \tilde{u}_-(t) \le \tilde{u}_+(t)$. □

THEOREM 2.6. *Suppose that $R(g) = 0$. Assume that $R(t,x) = R(t) \in C^\infty((-\infty,\infty))$ is a function such that*

$$-\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} < R(t) \le b^2 \quad \textit{for} \quad |t| > t_0,$$

where $t_0 > 0$ and $0 < c < 1$ are constants. Then equation (2.3) has a positive solution on $(-\infty,\infty)$.

Proof. Put $u_+(t) = (c_0 + t^2)^{\frac{1}{4}}$. Then

$$u_+''(t) = \frac{1}{2}(c_0+t^2)^{-\frac{3}{4}} - \frac{3}{4}t^2(c_0+t^2)^{-\frac{7}{4}}.$$

$$\begin{aligned}
&\frac{4n}{n+1}u''_+(t) - R(t)u_+(t) \\
&= \frac{4n}{n+1}[\frac{1}{2}(c_0+t^2)^{-\frac{3}{4}} - \frac{3}{4}t^2(c_0+t^2)^{-\frac{7}{4}}] - R(t)(c_0+t^2)^{\frac{1}{4}} \\
&= \frac{4n}{n+1}(c_0+t^2)^{-\frac{7}{4}}\frac{1}{4}[2(c_0+t^2)-3t^2] - R(t)(c_0+t^2)^{-\frac{7}{4}}(c_0+t^2)^2 \\
&\leq \frac{n}{n+1}(c_0+t^2)^{-\frac{7}{4}}[2c_0(1+c) + \frac{cc_0^2}{t^2} + (c-1)t^2].
\end{aligned}$$

Since $0 < c < 1$,

$$2c_0(1+c) + \frac{cc_0^2}{t^2} + (c-1)t^2 \leq 0, \quad t < -t_0.$$

Therefore $u_+(t)$ is our upper solution. And put $u_-(t) = e^{-\alpha|t|}$, where α is positive constant and will be determined later. Then $u''_-(t) = \alpha^2 e^{-\alpha|t|}$

$$\begin{aligned}
\frac{4n}{n+1}u''_-(t) - R(t)u_-(t) &= \frac{4n}{n+1}\alpha^2 e^{-\alpha|t|} - R(t)e^{-\alpha|t|} \\
&\geq e^{-\alpha|t|}(\frac{4n}{n+1}\alpha^2 - b^2) \\
&\geq 0
\end{aligned}$$

for large α and $t_0 < -t_0$. Since $|t| > t_0$, we can take α large so that $u_-(t)$ is a lower solution $0 < u_-(t) < u_+(t)$. □

COROLLARY 2.7. *Suppose that $R(g) = 0$. Assume that $R(t,x) = R(t) \in C^\infty((-\infty,\infty))$ is a function such that*

$$-\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} < R(t) \leqq 0 \quad \textit{for} \quad |t| > t_0,$$

where $t_0 > 0$ and $0 < c < 1$ are constants. Then equation (2.3) has a positive solution on $(-\infty,\infty)$ and on M the resulting Lorentzian warped product metric is a nonspacelike future (or past) geodesically complete metric of non-positive scalar curvature outside a compact set.

Proof. Since $R(g) = 0$ and $R(t,x) \leqq 0$, the lower solution $u_-(t) = c_-$ is a small positive constant and the upper solution $u_+(t) = (c_0 + t^2)^{\frac{1}{4}}$ as in Theorem 2.6. Therefore equation (2.3) has a positive solution $u(t) = f^{\frac{n+1}{2}}(t)$ such that $0 < u_-(t) \leq u(t) \leq u_+(t)$. Hence

$$\int_{t_0}^{\infty} \left(\frac{f(t)}{1+f(t)}\right)^{\frac{1}{2}} dt = \int_{t_0}^{\infty} \left(\frac{u(t)^{\frac{2}{n+1}}}{1+u(t)^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt$$

$$\geqq \int_{t_0}^{\infty} \left(\frac{c_-^{\frac{2}{n+1}}}{1+c_-^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt \longrightarrow \infty$$

and $\int_{t_0}^{\infty} f(t)^{\frac{1}{2}} dt = \int_{t_0}^{\infty} u(t)^{\frac{1}{n+1}} dt \geqq \int_{t_0}^{\infty} c_-^{\frac{1}{n+1}} dt \longrightarrow \infty$

$$\left(\int_{-\infty}^{t_0} \left(\frac{f(t)}{1+f(t)}\right)^{\frac{1}{2}} dt \geqq \int_{-\infty}^{t_0} \left(\frac{c_-^{\frac{2}{n+1}}}{1+c_-^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt \longrightarrow \infty\right.$$

and $\left.\int_{-\infty}^{t_0} f(t)^{\frac{1}{2}} dt \geq \int_{-\infty}^{t_0} c_-^{\frac{1}{n+1}} dt \longrightarrow \infty\right)$. □

REMARK 2.8. In case that $R(g) = 0$ and $0 < R(t) < b^2$ we do not know whether or not our resulting Lorentzian warped metric is a nonspacelike future geodesically complete one.

3. Fiber manifold in class (C)

In this section, we assume that the fiber manifold N of $M = (-\infty, \infty) \times_f N$ belongs to class (C). In this case, N admits a Riemannian metric of positive scalar curvature. If we let $u(t) = (c_0 + t^2)^{\frac{1}{4}}$, then we have

$$R(t,x) \geq -\frac{4n}{n+1}\frac{2c_0 - t^2}{4(c_0+t^2)^2}, \quad |t| > t_0,$$

where $c > 1$ and t_0 are positive constants. By the similar proof like as Proposition 2.4, we have the following:

THEOREM 3.1. *If $R(g)$ is positive, then there is no positive solution to equation (2.3) with*

$$R(t) \le -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t > t_0 \quad \text{or} \quad t \le -t_0.$$

where $c > 1$ and $t_0 > 0$ are constants.

If N belongs to (C), then any smooth function on N is the scalar curvature of some Riemannian metric. So we can take a Riemannian metric g_1 on N with scalar curvature $R(g_1) = \frac{4n}{n+1}k^2$, where k is a positive constant. Then equation (2.3) becomes

$$\frac{4n}{n+1}u''(t) + \frac{4n}{n+1}k^2 u(t)^{1-\frac{4}{n+1}} - R(t,x)u(t) = 0. \tag{3.1}$$

If $R(t,x)$ is the function of only t- variable, then we have the following theorem.

THEOREM 3.2. *Suppose that $R(g) = \frac{4n}{n+1}k^2$ and $R(t,x) = R(t) \in C^\infty((-\infty,\infty))$. Assume that for $|t| > t_0$ there exist an upper solution u_+ and a lower solution u_- such that $0 < u_- < u_+$. Then there exists a solution u of (3.1) such that $0 < u_- < u < u_+$ for $|t| > t_0$.*

Proof. We have only to show that there exist an upper solution $\tilde{u}_+(t)$ and a lower solution $\tilde{u}_-(t)$ such that for all $t \in (-\infty,\infty)$ $\tilde{u}_-(t) < \tilde{u}_+(t)$. Since $R(t) \in C^\infty(-\infty,\infty)$, there exists a positive constant b such that $|R(t)| \le \frac{4n}{n+1}b^2$ for $|t| \le t_0$. Since $1 - \frac{4}{n+1} < 1$ and $R(t)$ is a bounded function, small constant $c_0$is an upper solution for $|t| \le t_0$. Then put $\tilde{u}_+(t) = c_0$ for $t \in [-t_0, t_0]$ and $\tilde{u}_+(t) = u_+(t)$ for $|t| > t_0$, which is our desired (weak) upper solution such that $c_0 \le \tilde{u}_+(t)$ for all $t \in [-t_0, t_0]$. Put $\tilde{u}_-(t) = c_0 e^{-\alpha|t|}$ for $t \in [-t_0, t_0]$ and some large positive α, which will be determined later, and $\tilde{u}_-(t) = u_-(t)$ for $|t| > t_0$. Then, for $t \in [-t_0, t_0]$,

$$\begin{aligned}\frac{4n}{n+1}u''_-(t) + R(t)u_-(t) &\ge \frac{4n}{n+1}(u''_-(t) - b^2u_-(t) \\ &= \frac{4n}{n+1}c_0e^{-\alpha|t|}(\alpha^2 - b^2) \\ &\ge 0\end{aligned}$$

for large α. Thus $\tilde{u}_-(t)$ is our desired (weak) lower solution such that for all $t \in (-\infty, \infty)$ $0 < \tilde{u}_-(t) < \tilde{u}_+(t)$. □

LEMMA 3.3. *Let $u(t)$ be a positive smooth function on $(-\infty, \infty)$. If $u(t)$ satisfies*

$$\frac{u''(t)}{u(t)} \leq \frac{C}{t^2}$$

for some constant $C \geq 1$, then there exists $t_0 > 0$ such that for all $|t| > t_0$

$$u(t) \leq C_0 |t|^\epsilon$$

for some positive constant C_0 and $\epsilon > 1$.

Proof. In case that $t > t_0$, we also have the similar proof as in [J.]. Assume that

$$\frac{u''(t)}{u(t)} \leq \frac{C}{t^2} \quad \text{for} \quad t < -t_0.$$

Since $C \geq 1$, we can choose $\epsilon > 1$ such that $\epsilon(\epsilon - 1) = C$. Then from the hypothesis, we have

$$(-t)^\epsilon u''(t) \leq \epsilon(\epsilon - 1)(-t)^{\epsilon-2} u(t).$$

Upon integration from $t(< \tau < 0)$ to τ, and using integration by parts, we obtain

$$(-\tau)^\epsilon u'(\tau) - (-t)^\epsilon u'(t) + \epsilon \int_t^\tau (-s)^{\epsilon-1} u'(s) ds \leq C \int_t^\tau (-s)^{\epsilon-2} u(s) ds.$$

Therefore we have

$$(-\tau)^\epsilon u'(\tau) + \epsilon(-\tau)^{\epsilon-1} u(\tau) \leq (-t)^\epsilon u'(t) + \epsilon(-t)^{\epsilon-1} u(t). \tag{3.2}$$

We consider two following cases:

[Case 1] There exists $\tau < 0$ such that $u'(\tau) \geq 0$.
If there is a number $\tau < 0$ such that $u'(\tau) \geq 0$, then we have

$$(-t)^{\epsilon} u'(t) + \epsilon(-t)^{\epsilon-1} u(t) \geq 0.$$

This gives $\int_t^{\tau} \frac{u'(s)}{u(s)} ds \leq \int_t^{\tau} \frac{\epsilon}{s} ds$ and $(\ln |u(t)|) \leq (\ln \frac{u(\tau)}{|\tau|^{\epsilon}} |t|^{\epsilon})$ for all $0 > \tau > t$. Hence $|u(t)| \leq c_1 |t|^{\epsilon}$, where c_1 is a positive constant.

[Case 2] There does not exist $\tau < 0$ such that $u'(\tau) \geq 0$.

In other words, if $u'(t) < 0$ for all $t < \tau < 0$, then $u(t)$ is decreasing. Thus $u(\tau)$ is minimizing. Let c_2 be a positive constant such that

$$(-t)^{\epsilon} u'(t) + \epsilon(-t)^{\epsilon-1} u(t) \geq -c_2,$$

from equation(3.2). Thus for all $0 > \tau > t$

$$\frac{u'(t)}{u(t)} \geq \frac{\epsilon}{t} - \frac{c_2}{u(t)(-t)^{\epsilon}} \geq \frac{\tilde{\epsilon}}{t},$$

where $\tilde{\epsilon} > 1$. This gives $\int_t^{\tau} \frac{u'(s)}{u(s)} ds \leq \int_t^{\tau} \frac{\tilde{\epsilon}}{s} ds$ and $(\ln |u(t)|) \leq (\ln \frac{u(\tau)}{|\tau|^{\tilde{\epsilon}}} |t|^{\tilde{\epsilon}})$ for all $0 > \tau > t$. Hence $u(t) \leq C|t|^{\tilde{\epsilon}}$ for some positive constant C.

Thus from two cases we always find $t_0 > 0$ and a constant $c_0 > 0$ such that

$$u(t) \leq c_0 |t|^{\epsilon}$$

for all $|t| \geq t_0$. □

In Proposition 3.4, when $C \leq n(n-1)$, we can prove the following fact about the nonexistence of warping function, using the above Lemma.

PROPOSITION 3.4. *Suppose that N belongs to class* (C). *Let g be a Riemannian metric on N. We may assume that $R(g) = \frac{4n}{n+1} k^2$, where k is a positive constant. On $(-\infty, \infty) \times_f N$, there does not exist a warped product metric*

$$g' = -dt^2 + f(t)^2 g$$

with scalar curvature

$$0 < R(t,x) = R(t) \leq \frac{n(n-1)}{t^2}$$

for all $x \in N$ and $t > t_0$ or $t < -t_0$, where t_0 is a positive constants.

Proof. Assume that we can find a warped product metric on $(-\infty, \infty) \times_f N$ with

$$0 < R(t,x) = R(t) \leq \frac{n(n-1)}{t^2}$$

for all $x \in N$ and $t > t_0$ or $t < -t_0$. In case that $t > t_0$, we have similar proof as in [J.]. So we assume the case that $t < -t_0$. In equation (2.3), we have

$$\frac{4n}{n+1}\left[\frac{u''(t)}{u(t)} + \frac{k^2}{u(t)^{\frac{4}{n+1}}}\right] = R(t) \leq \frac{n(n-1)}{t^2}. \tag{3.3}$$

$$\frac{u''(t)}{u(t)} \leq \frac{\frac{(n-1)(n+1)}{4}}{t^2}. \tag{3.4}$$

In equation (3.4), we can apply Lemma 3.5 and take $\epsilon = \frac{n+1}{2}$. Hence we have $t_0 > a$ such that

$$u(t) \leq c_0(-t)^{\epsilon}$$

for some positive constants c_0 and all $t < -t_0$.

Then

$$\frac{k^2}{u(t)^{\frac{4}{n+1}}} \geq \frac{c'}{t^2},$$

where $0 < c' \leq \frac{k^2}{c_0^{\frac{4}{n+1}}}$ is a positive constant. Hence equation(3.3) gives

$$\frac{u''(t)}{u(t)} \leq \frac{(n+1)(n-1) - \delta}{4t^2},$$

where $4c' \geq \delta > 0$ is a constant. We can choose $\delta' > 0$ such that

$$\frac{(n+1)(n-1)-\delta}{4} = (\frac{n+1}{2} - \delta')(\frac{n-1}{2} - \delta')$$

for small positive δ. Applying the Lemma again, we have $t_1 > 0$ such that

$$u(t) \leq c_1(-t)^{\frac{n+1}{2}-\delta'}$$

for some $c_1 > 0$ and all $t < -t_1$ and

$$\text{(3.5)} \qquad \frac{k^2}{u^{\frac{4}{n+1}}} \geq \frac{c''}{(-t)^{2-\epsilon}},$$

where $\epsilon = \frac{4}{n+1}\delta'$ and $0 < c'' \leq \frac{k^2}{c_1^{\frac{4}{n+1}}}$. Thus equation (3.3)and (3.5) give

$$u''(t) \leq 0$$

for t negatively large. Hence $u(t) \leq c_2(-t)$ for some constant $c_2 > 0$ and negatively large t. From equation (3.3) we have

$$\frac{u''(t)}{u(t)} \leq \frac{-k^2}{(c_2(-t))^{\frac{4}{n+1}}} + \frac{(n+1)(n-1)}{4t^2} \leq \frac{c_3}{t}$$

for t negatively large enough, as $n \geq 3$. Here c_3 is a positive constant. Multiplying $u(t)$ and integrating from t to τ we have

$$u'(\tau) - u'(t) \leq c_3 \int_t^\tau \frac{u(s)}{s} ds, \quad t < \tau < -t_0.$$

We consider two following cases: [Case 1] There exists $\tau \leq \min\{t_0, t_1\}$ such that $u'(\tau) \geq 0$. Since $\int_t^\tau \frac{u(s)}{s} ds < 0,$ for $t < \tau < -t_0$, if $u'(\tau) \geq 0$ for some τ, then $u'(t) \geq c_4$ for some positive constant c_4. $\int_t^\tau u(s)ds \geq c_4 \int_t^s ds$. Hence $u(t) \leq 0$ for t negatively large enough, contradicting the fact that u is positive. [Case 2] There does not exist $\tau \leq \min\{t_0, t_1\}$ such that $u'(\tau) \geq 0$. In other words, if $u'(t) < 0$ for all t negatively large, then $u(t)$ is decreasing, hence

$$u'(t) - u'(\tau) \geq c_3 \int_t^\tau \frac{u(s)}{-s} ds \geq c_3 u(\tau) \int_t^\tau \frac{1}{-s} ds = c_3 u(\tau) \ln|\frac{t}{\tau}| \to \infty.$$

Thus $u'(t)$ has to be positive for some t negatively large, which is a contradiction to the hypothesis.

Therefore there does not exist such warped product metric. $\square$

THEOREM 3.5. *Assume that $R(t,x) = R(t) \in C^\infty((-\infty,\infty))$ is a positive function such that*

$$b|t|^s \geq R(t) \geq \frac{4n}{n+1}\frac{C}{|t|^\alpha} \quad for \quad |t| > t_0,$$

where $t_0 > 0$, $\alpha < 2$, C and b are positive constants. Then equation (3.1) has a positive solution on $(-\infty,\infty)$.

Proof. We let $u_+ = c_0 + t^{2m}$, where m is a positive integer. If we take m large enough so that $2m\frac{4}{n+1} > 2$, then we have

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2u_+(t)^{1-\frac{4}{n+1}} - R(t)u_+(t) \\
&\leq \frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2u_+(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1}\frac{C}{|t|^\alpha}u_+(t) \\
&= \frac{4n}{n+1}t^{2m}\left[\frac{2m(2m-1)}{t^2} + \frac{k^2}{t^{2m\frac{4}{n+1}}}\left(\frac{c_0}{t^{2m}}+1\right)^{1-\frac{4}{n+1}} - \frac{C}{|t|^\alpha}\left(\frac{c_0}{t^{2m}}+1\right)\right] \\
&\leq 0, \quad t \leq -t_0 \quad \text{for some large} \quad t_0,
\end{aligned}$$

which is possible for large fixed m since $\alpha < 2$. Therefore $u_+(t)$ is our upper solution. And put $u_-(t) = e^{-\alpha|t|}, \alpha > 0$, since $t < -t_0 < 0$,

$$\begin{aligned}
&\frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2u_-(t)^{1-\frac{4}{n+1}} - R(t)u_-(t) \\
&\geq \frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2u_-(t)^{1-\frac{4}{n+1}} - b^2u_-(t) \\
&= \frac{4n}{n+1}\alpha^2e^{-\alpha|t|} + \frac{4n}{n+1}k^2(e^{-\alpha|t|})^{1-\frac{4}{n+1}} - b^2e^{-\alpha|t|} \\
&= \frac{4n}{n+1}e^{-\alpha|t|}\left[\alpha^2 + k^2e^{\alpha|t|\frac{4}{n+1}} - \frac{n+1}{4n}b^2\right] \\
&\geq 0.
\end{aligned}$$

Thus we can take the lower solution $u_-(t)$ so that $0 < u_-(t) < u_+(t)$. So by Theorem 3.2., we obtain a positive solution. □

COROLLARY 3.6. *Assume that $R(t,x) = R(t) \in C^\infty((-\infty,\infty))$ is a positive function such that*

$$b|t|^s \geq R(t) \geq \frac{C}{t^2} \quad \text{for} \quad |t| > t_0,$$

where $t_0 > a$, b and C are positive constants. If $C > n(n-1)$, then equation (3.1) has a positive solution on $(-\infty,\infty)$.

Proof. In case that $C > n(n-1)$, we may take $u_+(t) = C_0 + C_+|t|^{\frac{n+1}{2}}$, where C_0, C_+ are positive constants. Then

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - R(t)u_+(t) \\
&\leq n(n-1)C_+|t|^{\frac{n-3}{2}} + \frac{4n}{n+1}k^2(C_0 + C_+|t|^{\frac{n+1}{2}})^{1-\frac{4}{n+1}} - \frac{C}{t^2}(C_0 + C_+|t|^{\frac{n+1}{2}}) \\
&\leq C_+|t|^{\frac{n-3}{2}}\left[n(n-1) - C + \frac{4n}{n+1}k^2 C_+^{\frac{-4}{n+1}}\left(\frac{C_0}{C_+|t|^{\frac{n+1}{2}}} + 1\right)^{1-\frac{4}{n+1}}\right] \\
&\leq 0,
\end{aligned}$$

which is possible if we take C_+ to be large enough since $n(n-1) - C < 0$. And since the exponent $1 - \frac{4}{n+1}$ is less than 1 and $R(t)$ is a bounded function, we can take $u_-(t) = e^{-\alpha|t|}$ as in Theorem 3.5. In this case, we also obtain a positive solution. □

COROLLARY 3.7. *Assume that $R(t,x) = R(t) \in C^\infty((-\infty,\infty))$ is a positive function such that*

$$b \geq R(t) \geq \frac{4n}{n+1}\frac{C}{|t|^\alpha} \quad \text{for} \quad |t| > t_0,$$

where $t_0 > 0$, $\alpha < 2$, C and b are positive constants. Then equation (3.1) has a positive solution on $(-\infty,\infty)$ and on M the resulting Lorentzian warped product metric is a nonspacelike future geodesically complete metric of positive scalar curvature outside a compact set.

Proof. We let $u_+(t) = c_0 + t^{2m}$ as in Theorem 3.5. And since the exponent $1 - \frac{4}{n+1}$ is less than 1 and $R(t)$ is a bounded function, we can take the lower solution $u_-(t) = c_-$ as a small positive constant such that $0 < u_-(t) \leq u_+(t)$,

$$\frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_-(t)^{1-\frac{4}{n+1}} - R(t)u_-(t) \geq 0,$$

which is possible since $u_+(t) = c_0 + t^{2m}$ has a positive minimum on $(-\infty, \infty)$. Therefore equation (2.3) has a positive solution $u(t) = f(t)^{\frac{n+1}{2}}$ such that $0 < u_-(t) \leq u(t) \leq u_+(t)$. Hence, similarly as in Corollary 2.7,

$$\int_{t_0}^{\infty} \left(\frac{f(t)}{1+f(t)}\right)^{\frac{1}{2}} dt = \int_{t_0}^{\infty} \left(\frac{u(t)^{\frac{2}{n+1}}}{1+u(t)^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt$$
$$\geqq \int_{t_0}^{\infty} \left(\frac{c_-^{\frac{2}{n+1}}}{1+c_-^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt \longrightarrow \infty$$

and

$$\int_{t_0}^{\infty} f(t)^{\frac{1}{2}} dt = \int_{t_0}^{\infty} u(t)^{\frac{1}{n+1}} dt \geqq \int_{t_0}^{\infty} c_-^{\frac{1}{n+1}} dt \longrightarrow \infty$$

$$\left(\int_{-\infty}^{t_0} \left(\frac{f(t)}{1+f(t)}\right)^{\frac{1}{2}} dt \geqq \int_{-\infty}^{t_0} \left(\frac{c_-^{\frac{2}{n+1}}}{1+c_-^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt \longrightarrow \infty\right.$$

and

$$\left.\int_{-\infty}^{t_0} f(t)^{\frac{1}{2}} dt \geqq \int_{-\infty}^{t_0} c_-^{\frac{1}{n+1}} dt \longrightarrow \infty\right),$$

which, by Remark 2.2, implies that the resulting warped product metric is a nonspacelike future geodesically complete one. □

COROLLARY 3.8. *Assume that* $R(t, x) = R(t) \in C^\infty((-\infty, \infty))$ *is a positive function such that*

$$b \geq R(t) \geq \frac{C}{t^2} \quad \textit{for} \quad t \geq t_0,$$

where $t_0 > a$, b *and* C *are positive constants. If* $C > n(n-1)$, *then equation* (3.1) *has a positive solution on* $(-\infty, \infty)$ *and on* M *the resulting Lorentzian warped product metric is a nonspacelike future geodesically complete metric of positive scalar curvature outside a compact set.*

REMARK 3.9. The result in Proposition 3.4, Theorem 3.5 and Corollary 3.6 are almost sharp as we can get as close to $\frac{n(n-1)}{t^2}$ as possible. For example, let $R(g) = \frac{4n}{n+1}k^2$ and $u(t) = 1 + |t|^{\frac{n+1}{2}}$. Then we have

$$R \geq \frac{n(n-1)|t|^{\frac{n+1}{2}-2}}{1+|t|^{\frac{n+1}{2}}} = \frac{n(n-1)}{|t|^2}\frac{|t|^{\frac{n+1}{2}}}{1+|t|^{\frac{n+1}{2}}},$$

which converges to $\frac{n(n-1)}{|t|^2}$ as $|t|$ goes to ∞.

References

[1] P. Aviles and R. McOwen, *Conformal deformation to constant negative scalar curvature on noncompact Rimannian manifolds*, Diff. Geom. **27** (1988), 225-239.

[2] A. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin-New York, 1987.

[3] J. K. Beem and P. E. Ehrlich, *Global Lorentzian Geometry*, Pure and Applied Mathematics Dekker, New York **67** (1981).

[4] J. K. Beem, P. E. Ehrlich and Th. G. Powell, *Warped product manifolds in relativity*, Selected Studies (Th. M. Rassias, G.M. Rassias, eds.), North-Holland, 1982, pp. 41-56.

[5] J. Bland and M. Kalka, *Negative scalar curvature metrics on non-compact manifolds*, Trans. A. M. S. **316** (1989), 433-446.

[6] F. Dobarro and E. Lami Dozo, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Math. I.H.E.S. **58** (1983), 295-408.

[7] P. E. Ehrlich, Yoon-Tae Jung and Seon-Bu Kim, *Constant Scalar Curvatures on Warped Product Manifolds*, Tsukuba J. Math. **20** (1996), no. 1, 239-256.

[8] M. Gromov and H. B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Math. I.H.E.S. **58**.(1983), 295-408.

[9] Yoon-Tae Jung, *Partial Differential Equations on Semi-Riemannian manifolds*, preprint.
[10] J. L. Kazdan and F. W. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, J. Diff. Geo. **10** (1975), 113-134.
[11] ______, *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature*, Ann. of Math. **101** (1975), 317-331.
[12] ______, *Curvature functions for compact 2 - manifolds*, Ann. of Math. **99** (1974), 14-74.
[13] M. C. Leung, *Conformal scalar curvature equations on complete manifolds*, Comm. in P. D. E. **20** (1995), 367-417.
[14] ______, *Conformal deformation of warped products and scalar curvature functions on open manifolds*, preprint.
[15] H. B. Lawson and M. Michelsohn, *Spin Geometry*, Princeton University Press, Princeton, 1989.
[16] T. G. Powell, *Lorentzian manifolds with non-smooth metrics and warped products*, Ph.D thesis, Univ. of Missouri-Columbia, 1982.

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