A PROPAGATION OF QUADRATICALLY HYPERNORMAL WEIGHTED SHIFTS

YONG BIN CHOI

ABSTRACT. In this note we answer to a question of Curto: Non-first two equal weights in the weighted shift force subnormality in the presence of quadratic hyponormality. Also it is shown that every hyponormal weighted shift with two equal weights cannot be polynomially hyponormal without being flat.

Let \( \mathcal{H} \) be a complex Hilbert space and let \( \mathcal{L}(\mathcal{H}) \) be the algebra of bounded operators on \( \mathcal{H} \). An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to be normal if \( T^*T = TT^* \), hyponormal if \( T^*T \geq TT^* \), and subnormal if \( T = N|_{\mathcal{H}} \), where \( N \) is normal on some Hilbert space \( \mathcal{K} \supseteq \mathcal{H} \). If \( T \) is subnormal then \( T \) is also hyponormal. Recall that given a bounded sequence of positive numbers \( \alpha : \alpha_0, \alpha_1, \cdots \) (called weights), the (unilateral) weighted shift \( W_\alpha \) associated with \( \alpha \) is the operator on \( \ell^2(\mathbb{Z}_+) \) defined by \( W_\alpha e_n := \alpha_n e_{n+1} \) for all \( n \geq 0 \), where \( \{e_n\}_{n=0}^{\infty} \) is the canonical orthonormal basis for \( \ell^2 \). It is straightforward to check that \( W_\alpha \) is hyponormal if and only if \( \alpha_n \leq \alpha_{n+1} \) for all \( n \geq 0 \). The Bram-Halmos criterion for subnormality states that an operator \( T \) is subnormal if and only if

\[
\sum_{i,j} (T^i x_j, T^j x_i) \geq 0
\]

for all finite collections \( x_0, x_1, \cdots, x_k \in \mathcal{H} \) ([2],[3, III.1.9]). If we denote by \([A, B] := AB - BA\) the commutator of two operators \( A \) and \( B \), and if we define \( T \) to be \( k \)-hyponormal whenever the \( k \times k \) operator matrix

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$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k$ is positive, then the Bram-Halmos criterion can be rephrased as saying that $T$ is subnormal if and only if $T$ is $k$-hyponormal for every $k \geq 1$ ([6]). Recall ([1],[6]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be weakly $k$-hyponormal if $T + \alpha_1 T^2 + \cdots + \alpha_{k-1} T^k$ is hyponormal for every $\alpha_1, \cdots, \alpha_{k-1} \in \mathbb{C}$. If $k = 2$ then it is said to be quadratically hyponormal. Also $T \in \mathcal{L}(\mathcal{H})$ is said to be polynomially hyponormal if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that $k$-hyponormal $\Rightarrow$ weakly $k$-hyponormal, and the converse is not true in general (cf. [4],[5]).

J. Stampfli [7] showed that for subnormal weighted shifts $W_\alpha$, a propagation phenomenon occurs which forces the flatness of $W_\alpha$ whenever two equal weights are present.

**Proposition of Subnormality ([7, Theorem 6]).** Let $W_\alpha$ be a subnormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$.

In [4] it was shown that Stampfli’s propagation for subnormality can be extended for 2-hyponormality.

**Proposition of 2-Hyponormality ([4, Corollary 6]).** Let $W_\alpha$ be a 2-hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., $W_\alpha$ is subnormal.

On the other hand, it was shown in [4, Theorem 2] that a hyponormal weighted shift with three equal weights cannot be quadratically hyponormal without being flat: If $W_\alpha$ is a quadratically hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$ and if $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., $W_\alpha$ is subnormal.

Furthermore, in [4, Proposition 11], it was shown that, in the presence of quadratic hyponormality, two consecutive pairs of equal weights again force subnormality: If $W_\alpha$ is a quadratically hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$, and if $\alpha_n = \alpha_{n+1}$ and $\alpha_{n+2} = \alpha_{n+3}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., $W_\alpha$ is subnormal.

In [4] the following question was raised: Whether two non-consecutive pairs of equal weights force subnormality in the presence of quadratic hyponormality? We now answer it affirmatively.
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**Theorem 1.** Let $W_\alpha$ be a weighted shift with weight sequence 
$\{\alpha_n\}_{n=0}^\infty$, and assume that $W_\alpha$ is quadratically hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., $W_\alpha$ is subnormal.

**Proof.** Without loss of generality (the restriction of a quadratically hyponormal operator to an invariant subspace is also quadratically hyponormal), we may assume that $n = 1$ and $\alpha_1 = \alpha_2 = 1$. We will show that either $\alpha_0 = 1$ or $\alpha_3 = 1$. Then three equal weights are present, so that by [4, Theorem 2], we have that $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$. For this we need to consider the selfcommutator $[(W_\alpha + s W_\alpha^2)^*, W_\alpha + s W_\alpha^2]$. Let $W_\alpha$ be a hyponormal weighted shift. For $s \in \mathbb{C}$, we write

$$D(s) := [(W_\alpha + s W_\alpha^2)^*, W_\alpha + s W_\alpha^2]$$

and we let

$$D_n(s) := P_n [(W_\alpha + s W_\alpha^2)^*, W_\alpha + s W_\alpha^2] P_n$$

where $P_n$ is the orthogonal projection onto the subspace generated by $\{e_0, \cdots, e_n\}$,

$$q_n := u_n + |s|^2 v_n, \quad r_n := s \sqrt{w_n},$$

$$u_n := \alpha_n^2 - \alpha_{n+1}^2,$$

$$v_n := \alpha_n^2 \alpha_{n+1}^2 - \alpha_{n-1}^2 \alpha_n^2,$$

$$w_n := \alpha_n^2 (\alpha_{n+1}^2 - \alpha_{n-1}^2)^2,$$

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} = 0$. Clearly, $W_\alpha$ is quadratically hyponormal if and only if $D_n(s) \geq 0$ for all $s \in \mathbb{C}$ and all

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n ≥ 0. Let \( d_n(\cdot) := \det(D_n(\cdot)) \). Then \( d_n \) satisfies the following 2-step recursive formula:

\[
d_0 = q_0, \quad d_1 = q_0q_1 - |r_0|^2, \quad d_{n+2} = q_{n+2}d_{n+1} - |r_{n+1}|^2d_n;
\]

if we let \( t := |s|^2 \), we observe that \( d_n \) is a polynomial in \( t \) of degree \( n + 1 \). For our purpose we assume \( s \in \mathbb{R} \). Then a straightforward calculation shows that if we let \( t := s^2 \) and \( d_n = \sum_{i=0}^{n+1} c(n, i)t^i \), then

\[
d_0(t) = \alpha_0^2 + \alpha_3^2 t;
\]

\[
d_1(t) = (\alpha_0^2 - \alpha_0^4) + c(1, 1)t + c(1, 2)t^2;
\]

\[
d_2(t) = \alpha_0^2(1 - \alpha_0^2)(\alpha_3^2 - 1)t + c(2, 2)t^2 + c(2, 3)t^3;
\]

\[
d_3(t) = \alpha_0^2(1 - \alpha_0^2)(\alpha_3^2 - 1)(\alpha_3^2\alpha_4^2 - 1)t^2 + c(3, 3)t^3 + c(3, 4)t^4;
\]

\[
d_4(t) = \alpha_0^2\alpha_4^2(\alpha_0^2 - 1)(\alpha_3^2 - 1)^3t^2 + c(4, 3)t^3 + c(4, 4)t^4 + c(4, 5)t^5,
\]

so that

\[
\lim_{t \to 0^+} \frac{d_4(t)}{t^2} = \alpha_0^2\alpha_4^2(\alpha_0^2 - 1)(\alpha_3^2 - 1)^3.
\]

Note that \( \alpha_0^2 \leq 1 \leq \alpha_3^2 \). But since \( d_4(t) \geq 0 \) for all \( t \geq 0 \), we must have that either \( \alpha_0 = 1 \) or \( \alpha_3 = 1 \). This completes the proof.

As a corollary of Theorem 1, we can see that two pairs of equal weights force subnormality because the condition “two pairs of equal weights” guarantees \( \alpha_n = \alpha_{n+1} \) for some \( n \geq 1 \), and in turn, by Theorem 1, subnormality. However, in Theorem 1, note that the condition “\( n \geq 1 \)” cannot be relaxed to the condition “\( n \geq 0 \)”.

For example, if (cf. [4, Proposition 7])

\[
(1.1.1) \quad \alpha_0 = \alpha_1 = \sqrt{\frac{2}{3}}, \quad \alpha_n = \sqrt{\frac{n+1}{n+2}} \quad (n \geq 2),
\]

then \( W_\alpha \) is quadratically hyponormal but not subnormal.

Also we have:
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**Theorem 2.** If $W_\alpha$ is a polynomially hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$ with $\alpha_0 = \alpha_1 = 1$, then $\alpha_0 = \alpha_1 = \alpha_2 = \ldots = 1$.

**Proof.** Without loss of generality we may assume $\alpha_0 = \alpha_1 = 1$. We first claim that if $W_\alpha$ is a hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$ and if $\alpha_0 = \alpha_1 = 1$, then

$$W_\alpha \text{ is weakly } k\text{-hyponormal } \implies (2 - \alpha_{k-1}^2) \alpha_k^2 \geq 1 \text{ for all } k \geq 3. \tag{2.1}$$

For (2.1) suppose $W_\alpha$ is weakly $k$-hyponormal. Then $T_k := W_\alpha + sW_\alpha^k$ is hyponormal for every $s \in \mathbb{R}$. For $k \geq 3$ we have

$$D_k := P_k[T_k, T_k^*]P_k = \begin{pmatrix}
q_{k,0} & 0 & 0 & \cdots & r_{k,0} & 0 \\
0 & q_{k,1} & 0 & \cdots & 0 & r_{k,1} \\
0 & 0 & q_{k,2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
r_{k,0} & 0 & 0 & \cdots & q_{k,k-1} & 0 \\
0 & r_{k,1} & 0 & \cdots & 0 & q_{k,k}
\end{pmatrix},$$

where

$$q_{k,j} = \begin{cases} (\alpha_j^2 - \alpha_{j-1}^2) + s^2 (\alpha_{k+j-1}^2 \alpha_{k+j-2}^2 \cdots \alpha_j^2), & (0 \leq j \leq k-1) \\
(\alpha_k^2 - \alpha_{k-1}^2) + s^2 (\alpha_{2k-1}^2 \alpha_{2k-2}^2 \cdots \alpha_k^2 - \alpha_{k-1}^2 \alpha_{k-2}^2 \cdots \alpha_0^2), & (j = k) \end{cases};$$

$$r_{k,0} = s \alpha_0 \alpha_1 \cdots \alpha_{k-2} \alpha_{k-1};$$

$$r_{k,1} = s \alpha_0 \alpha_1 \cdots \alpha_{k-1} (\alpha_k^2 - \alpha_0^2).$$

Thus

$$\det D_k = \begin{cases} (q_{k,k}q_{k,1} - r_{k,1}^2)(q_{k,k-1}q_{k,0} - r_{k,0}^2)q_{k,k-2}q_{k,k-3} \cdots q_{k,2}, & (k \geq 4) \\
(q_{3,3}q_{3,1} - r_{3,1}^2)(q_{3,2}q_{3,0} - r_{3,0}^2), & (k = 3). \end{cases}$$

If $\alpha_0 = \alpha_1 = 1$ and if we let $t := s^2$ then by a straightforward calculation shows that

$$\lim_{t \to 0^+} \frac{\det D_k}{t^k} = (2\alpha_k^2 - \alpha_{k-1}^2 \alpha_k^2 - 1) \prod_{j=2}^{k-1} \frac{\alpha_j^2(\alpha_j^2 - \alpha_{j-1}^2)}{\alpha_j^2}.$$
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Since $W_\alpha$ is weakly $k$-hyponormal then $\det D_k \geq 0$, it follows that $(2 - \alpha_{k-1}^2) \alpha_k^2 - 1 \geq 0$, which proves (2.1). Suppose $W_\alpha$ is polynomially hyponormal. Since $\{\alpha_n\}$ is bounded and increasing it follows that $\lim \alpha_n = \alpha$ for some $\alpha \in \mathbb{C}$. Therefore taking $\lim_n$ on both sides of the inequality in (2.1), we have that $(\alpha - 1)^2 \leq 0$, i.e., $\alpha = 1$, which forces $\alpha_0 = \alpha_1 = \alpha_2 = \cdots = 1$. This completes the proof. $\Box$

**Corollary 3.** If $W_\alpha$ is the weighted shift with weight sequence $\{\alpha_n\}$ which is polynomially hyponormal but not subnormal then the weight sequence $\{\alpha_n\}$ is strictly increasing.

*Proof.* This immediately follows from Theorem 2. $\Box$

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**References**


**Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea**