ON DOUBLY STOCHASTIC $k$-POTENT MATRICES AND REGULAR MATRICES

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ABSTRACT. In this paper, we determine the structure of $k$-potent elements and regular elements of the semigroup $\Omega_n$ of doubly stochastic matrices of order $n$. In connection with this, we find the structure of the matrices $X$ satisfying the equation $AXA = A$. From these, we determine a condition of a doubly stochastic matrix $A$ whose Moore-Penrose generalized inverse is also a doubly stochastic matrix.

0. Introduction

Let $\mathbb{R}^n$ denote the set of all $n$-dimensional real column vectors, and let $e$ denote the vector in $\mathbb{R}^n$ all of whose entries are 1. If all of the entries of a real matrix $A$ are nonnegative, $A$ is called nonnegative matrix and is denoted by $A \geq 0$. An $n \times n$ nonnegative matrix $D$ is said to be doubly stochastic if $De = e, e^T D = e^T$. Let $J_n$ denote the $n \times n$ doubly stochastic matrix all of whose entries are $1/n$. As usual, we denote the set of all $n \times n$ doubly stochastic matrices by $\Omega_n$. The set $\Omega_n$ has very rich and interesting combinatorial and geometric properties. For instance, there are a lot of results for permanent of elements in $\Omega_n$ [6,7,8], and it is very well known that $\Omega_n$ is a convex polyhedron whose vertices are all permutation matrices [See (1) chapter 2]. In addition, $\Omega_n$ is known to be a compact semigroup under the ordinary matrix multiplication with respect to the natural topology [4], and is very closely related to the theory of majorization [5].

In earlier results, it was proved that there is a finite number of maximal subgroups of the semigroup $\Omega_n$ [1], and all idempotent elements are

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obtained in [2]. In 1983, J. S. Montague and R. J. Plemmons gave several equivalence relations for regular elements in \( \Omega_n \), and characterized the Green's relation on the semigroup \( \Omega_n \) [9]. S. Schwarz [11] and H. K. Farahat [3] have shown that the maximal subgroups of \( \Omega_n \) are direct sum of symmetric groups.

In this paper, we explicitly determine the structure of regular elements and \( k \)-potent elements of the semigroup \( \Omega_n \) by using Perron-Frobenius Theorem.

1. Preliminaries

An \( n \times n \) matrix \( A \) is said to be reducible if there exists a permutation matrix \( P \) such that

\[
P^T AP = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}
\]

where \( A_{11} \) and \( A_{22} \) are nonvacuous square matrices, otherwise it is called irreducible.

For each square matrix \( B \), there exists a permutation matrix \( Q \) such that

\[
Q^T B Q = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ O & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_{kk} \end{bmatrix}
\]

where \( B_{ii} \) is an irreducible square matrix for each \( i = 1, \cdots, k \). The matrices \( B_{ii} \) are irreducible components of \( B \). If a doubly stochastic matrix \( D \) is reducible, then it is permutation similar to a direct sum of irreducible doubly stochastic matrices.

The following theorem about the spectral radius of nonnegative irreducible matrices is very well known. For a real square matrix \( A \), let \( \rho(A) \) denote the spectral radius of \( A \).

**Theorem 1** (Perron-Frobenius). If \( A \) is a square nonnegative and irreducible matrix, then

(a) \( \rho(A) \) is an eigenvalue of \( A \),

(b) There is a positive eigenvector corresponding to the eigenvalue \( \rho(A) \),

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(c) Algebraic multiplicity of $\rho(A)$ as an eigenvalue of $A$ is 1.

By Theorem 1 and Geršgorin Theorem, 1 is the simple and maximal eigenvalue of all irreducible doubly stochastic matrix. Since a reducible doubly stochastic matrix is permutation similar to a direct sum of doubly stochastic matrices, we get the following.

**Lemma 2.** Algebraic multiplicity of the eigenvalue 1 of a doubly stochastic matrix is the number of its irreducible components.

2. $k$-potent elements and regular elements of $\Omega_n$

An element $E$ in $\Omega_n$ is said to be *idempotent* if $E^2 = E$, and it is said to be *$k$-potent* if $E^k = E$ but $E^m \neq E$ for $m = 2, \ldots, k - 1$. It is very well known that an idempotent matrix is diagonalizable, and each of its eigenvalues is either 1 or 0. Therefore the rank of an idempotent matrix is the same as its trace. Note $J_n$ is the only doubly stochastic matrix of rank 1. From these facts and from Lemma 2, we obtain the following. The following theorem due to J. L. Doob [2].

**Theorem 3 (Doob).** An $n \times n$ doubly stochastic matrix $A$ is idempotent if and only if there exists a permutation matrix $P$ such that

$$P^T AP = J_{k_1} \oplus J_{k_2} \oplus \cdots \oplus J_{k_r},$$

where $k_1 + k_2 + \cdots + k_r = n$.

For an $r \times r$ permutation matrix $P$ and square matrices $A_1, A_2, \cdots, A_r$ we denote by $\otimes(P; A_1, \cdots, A_r)$ the matrix obtained from $P$ replacing the 1 in the $i$th row by $A_i$ for each $i = 1, 2, \cdots, n$, and each of the zero entries by a zero matrix of suitable size. For example, if

$$P = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix},$$

then

$$\otimes(P; A_1, A_2, A_3) = \begin{bmatrix}
O & A_1 & O \\
O & O & A_2 \\
A_3 & O & O
\end{bmatrix}. $$

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**THEOREM 4.** An $n \times n$ doubly stochastic matrix $A$ is $k$-potent if and only if there exist permutation matrices $P$, $Q$ such that

$$P^T AP = \otimes(Q; J_{k_1}, J_{k_2}, \cdots, J_{k_r})$$

where $k_1 + k_2 + \cdots + k_r = n$ and $Q$ is $k$-potent in which all the irreducible components of $Q$ are of the same order.

**Proof.** Let $k \geq 3$ be an integer such that $A^k = A$. Then $A^{k-1}A = A, AA^{k-1} = A$. This says that all column vectors (row vectors) of $A$ are right (left, respectively) eigenvectors of $A^{k-1}$ corresponding to eigenvalue 1. If $A^{k-1}$ is irreducible, then $e$ is the only eigenvector of $A^{k-1}$ corresponding to the eigenvalue 1. So, $A$ must be $J_n$, an idempotent matrix, and we are done in this case.

Suppose that, for some permutation matrix $P$, $P^T A^{k-1}P$ is of the form

$$P^T A^{k-1}P = \begin{bmatrix} B_{11} & O & \cdots & O \\ O & B_{22} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_{rr} \end{bmatrix}$$

where each $B_{ii}$ is an irreducible component of $A^{k-1}$ for $i = 1, \cdots, r$. Let $P^T AP$ be partitioned as

$$P^T AP = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix},$$

where $A_{ii}$ is of the same size as $B_{ii}$ for each $i = 1, 2, \cdots, r$.

Since all column vectors and row vectors of $A$ are right and left eigenvectors of $A^{k-1}$ corresponding to eigenvalue 1, $B_{ii}A_{ij} = A_{ij}$ and $A_{ij}B_{jj} = A_{ij}$ for all $i, j = 1, 2, \cdots, r$. From the irreducibility of $B_{ii}$, all the entries of each of the blocks $A_{ij}$ are the same for $i, j = 1, 2, \cdots, r$.

Now we partition $P^T A^{k-2}P$ as the same way of $P^T AP$, 

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$$P^T A^{k-2} P = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ C_{21} & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rr} \end{bmatrix}$$

where each $C_{ii}$ is of the same size as $A_{ii}$ for each $i = 1, 2, \cdots, r$.

Since $AA^{k-2} = A^{k-2} A = A^{k-1}$, we have

$$P^T A A^{k-2} P = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ C_{21} & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rr} \end{bmatrix}$$

$$= \begin{bmatrix} B_{11} & O & \cdots & 0 \\ O & B_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_{rr} \end{bmatrix} = P^T A^{k-1} P.$$

Since each $A_{ij}$ is either a positive matrix or $O$ matrix and since $C_{ij}$ is a nonnegative matrix and $\sum_{i=1}^{r} A_{ii} C_{ij} = O$ for $i \neq j$, we get that $A_{ii} C_{ij} = O$ if $i \neq j$.

Suppose that there are two nonzero blocks in the same block row of the above partition $P^T A P$, say $A_{11}$ and $A_{12}$ are nonzero blocks, then all of $C_{ij}$ are $O$ for $i = 1, 2$, $j = 2, \cdots, r$, i.e.

$$P^T A^{k-2} P = \begin{bmatrix} C_{11} & O & O & \cdots & O \\ C_{21} & O & O & \cdots & O \\ C_{31} & C_{32} & C_{33} & \cdots & C_{3r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & C_{r3} & \cdots & C_{rr} \end{bmatrix}$$

Contradicting to that $A^{k-2}$ is doubly stochastic. Therefore each row of $P^T A P$ has exactly one nonnegative block entry, telling us that each nonzero block is a doubly stochastic matrix, so that it is a square matrix. Let $A_i$ denote the nonzero block in the $i$th row of $P^T A P$. Then $P^T A P$ can be expressed as $\otimes(Q; A_1, A_2, \cdots, A_r)$ for some permutation matrix $Q$. 

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Now we determine the size of the each nonzero block. First, we can easily observe that $A_i$ and $A_j$ have the same size if $A_i$ is placed in the position $(i, j)$ of $Q$. There is a well known fact that a matrix $A = (a_{ij})_{n \times n}$ is irreducible if and only if for every pair of distinct integers $p, q$ with $1 \leq p, q \leq n$ there is a sequence of distinct integers $k_1, k_2, \cdots, k_{m-1}, k_m = q$ such that all of the matrix entries $a_{pk_1}, a_{k_1k_2}, \cdots, a_{k_{m-1}q}$ are nonzero.

Thus nonzero blocks in $\otimes(Q; A_1, \cdots, A_r)$ which came from 1’s in each irreducible component of $Q$ have the same size.

The converse is obvious. \hfill \Box

In the semigroup $\Omega_n$, an elements $E \in \Omega_n$ is called a regular element if $EXE = E$ for some $X \in \Omega_n$. If $XEX = X$ in addition, then $E$ and $X$ are said to be semi-inverses each other. In this case, $EX$ and $XE$ are idempotent elements of $\Omega_n$.

In 1973, Montague and Plemons proved that there are only a finite number of regular elements in $\Omega_n$. What are the regular elements of $\Omega_n$? In a survey paper ([7], see p 253) of H. Minc, it is noted that Sinkhorn [unpublished] showed that $A$ is regular if and only if $A$ is permutation equivalent to $J_{n_1} \oplus J_{n_2} \oplus \cdots \oplus J_{n_r}$ for some positive integers $n_1, n_2, \cdots, n_r$. In the following we give a proof of this fact by an argument similar to that used in the proof of Theorem 4.

If $E \in \Omega_n$ is a regular element, then all column vectors of $E$ are eigenvectors of certain doubly stochastic matrix corresponding to the eigenvalue 1.

Now, we are ready to prove the following theorem.

THEOREM 5. An $n \times n$ matrix $A$ in $\Omega_n$ is regular if and only if there exist permutation matrices $P, Q$ such that

$$PAQ = J_{k_1} \oplus J_{k_2} \oplus \cdots \oplus J_{k_r}$$

where each $k_i$’s are positive integers with $k_1 + k_2 + \cdots + k_r = n$.

Proof. Suppose $AXA = A$ where $A, X \in \Omega_n$. Then, by Theorem 3, there exist permutation matrices $P_1, P_2$ such that

(1) \hspace{1cm} $P_1^TAXP_1 = J_{k_1} \oplus \cdots \oplus J_{k_r}$,

(2) \hspace{1cm} $P_2^TXAP_2 = J_{l_1} \oplus \cdots \oplus J_{l_m}$.

Since $AX$ and $XA$ have the same eigenvalues counting multiplicities, we have $r = m$. 406
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Let $P_1^T AP_2$ be partitioned as

$$
P_1^T AP_2 = \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1r} \\
A_{21} & A_{22} & \cdots & A_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
A_{r1} & A_{r2} & \cdots & A_{rr}
\end{bmatrix}
$$

where each of $A_{ij}$'s is of size $k_i \times l_j$ for $i, j = 1, 2, \cdots, r$. Since all column vectors (row vectors) of $A$ is right (left, respectively) eigenvectors of $AX$ ($XA$, respectively) corresponding to the eigenvalue 1, each $A_{ij}$ is either a positive matrix or $O$. Let $P_2^T XP_1$ be partitioned as that of $(P_1^T AP_2)^T$. Then (1) and (2) can be expressed as

$$
P_1^T AP_2 P_2^T XP_1 = \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1r} \\
A_{21} & A_{22} & \cdots & A_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
A_{r1} & A_{r2} & \cdots & A_{rr}
\end{bmatrix}
\begin{bmatrix}
C_{11} & C_{12} & \cdots & C_{1r} \\
C_{21} & C_{22} & \cdots & C_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
C_{r1} & C_{r2} & \cdots & C_{rr}
\end{bmatrix}
\begin{bmatrix}
J_{k_1} & 0 & \cdots & 0 \\
0 & J_{k_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J_{k_r}
\end{bmatrix}
$$

(3)

and

$$
P_2^T XP_1 P_1^T AP_2 = \begin{bmatrix}
C_{11} & C_{12} & \cdots & C_{1r} \\
C_{21} & C_{22} & \cdots & C_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
C_{r1} & C_{r2} & \cdots & C_{rr}
\end{bmatrix}
\begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1r} \\
A_{21} & A_{22} & \cdots & A_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
A_{r1} & A_{r2} & \cdots & A_{rr}
\end{bmatrix}
\begin{bmatrix}
J_{k_1} & 0 & \cdots & 0 \\
0 & J_{k_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J_{k_r}
\end{bmatrix}
$$

(4)

Since the block multiplication is well defined, and since all entries of each $A_{ij}$ are the same, we see that each of the $A_{ij}$'s is either positive or $O$. Now suppose that there are two nonzero blocks in a same block rows of $P_1^T AP_2$, say $A_{11} \neq O$ and $A_{12} \neq O$, then $P_2^T XP_1$ is of the form
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\[
P_2^T X P_1 = \begin{bmatrix}
C_{11} & O & O & \cdots & O \\
C_{21} & O & O & \cdots & O \\
C_{31} & C_{32} & C_{33} & \cdots & C_{3r} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_{r1} & C_{r2} & C_{r3} & \cdots & C_{rr}
\end{bmatrix}.
\]

From (4) and (5), \( C_{21} A_{11} = O \), yielding \( C_{21} = O \), a contradiction. Thus each block row of \( P_1^T A P_2 \) has exactly 1 nonzero block. Similarly, there is only one nonzero block in block column of \( P_1^T A P_2 \). This implies our assertion.

Converse is obvious. \( \square \)

From the fact that \( J_n A = A J_n = J_n \) and from Theorem 5, we have the following.

**Corollary 1.** Let \( A \in \Omega_n \). If the equation \( AXA = A \) is solvable in \( \Omega_n \), then

1. \( P A Q = J_{k_1} \oplus \cdots \oplus J_{k_r} \) for some permutation matrices \( P \) and \( Q \)
2. \( Q^T X P^T = B_1 \oplus \cdots \oplus B_r \) where \( B_i \in \Omega_{k_i} \) for \( i = 1, \cdots, r \).

For an \( m \times n \) real matrix \( Y \), let \( Y^+ \) denote the *Moore-Penrose generalized inverse* of \( Y \), that is, the unique \( n \times m \) matrix satisfying \( YY^+ Y = Y \), \( Y^+ Y Y^+ = Y^+ \), \( (Y Y^+)^T = Y Y^T \) and \( (Y^+ Y)^T = Y^+ Y \).

If \( Y \) is a square nonsingular matrix, then clearly \( Y^+ = Y^{-1} \). Furthermore, if all column vectors of \( Y \) are linearly independent, then \( Y^+ = (Y^T Y)^{-1} Y^T \).

The following corollary directly comes from previous results.

**Corollary 2.** Let \( A \in \Omega_n \). Then the following statements are equivalent.

1. \( A \) is regular.
2. \( P A Q = J_{k_1} \oplus \cdots \oplus J_{k_r} \) for some permutation matrices \( P \) and \( Q \)
3. \( A^T \) is the unique semi-inverse of \( A \).
4. \( A^+ \) is nonnegative.
5. \( A^+ = A^T \).

The equivalence of (1), (3), (4), (5) in Corollary 7 is proved in [9].

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References


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