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A NON-UNICELLULAR OPERATOR

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Abstract. In this paper, we want to give an operator which is not unicellular. We try to prove the non-unicellularity of the operator by using the method given in [8].

1. Introduction

The study of invariant subspaces is one of the most important, most difficult, and most exasperating problems of operator theory. One of the questions about invariant subspaces is the following : is there an operator whose lattice of invariant subspaces is isomorphic to the positive integers? In other words, is there an operator for which there is an one-to-one and order-preserving correspondence $n \longmapsto M_n$, for each $n = 0, 1, 2, \cdots, \infty$, between the indicated integers(including ∞) and all invariant subspaces? An operator satisfying the above condition is called to be unicellular and we have such well-known operators : Donoghue weighted shift operator, Volterra operator, etc. And there are many ways to solve the problem. We have investigated and found a sufficient and necessary condition which a strictly lower triangular operator can be unicellular and showed the unicellularity of the Donoghue weighted shift operator under a certain condition in [8].

In this paper, we want to give an operator which is not unicellular. This investigation will give an information under which condition a strictly lower triangular operator can be unicellular or not.

We first introduce some definitions and a theorem. Let $\mathcal{H}$ be a Hilbert space and A an operator on $\mathcal{H}$. Let M denote a subspace of $\mathcal{H}$. M is invariant under A means that $Ax \in M$ for all $x \in M$.

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The collection of all subspaces of $\mathcal{H}$ under A is denoted by Lat$A$. An operator A is unicellular if the collection Lat$A$ is totally ordered by inclusion. Let A be a bounded operator with $\| A \| < 1$ on ℓ^2, and let $\{e_0, e_1, \cdots\}$ denote the standard basis for ℓ^2. Let x be a column vector in ℓ^2. Then $A^n x$ is a column vector in ℓ^2 for each $n = 1, 2, \cdots$, and we have an infinite matrix $[x, Ax, A^2x, \cdots]^t$ which will be denoted by $S_x(A)$. The matrix $S_x(A)$ is a bounded linear transformation on ℓ^2.

THEOREM 1([8]). *Let A be a strictly lower triangular operator with $\| A \| < 1$ and U the unilateral shift on ℓ^2. Then A is unicellular if and only if for any $x = (1, x_1, \cdots)^t \in \ell^2$, $S_x(U^{*N} A U^N)$ is one-to-one for every $N = 0, 1, 2, \cdots$.*

2. An Example

Let W_1 and W_2 be operators on ℓ^2 defined by the following: $(W_1)_{k\ k+1} = r^{2k+1}$ and $(W_2)_{k\ k+2} = r^{k+1}$ for $k = 0, 1, 2, \cdots$ and the other entries are zero, where $0 < r < 1$.
Then, from easy computation, we have the following facts.
i) $(W_1^n)_{k\ n+k} = r^{2k+1} r^{2k+3} \cdots r^{2k+2n-1} = r^{n(n+2k)}, k = 0, 1, 2, \cdots$.
ii) $(W_2^n)_{k\ 2n+k} = r^{k+1} r^{k+3} \cdots r^{k+2n-1} = r^{n(n+k)}, k = 0, 1, 2, \cdots$.
iii) $(W_1^{n-j} W_2^j)_{k\ n+j+k} = r^{(n-j)(n-j+2k)+j(n+k)}, k = 0, 1, 2, \cdots$.
iv) $(W_2 W_1)_{n\ m} / (W_1 W_2)_{n\ m} = r^3$ for any $n = 0, 1, 2, \cdots$ and $m = n+3$.

Let $B = W_1 + W_2$. Then

$$\begin{aligned} B^n &= (W_1 + W_2)^n \\ &= W_1^n + W_1^{n-1} W_2 + W_1^{n-2} W_2 W_1 + \cdots + W_2 W_1^{n-1} \\ &+ W_1^{n-2} W_2^2 + W_1^{n-3} W_2^2 W_1 + \cdots + W_2^2 W_1^{n-2} + \cdots \\ &+ W_1^{n-j} W_2^j + W_1^{n-j-1} W_2^j W_1 + \cdots + W_2^j W_1^{n-j} + \cdots \\ &+ W_1 W_2^{n-1} + W_2 W_1 W_2^{n-2} + \cdots + W_2^{n-1} W_1 + W_2^n. \end{aligned}$$

From the expansion of $(W_1 + W_2)^n$, we look at all terms in the above expansion which contain exactly $n - j$ W_1's and j W_2's for $0 \leq j \leq n$.

We let $a_k(n,j)$ denote the number of these terms requiring exactly k-interchanges of W_1 with W_2 to obtain $W_1^{n-j}W_2^j$. Let $P(n-j,j,k)$ be the number of partitions of $k=(k_1,k_2,\cdots,k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m$, $m \leq j$ and $k_i \leq n-j$ for $i=1,2,\cdots,m$. Then

$$P(n-j,j,k) = \begin{cases} 0 \text{ if } k > j(n-j) \\ 1 \text{ if } k = j(n-j). \end{cases}$$

LEMMA 2. *For positive integers n,j, and k, $P(n-j,j,k) = a_k(n,j)$.*

Proof. Let $(k_1,k_2,\cdots,k_m)$ be a partition of k such that $\sum_{i=1}^m k_i = k$, $0 < k_m \leq k_{m-1} \leq \cdots \leq k_1 \leq n-j$, and $m \leq j$. This partition corresponds to a term having $n-j$ W_1's and j W_2's by the following procedure. Start from the term $W_1^{n-j}W_2^j = \underbrace{W_1W_1\cdots W_1}_{n-j}\underbrace{W_2W_2\cdots W_2}_{j}$.

Interchange the first W_2 (from the left side) with W_1 k_1 times.
Interchange the second W_2 with W_1 k_2 times.

⋮

Interchange the m-th W_2 with W_1 k_m times. Then the term given by the above procedure requires exactly k-interchaning of W_1 with W_2 to get $W_1^{n-j}W_2^j$.

Reversing this procedure each term having $n-j$ W_1's and j W_2's which needs exactly k-interchanges of W_1 with W_2 to get $W_1^{n-j}W_2^j$, determines a partition of k into at most j parts, each $\leq n-j$. So, $a_k(n,j) = P(n-j,j,k)$. □

THEOREM 3. *Let $B = W_1 + W_2$.*

i) $B^n_{0\ n+j} = (W_1^{n-j}W_2^j)_{0\ n+j}(1 + a_1(n,j)r^3 + a_2(n,j)r^6 + \cdots + a_{j(n-j)}(n,j)\ r^{3j(n-j)})$ *for all* $0 \leq j \leq n$, *where* $a_0(n,j)=1$.

ii) $B^n_{0\ n+j} = B^n_{0\ n+(n-j)}$ *for each* $0 \leq j < n_0$, *where*

$$n_0 = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

iii) $\frac{B^n_{0\ n+j}}{B^n_{0\ n+j+1}} < r$ *for all* $0 \leq j < n_0$.

iv) $\frac{B^n_{0\ n+j}}{B^n_{0\ n+n_0}} < r^{n+n_0-(n+j)}$ for $0 \leq j < n_0$.

Proof. i) $B^n_{0\ n+j} = \left(W_1^{n-j}W_2^j + W_1^{n-j-1}W_2W_1W_2^{j-1} + \cdots + W_2^jW_1^{n-j}\right)_{0\ n+j}$. By the definition of $a_k(n,j)$,

$$B^n_{0\ n+j} = (W_1^{n-j}W_2^j)_{0\ n+j}\left(1 + a_1(n,j)r^3 + a_2(n,j)r^6 + \cdots + a_{j(n-j)}(n,j)r^{3j(n-j)}\right).$$

ii) $(W_1^{n-j}W_2^j)_{0\ n+j} = r^{n^2-nj+j^2} = (W_1^jW_2^{n-j})_{0\ 2n-j}$ for $0 \leq j < n$. Since $a_k(n,j)$ is the number of terms requiring exactly k-interchanging of W_1 with W_2 to obtain $W_1^{n-j}W_2^j$, and since $a_k(n,n-j)$ is equal to the number of terms requiring k-interchanging of W_2 with W_1 to get $W_1^jW_2^{n-j}$, we have $a_k(n,j) = a_k(n,n-j)$. Hence,

$$\begin{aligned} &B^n_{0\ n+j} \\ &= (W_1^{n-j}W_2^j)_{0\ n+j}(1 + a_1(n,j)r^3 + \cdots + a_{j(n-j)}(n,j)r^{3j(n-j)}) \\ &= (W_1^jW_2^{n-j})_{0\ 2n-j}(1 + a_1(n,n-j)r^3 + \cdots + a_{j(n-j)}(n,n-j)r^{3j(n-j)}) \\ &= B^n_{0\ 2n-j}. \end{aligned}$$

iii) By i) and Lemma 2,

$$\begin{aligned} &B^n_{0\ n+j} \\ &= (W_1^{n-j}W_2^j)_{0\ n+j}\Big(1 + P(n-j,j,1)r^3 + \cdots + P\big(n-j,j,j(n-j)\big) \\ &\qquad r^{3j(n-j)}\Big) \\ &= (W_1^{n-j}W_2^j)_{0\ n+j}\sum_{k=0}^{\infty} P(n-j,j,k)r^{3k}, \end{aligned}$$

since $P(n-j,j,k) = 0$ if $k = 0$ or $k > j(n-j)$. By Theorem 3.1 in [1, p.33],

$$\sum_{k=0}^{\infty} P(n-j,j,k)r^{3k} = \frac{(r^3)_n}{(r^3)_{n-j}(r^3)_j},$$

where $(r^3)_k = (1-r^3)^k(1-r^3)^{k-1}\cdots(1-r^3)$ for $k = 0, 1, \cdots$. Hence,

$$B^n_{0\ n+j} = (W_1^{n-j}W_2^j)_{0\ n+j}\frac{(r^3)_n}{(r^3)_{n-j}(r^3)_j}$$

and

$$B^n_{0\ n+j+1} = (W_1^{n-j-1}W_2^{j+1})_{0\ n+j+1}\frac{(r^3)_n}{(r^3)_{n-j-1}(r^3)_{j+1}}.$$

Thus,

$$\begin{aligned}\frac{B^n_{0\ n+j}}{B^n_{0\ n+j+1}} &= \frac{(W_1^{n-j}W_2^{j})_{0\ n+j}(1-(r^3)^{j+1})}{(W_1^{n-j-1}W_2^{j+1})_{0\ n+j+1}(1-(r^3)^{n-j})} \\ &= \frac{r^{n^2-nj+j^2}}{r^{n^2-n(j+1)+(j+1)^2}}\frac{1-(r^3)^{j+1}}{1-(r^3)^{n-j}}.\end{aligned}$$

Since $0 \le j \le n_0 - 1$, $j+1 \le n-j$. Hence, $\frac{1-(r^3)^{j+1}}{1-(r^3)^{n-j}} \le 1$. And $r^{n-2j+1} < r$ for all $0 \le j < n_0$. Thus $\frac{B^n_{0\ n+j}}{B^n_{0\ n+j+1}} < r$.

iv) $\frac{B^n_{0\ n+j}}{B^n_{0\ n+n_0}} = \frac{B^n_{0\ n+j}}{B^n_{0\ n+j+1}}\frac{B^n_{0\ n+j+1}}{B^n_{0\ n+j+2}}\cdots\frac{B^n_{0\ n+n_0-1}}{B^n_{0\ n+n_0}} < r^{n_0-j}$ for all $0 \le j < n_0$. □

Now, we will show that the operator $A = B^*$ is not unicellular.
We need show that r can be chosen so that $S_e(A)$ is not one-to-one.

$$S_e(A) = \begin{pmatrix} 1 & 0 & 0 & & & & \\ 0 & {B^1}_{0\ 1} & {B^1}_{0\ 2} & 0 & & & \\ & 0 & {B^2}_{0\ 2} & {B^2}_{0\ 3} & {B^2}_{0\ 4} & 0 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let D be a diagonal operator defined by $D_{n\ n} = B^n_{0\ n+n_0}$, where

$$n_0 = \begin{cases} \frac{n}{2} & \text{if} \quad n \quad \text{is} \quad \text{even} \\ \frac{n-1}{2} & \text{if} \quad n \quad \text{is} \quad \text{odd.}\end{cases}$$

Then $S_e(A) = D(Q_1 + Q_2)$, where Q_1 is defined by

$$(Q_1)_{n\ k} = \begin{cases} \frac{{B^n}_{0\ k}}{{B^n}_{0\ n+n_0}} & n+1 \le k \le 2n-1,\ n \ge 2 \\ 1 & n = k, \quad n = 0 \quad \text{or} \quad 1 \\ 0 & \text{otherwise}\end{cases}$$

and Q_2 is defined by

$$(Q_2)_{n\ k} = \begin{cases} \frac{B^n{}_{0\ k}}{B^n{}_{0\ n+n_0}} & k = n, \quad \text{or} \quad k = 2n \quad n \geq 2 \\ 1 & k = 2, \quad n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let Q be defined by

$$Q_{n\ k} = \begin{cases} 1 & \text{if} \quad n = k = 0 \quad \text{or} \quad 1 \\ 1 & \text{if} \quad n \geq 2,\ n \text{ is even},\ k - n = \frac{n}{2} \\ 1 & \text{if} \quad n \geq 2,\ n \text{ is odd},\ k - n = \frac{n-1}{2} \text{ or } \frac{n-1}{2} + 1 \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 4. *For the above operators, we have the following.*
1) *Q is a semi-Fredholm operator with index $Q \geq 1$.*
2) *$Q_1 - Q$ is bounded and $||Q_1 - Q|| < \frac{2r}{1-r}$.*
3) *Q_2 is Hilbert-Schmidt.*

Proof. 1) Q is surjective. Since $Q(x) = 0$ for all $x = (0, 0, x_2, 0, \cdots)^t$ in ℓ^2, $\{e_2\} \subset \text{Ker}Q$. And $\text{Ker}Q^* = \{0\}$. Hence Q is a semi-Fredholm operator with index$Q \geq 1$.
2) For $n \geq 4$,

$$\sum_{k=0}^{\infty}(Q_1 - Q)_{n\ k} = \begin{cases} \displaystyle\sum_{\substack{k=n+1 \\ k-n\neq n_0}}^{2n-1} \frac{B^n_{0\ k}}{B^n{}_{0\ n+n_0}} & \text{if } n \text{ is even} \\ \displaystyle\sum_{\substack{k=n+1 \\ k-n\neq n_0 \\ k-n\neq n_0+1}}^{2n-1} \frac{B^n_{0\ k}}{B^n{}_{0\ n+n_0}} & \text{if } n \text{ is odd} \end{cases}$$

$$\begin{aligned} &< 2\sum_{k=n_0+1}^{2n-1} r^{n+n_0-k} && \text{by Theorem 3, iv)} \\ &< 2(r+r^2+\cdots) && \text{since } n_0 \geq 2 \\ &< \frac{2r}{1-r}. \end{aligned}$$

Thus $\sum_{k=0}^{\infty}(Q_1-Q)_{n\ k} < \frac{2r}{1-r}$ for all $n = 0,1,2,\cdots$.

Consider $k \geq 3$, let $k_0 = \begin{cases} [\frac{2(k+1)}{3}] & \text{if } k = 4+3d,\ d = 0,1,\cdots \\ [\frac{2k}{3}] & \text{otherwise} \end{cases}$

where $[\frac{2k}{3}]$ is the greatest positive integer that does not exceed $\frac{2k}{3}$. Then

$$\sum_{k=0}^{\infty}(Q_1-Q)_{n\ k} = \begin{cases} \displaystyle\sum_{\substack{n=\frac{k}{2}+1 \\ n\neq k_0}}^{k-1} \frac{B^n{}_{0\ k}}{B^n{}_{0\ n+n_0}} & \text{if } k \text{ is even} \\ \displaystyle\sum_{\substack{n=\frac{k+1}{2} \\ n\neq k_0}}^{k-1} \frac{B^n{}_{0\ k}}{B^n{}_{n+n_0}} & \text{if } k \text{ is odd} \end{cases}$$

$$\begin{aligned} &< 2\sum_{n=k_0+1}^{k-1} r^{n+n_0-k} \\ &< 2(r+r^2+\cdots), \quad n \geq 2, \quad \text{since} \quad k \geq 3 \\ &< \frac{2r}{1-r}. \end{aligned}$$

Then $\sum_{n=0}^{\infty}(Q_1-Q)_{n\ k} < \dfrac{2r}{1-r}$ for all $k = 0,1,2,\cdots$.

By the Schur test, $Q_1 - Q$ is bounded, and $||Q_1 - Q|| < \frac{2r}{1-r}$.

3) $\frac{B^n{}_{0\ n}}{B^n{}_{0\ n+n_0}} = \frac{B^n{}_{0\ 2n}}{B^n{}_{0\ n+n_0}} < r^{n_0}$ for each $n = 2,3,\cdots$. So, Q_2 is Hilbert-Schmidt. □

Thus if $r > 0$ is small enough, then $||Q_1 - Q||$ can be made arbitrarily small. Then $Q = (Q_1 - Q) + Q$ is a semi-Fredholm operator with index$Q$ =index$Q_1 \geq 1$. Since Q_2 is compact, index$(Q_1 + Q_2)$ =index$Q_1 \geq 1$. Thus dim Ker$(Q_1 + Q_2) \geq 1$, so $Q_1 + Q_2$ is not one-to-one. Therefore $S_e(A)$ is not one-to-one.

EXAMPLE. Let $A = B^*$, where r is sufficiently small so that $\frac{2r}{1-r}$ is small enough. Then A is not unicellular.

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