MAYER-VIETORIS SEQUENCE
AND TORSION THEORY

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ABSTRACT. This work presents a new construction of Mayer-Vietoris sequence using techniques from torsion theory and including the classical case as an example.

1. Introduction

Any reader with a basic grounding in local cohomology will recall the important role that the Mayer-Vietoris sequence can play in that subject. There is an analogue of the Mayer-Vietoris sequence in torsion theory. It is our intention in this paper to present the basic theory of the Mayer-Vietoris sequence in torsion theory. In the following $R$ will always be a commutative Noetherian ring and we denote by $\mathcal{C}(R)$ the category of modules over $R$. We will begin with a quick introduction to torsion theories, looking for the tools we will use in our construction. In section 3 we construct the Mayer-Vietoris sequence relative to a pair of torsion functors $\sigma$ and $\tau$.

2. Preliminaries

A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}(R)$ is a pair of non-empty classes of $R$-modules satisfying:

1. $\text{Hom}_R(M, N) = 0$ for all $M \in \mathcal{T}$ and each $N \in \mathcal{F}$;
2. if $\text{Hom}_R(X, N) = 0$ for all $N \in \mathcal{F}$, then $X \in \mathcal{T}$;
3. if $\text{Hom}_R(M, Y) = 0$ for all $M \in \mathcal{T}$, then $Y \in \mathcal{F}$;
4. $\mathcal{T}$ is closed under submodules.

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The $R$-modules in $\mathcal{T}$ are called torsion and those in $\mathcal{F}$ are called torsion free. From the definition, it follows that $\mathcal{T}$ is closed under quotients, direct sums, extensions and submodules and $\mathcal{F}$ is closed under submodules, direct products, extensions and essential extensions.

It is possible to define a torsion theory from a single non-empty class $\mathcal{T}$ of $R$-modules, closed under quotients, direct sums, extensions and submodules (such a class is called torsion class), by setting

$$\mathcal{F} = \{ N \in \mathcal{C}(R) : \text{Hom}(M, N) = 0 \text{ for all } M \in \mathcal{T} \}.$$ 

The pair $(\mathcal{T}, \mathcal{F})$ is then a torsion theory. In a similar way, we can obtain a torsion theory from a non-empty torsion free class, i.e., a class closed under submodules, direct products, extension and essential extension.

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory, then we can consider for each $R$-module $M$ the submodule

$$\sigma(M) = \sum \{ N \subseteq M : N \in \mathcal{T} \}.$$ 

It is easy to prove that $\sigma(-)$ defines a subfunctor of the identity functor in $\mathcal{C}(R)$ satisfying the following properties: for any $R$-module $M$ and any submodule $N$ of $M$ we have $\sigma(N) = N \cap \sigma(M)$ and $\sigma(M/\sigma(M)) = 0$. Such a functor is called torsion functor. On the other hand, given a torsion functor $\tau$, the classes

$$\mathcal{T}_\tau = \{ M \in \mathcal{C}(R) : \tau(M) = M \}$$

and

$$\mathcal{F}_\tau = \{ M \in \mathcal{C}(R) : \tau(M) = 0 \}$$

define a torsion theory. With this development it is possible to prove that there exists a bijective correspondence between torsion theories and torsion functors.

Moreover there is a one-to-one correspondence between torsion theories and idempotent filters [4, (0.4)]. We note that Gabriel’s definition of idempotent filter [4, Page 7] becomes shorter when one is working in commutative algebra. We call the set of ideals $\Delta$ an idempotent filter over $R$ if it satisfies the following conditions:

1. $R \in \Delta$;
2. if $I \in \Delta$ and $I \subseteq J$, then $J \in \Delta$;
3. if $I \in \Delta$ and $J$ is an ideal of $R$ such that $(J : a) \in \Delta$ for all $a \in I$, then $I \cap J \in \Delta$.

To any torsion theory $(\mathcal{T}, \mathcal{F})$ we associated the set

$$\Delta = \{ I \text{ an ideal of } R : R/I \in \mathcal{T} \}.$$
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Then $\Delta$ is an idempotent filter. Conversely, given any idempotent filter $\Delta$, we call the $R$-module $M$ torsion if $(0 :_R m) \in \Delta$ for all $m \in M$. In this way we obtain a one-to-one correspondence between torsion theories and idempotent filters. It is easy to see that, if $\sigma$ is a torsion functor and $\Delta$ is the corresponding idempotent filter, then, when $M$ is an $R$-module,

$$\sigma(M) = \{ x \in M : Ix = 0 \text{ for some } I \in \Delta \}.$$ 

There is a one-to-one correspondence between torsion theories in $\mathcal{C}(R)$ and partitions of $\text{Spec}(R)$ into two sets, one of them closed under specialization (see [1, Lemma (1.1)]). By saying that $T \subseteq \text{Spec}(R)$ is closed under specialization, we mean that, whenever $p, q \in \text{Spec}(R)$ and $p \subseteq q$ then $p \in T$ implies that $q \in T$. If $(T, F)$ is a torsion theory in $\mathcal{C}(R)$, then the associated partition $(T, F)$ of $\text{Spec}(R)$ is given by

$$T = \{ p \in \text{Spec}(R) : R/p \in T \}$$

and $F = \text{Spec}(R) - T$. The prime ideals in $T$ are called torsion-prime and the prime ideals in $F$ are called free-prime. To every partition $(T, F)$ of $\text{Spec}(R)$ in which $T$ is closed under specialization, we assign the torsion theory in which an $R$-module $M$ is torsion if and only if $\text{Ass}_R(M) \subseteq T$.

Let $(T, F)$ be a torsion theory in $\mathcal{C}(R)$ and $\sigma$ its corresponding torsion functor. It can easily be shown that $\sigma$ is an additive, covariant, $R$-linear and left exact functor. Hence the right derived functors of $\sigma$ may be formed. For each integer $i \geq 0$, we denote the $i$-th right derived functor of $\sigma$ by $H^i_\sigma$.

3. Mayer-Vietoris sequence

Throughout the paper, $R$ will always denote a commutative Noetherian ring with identity. The Mayer-Vietoris sequence in torsion theory involves two torsion functors, and so, throughout this paper, we fix our notations as following. Let $(T, F)$ and $(T_0, F_0)$ be two torsion theories in $\mathcal{C}(R)$. Suppose that $\sigma$, $\tau$ and $(T, F)$, $(U, V)$, where $T$ and $U$ are closed under specialization, denote respectively torsion functors and partitions of $\text{Spec}(R)$ corresponding to $(T, F)$ and $(T_0, F_0)$.

Set $T' = T \cap U$ and $T'' = T \cup U$. It is clear that $T'$ and $T''$ are closed under specialization. Let $(T', F')$ and $\sigma'$ be torsion theory and torsion functor corresponding to partition $(T', F')$ of $\text{Spec}(R)$. Also, we denote
torsion theory and torsion functor corresponding to partition \((T'', F'')\) of \(\text{Spec}(R)\) by \((T'', F'')\) and \(\sigma''\).

**Lemma 3.1.** Let \(M\) be an \(R\)-module. Then \(\sigma(M)\) and \(\tau(M)\) are submodules of \(\sigma''(M)\) and \(\sigma'(M)\) is a submodule of \(\sigma(M) \cap \tau(M)\).

**Proof.** Let \(M\) be an \(R\)-module and \(x \in \sigma(M)\). As we mentioned earlier, there exists an ideal \(I\) in \(L(\sigma) = \{I\text{ an ideal of } R: R/I \in T\}\) such that \(Ix = 0\). Since \(R/I \in T\) thus \(\text{Ass}_R(R/I) \subseteq T\) (see [3, Proposition 1.4]). Now, it is clear that \(\text{Ass}_R(R/I) \subseteq T \cup U = T''\). Hence \(R/I \in T''\) and \(x \in \sigma''(M)\). As above we can show that \(\tau(M)\) is a submodule of \(\sigma''(M)\). In order to prove the second part, let \(x \in \sigma'(M)\). Therefore, there exists \(J \in L(\sigma')\) such that \(Jx = 0\). Since \(L(\sigma') \subseteq L(\sigma) \cap L(\tau)\) thus \(J \in L(\sigma) \cap L(\tau)\) and so \(x \in \sigma(M) \cap \tau(M)\). \(\square\)

In order to prove Propositions 3.3 and 3.4, we need to following remark.

**Remark 3.2.** Let \(0 \rightarrow M \xrightarrow{\alpha} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \rightarrow E^i \xrightarrow{d^i} \cdots\) be a minimal injective resolution for \(R\)-module \(M\). Then

\[
0 \rightarrow \frac{\sigma(E^0)}{\sigma'(E^0)} \xrightarrow{f^0} \frac{\sigma(E^1)}{\sigma'(E^1)} \xrightarrow{f^1} \cdots \rightarrow \frac{\sigma(E^i)}{\sigma'(E^i)} \xrightarrow{f^i} \cdots
\]

and

\[
0 \rightarrow \frac{\sigma''(E^0)}{\tau(E^0)} \xrightarrow{g^0} \frac{\sigma''(E^1)}{\tau(E^1)} \xrightarrow{g^1} \cdots \rightarrow \frac{\sigma''(E^i)}{\tau(E^i)} \xrightarrow{g^i} \cdots
\]

are two complexes of \(R\)-modules and \(R\)-homomorphisms, where for all \(x \in \sigma(E^i)\) and \(y \in \sigma''(E^i)\)

\[
f^i : \sigma(E^i)/\sigma'(E^i) \rightarrow \sigma(E^{i+1})/\sigma'(E^{i+1})
\]

\[
f^i(x + \sigma'(E^i)) = \sigma(d^i)(x) + \sigma'(E^{i+1})
\]

\[
g^i : \sigma''(E^i)/\tau(E^i) \rightarrow \sigma''(E^{i+1})/\tau(E^{i+1})
\]

\[
g^i(y + \tau(E^i)) = \sigma''(d^i)(y) + \tau(E^{i+1}).
\]

**Proposition 3.3.** Let \(E^* : 0 \rightarrow M \xrightarrow{\alpha} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \rightarrow E^i \xrightarrow{d^i} \cdots\) be a minimal injective resolution for \(R\)-module \(M\). Then

\[
0 \rightarrow \sigma'(E^*) \rightarrow \sigma(E^*) \rightarrow \sigma(E^*)/\sigma'(E^*) \rightarrow 0
\]
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is an exact sequence of complexes which results the following long exact sequence

\[
\begin{align*}
0 & \longrightarrow H^0_\sigma(M) \longrightarrow H^0_\sigma(M) \longrightarrow \ker f^0 \longrightarrow H^1_\sigma(M) \longrightarrow \ldots \\
& \longrightarrow H^1_\sigma(M) \longrightarrow H^1_\sigma(M) \longrightarrow \ker f^i / Im f^{i-1} \longrightarrow H^{i+1}_\sigma(M) \longrightarrow \ldots.
\end{align*}
\]

**Proof.** We are concerned with a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \sigma'(E^0) \subseteq \sigma(E^0) \xrightarrow{\text{nat}} \sigma(E^0)/\sigma'(E^0) \longrightarrow 0 \\
\downarrow \sigma'(d^0) & & \downarrow f^0 \\
0 & \longrightarrow & \sigma'(E^1) \subseteq \sigma(E^1) \xrightarrow{\text{nat}} \sigma(E^1)/\sigma'(E^1) \longrightarrow 0 \\
\downarrow & & \ldots \\
0 & \longrightarrow & \sigma'(E^i) \subseteq \sigma(E^i) \xrightarrow{\text{nat}} \sigma(E^i)/\sigma'(E^i) \longrightarrow 0 \\
\downarrow \sigma'(d^i) & & \downarrow f^i \\
0 & \longrightarrow & \sigma'(E^{i+1}) \subseteq \sigma(E^{i+1}) \xrightarrow{\text{nat}} \sigma(E^{i+1})/\sigma'(E^{i+1}) \longrightarrow 0 \\
\downarrow & & \ldots \\
\end{array}
\]

in which the rows are exact and the columns are 0-sequences. Thus by [5, Theorem 4.6.5] we get the long exact sequence

\[
\begin{align*}
0 & \longrightarrow H^0_\sigma(M) \longrightarrow H^0_\sigma(M) \longrightarrow \ker f^0 \longrightarrow H^1_\sigma(M) \longrightarrow \ldots \\
& \longrightarrow H^i_\sigma(M) \longrightarrow H^i_\sigma(M) \longrightarrow \ker f^i / Im f^{i-1} \longrightarrow H^{i+1}_\sigma(M) \longrightarrow \ldots \square
\end{align*}
\]

**Proposition 3.4.** Let \(E^* : 0 \longrightarrow M \xrightarrow{\alpha} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \ldots \longrightarrow E^i \xrightarrow{d^i} \ldots\) be a minimal injective resolution for \(R\)-module \(M\). Then

\[
\begin{align*}
0 & \longrightarrow \tau(E^*) \longrightarrow \sigma''(E^*) \longrightarrow \sigma''(E^*)/\tau(E^*) \longrightarrow 0
\end{align*}
\]

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is an exact sequence of complexes which results the following long exact sequence

\[ 0 \rightarrow H^0_\tau(M) \rightarrow H^0_\sigma(M) \rightarrow \ker g^0 \rightarrow H^1_\tau(M) \rightarrow \ldots \]
\[ \rightarrow H^i_\tau(M) \rightarrow H^i_\sigma(M) \rightarrow \ker g^i/\text{Im}g^{i-1} \rightarrow H^{i+1}_\tau(M) \rightarrow \ldots \]

**Proof.** The proof of (3.4) is similar to that of Proposition 3.3. \qed

In order to prove our main result we need the following useful lemma.

**Lemma 3.5.** Let \( E \) be an injective \( R \)-module. If \( x \in \sigma''(E) \) then there exist \( y \in \sigma(E) \) and \( z \in \tau(E) \) such that \( x = y + z \).

**Proof.** Let \( E = \bigoplus_{p \in \text{Spec}(R)} E(R/p) \) be a decomposition of \( E \) as a direct sum of indecomposable modules. As we mentioned the class \( \mathcal{F}' \) is closed under passage to injective envelopes and direct sums; hence, if \( p \in \mathcal{F}' \) then \( E(R/p) \in \mathcal{F}' \). So that \( \sigma''(E) = \bigoplus_{p \in \mathcal{T} \cup \mathcal{U}} E(R/p) \) which is a submod-ule of \( \bigoplus_{p \in \mathcal{T}} E(R/p) \bigoplus \bigoplus_{p \in \mathcal{U}} E(R/p) = \sigma(E) \bigoplus \tau(E) \). Now, for each \( x \in \sigma''(E) \) there exist \( y \in \sigma(E) \) and \( z \in \tau(M) \) such that \( x = y + z \). \qed

We now come to the main theorem of this paper.

**Theorem 3.6.** For any \( R \)-module \( M \), there is a long exact sequence (called the Mayer-Vietoris sequence for \( M \) with respect to \( \sigma \) and \( \tau \))

\[ 0 \rightarrow H^0_\sigma(M) \rightarrow H^0_\sigma(M) \bigoplus H^0_\tau(M) \rightarrow H^0_\sigma(M) \rightarrow H^1_\sigma(M) \rightarrow \ldots \]
\[ \rightarrow H^i_\sigma(M) \rightarrow H^i_\sigma(M) \bigoplus H^i_\tau(M) \rightarrow H^i_\sigma(M) \rightarrow H^{i+1}_\sigma(M) \rightarrow \ldots \]

**Proof.** Let \( i \geq 0 \). We define

\[ h_i : \sigma(E^i)/\sigma'(E^i) \rightarrow \sigma''(E^i)/\tau(E^i) \]
\[ h_i(x + \sigma'(E^i)) = x + \tau(E^i) \]

for all \( x \in \sigma(E^i) \). We show that \( h_i \) is an isomorphism. It is clear that \( h_i \) is an \( R \)-homomorphism. Let \( x \in \sigma''(E^i) \). By Lemma 3.5 there exist \( y \in \sigma(E^i) \) and \( z \in \tau(E^i) \) such that \( x = y + z \) also

\[ h_i(y + \sigma'(E^i)) = y + \tau(E^i) = y + z + \tau(E^i) = x + \tau(E^i) \]

thus \( h_i \) is an epimorphism. Now, let \( x \in \sigma(E^i) \) and

\[ h_i(x + \sigma'(E^i)) = x + \tau(E^i) = 0. \]

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Thus there are $I \in L(\sigma)$ and $J \in L(\tau)$ such that $Ix = Jx = 0$. Since $R/I \in \mathcal{T}$ and $R/J \in \mathcal{T}_0$ we have $V(I) \subseteq T$ and $V(J) \subseteq U$. Hence $V(I + J) \subseteq U \cap T$ and so $R/I + J \in \mathcal{T}'$. Therefore, from $I + J \in L(\sigma')$ and $(I + J)x = 0$ it follows $x \in \sigma'(E')$, which is the required, i.e. $h_i$ is an isomorphism. Furthermore, by using the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \sigma'(E') & \longrightarrow & \sigma(E') & \longrightarrow & \sigma(E')/\sigma'(E') & \longrightarrow & 0 \\
\downarrow & & \downarrow \subseteq & & \downarrow \subseteq & & \downarrow h_i & & \\
0 & \longrightarrow & \sigma'(E'^{i+1}) & \longrightarrow & \sigma(E'^{i+1}) & \longrightarrow & \sigma(E'^{i+1})/\sigma'(E'^{i+1}) & \longrightarrow & 0 \\
\downarrow & & \downarrow \subseteq & & \downarrow \subseteq & & \downarrow h_i & & \\
0 & \longrightarrow & \tau(E') & \longrightarrow & \sigma''(E') & \longrightarrow & \sigma''(E')/\tau(E') & \longrightarrow & 0 \\
\downarrow & & \downarrow \subseteq & & \downarrow \subseteq & & \downarrow h_{i+1} & & \\
0 & \longrightarrow & \tau(E'^{i-1}) & \longrightarrow & \sigma''(E'^{i+1}) & \longrightarrow & \sigma''(E'^{i+1})/\tau(E'^{i-1}) & \longrightarrow & 0
\end{array}
\]

we obtain the following commutative diagram

\[
\begin{array}{cccccccc}
0 & \longrightarrow & H^0_{\sigma}(M) & \longrightarrow & H^0_{\sigma}(M) & \longrightarrow & \ker f^0 & \longrightarrow & H^1_{\sigma}(M) & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow \bar{h}_0 & & \downarrow & & \\
0 & \longrightarrow & H^0_{\sigma}(M) & \longrightarrow & H^0_{\sigma}(M) & \longrightarrow & \ker g^0 & \longrightarrow & H^1_{\sigma}(M) & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow f^i/\text{Im} f^{i-1} & & \downarrow h_i & & \\
0 & \longrightarrow & H^i_{\sigma}(M) & \longrightarrow & H^i_{\sigma}(M) & \longrightarrow & \ker g^0/\text{Im} g^{i-1} & \longrightarrow & H^{i+1}_{\sigma}(M) & \longrightarrow & \ldots
\end{array}
\]

with exact rows and $\bar{h}_i$ is an isomorphism, for each $i \geq 0$. Now, the reader will have no difficulty in showing that

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^0_{\sigma}(M) & \longrightarrow & H^0_{\sigma}(M) \oplus H^0_{\tau}(M) & \longrightarrow & H^0_{\sigma}(M) & \longrightarrow & H^1_{\sigma}(M) & \longrightarrow & \ldots \\
\longrightarrow & & & \longrightarrow & & & \longrightarrow & & & \longrightarrow & \\
0 & \longrightarrow & H^i_{\sigma}(M) & \longrightarrow & H^i_{\sigma}(M) \oplus H^i_{\tau}(M) & \longrightarrow & H^i_{\sigma}(M) & \longrightarrow & H^{i+1}_{\sigma}(M) & \longrightarrow & \ldots
\end{array}
\]

is an exact sequence. \(\square\)

Theorem 3.6 has some immediate consequences we record here.

**Corollary 3.7.** Let $I$ and $J$ be ideals of $R$. Then for any $R$-module $M$, there is a long exact sequence (called Mayer-Vietoris sequence with
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respect to $I$ and $J$)

$$
0 \rightarrow H^0_{I+J}(M) \rightarrow H^0_{I}(M) \bigoplus H^0_{J}(M) \rightarrow H^0_{I\cap J}(M) \rightarrow H^1_{I+J}(M) \rightarrow \ldots
$$

$$
\rightarrow H^1_{I+J}(M) \rightarrow H^1_{I}(M) \bigoplus H^1_{J}(M) \rightarrow H^1_{I\cap J}(M) \rightarrow H^{1+1}_{I+J}(M) \rightarrow \ldots
$$

**Proof.** Let $I$ be an ideal of $R$. Then $V(I)$ is closed under specialization; hence, the partition $(V(I), \text{Spec}(R) - V(I))$, determines a torsion theory in $\mathcal{C}(R)$ which is denote by $(\mathcal{T}_I, \mathcal{F}_I)$. On the other hand $\Gamma_I$, the local cohomology functor with respect to $I$ corresponds to the same partition of $\text{Spec}(R)$. Hence $\Gamma_I$ is the torsion functor corresponding to $(\mathcal{T}_I, \mathcal{F}_I)$ (see [1, Remark (3.3)]). Therefore, if we consider $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{T}_0, \mathcal{F}_0)$ as torsion theories corresponding to partitions $(V(I), \text{Spec}(R) - V(I))$ and $(V(J), \text{Spec}(R) - V(J))$ of $\text{Spec}(R)$, then it is easy to see that $(\mathcal{T}', \mathcal{F}')$ and $(\mathcal{T}'', \mathcal{F}'')$ are corresponding to partitions $(V(I + J), \text{Spec}(R) - V(I + J))$ and $(V(I \cap J), \text{Spec}(R) - V(I \cap J))$ of $\text{Spec}(R)$. Now, the statement follows by Theorem 3.6. \qed

**Corollary 3.8.** Let $M$ be an $R$-module. Then

$$
\sigma' - \text{depth}_R M \geq \max\{\sigma - \text{depth}_R M, \tau - \text{depth}_R M\}
$$

and

$$
\sigma'' - \text{depth}_R M = \min\{\sigma - \text{depth}_R M, \tau - \text{depth}_R M\}.
$$

**Proof.** The first inequality follows by definition of $\sigma' - \text{depth}_R M$ (see [2, Definition 1.1]) and $\sigma'(M)$ is a submodule of $\sigma(M) \cap \tau(M)$. The second part follows by Theorem 3.6 and

$$
\sigma'' - \text{depth}_R M = \inf\{i \geq 0 : H^i_{\sigma'}(M) \neq 0\}
$$

for an $R$-module $M$ (see [1, Lemma (1.3)]). \qed

**Definition and Remark 3.9.** The non-empty set $\Phi$ of ideals of $R$ is said to be a *system of ideals* if whenever $a, b \in \Phi$, then there is an ideal $c \in \Phi$ such that $c \subseteq ab$.

Let $\Phi$ be a system of ideals of $R$. Certainly the set of all ideals of $R$ is an idempotent filter containing $\Phi$. Let $(\Delta_i)_{i \in I}$ be the family of all idempotent filters each of which contains $\Phi$ and put $\Delta = \bigcap_{i \in I} \Delta_i$. Then $\Phi \subseteq \Delta$ and it is clear that $\Delta$ is an idempotent filter. We call $\Delta$ the...
idempotent filter generated by $\Phi$. In [1, Proposition (3.11)] it is shown that

$$\Delta = \{ I \text{ an ideal of } R : J \subseteq I \text{ for some ideal } J \in \Phi \}$$

and $\Gamma_\Phi = \Gamma_\Delta$, where $\Gamma_\Phi$ is the general local cohomology functor and for any $R$-module $M$,

$$\Gamma_\Phi(M) = \{ m \in M : Im = 0 \text{ for some } I \in \Phi \}.$$  

Moreover, in [1, Proposition (3.11)] it is shown that $\Delta$ is the set of dense ideals corresponding to the torsion theory defined by $\Gamma_\Phi$. That is

$$\Delta = \{ I \text{ an ideal of } R : \Gamma_\Phi(R/I) = R/I \} = \{ I \text{ an ideal of } R : V(I) \subseteq T \},$$

where $T = \{ p \in \text{Spec}(R) : \Gamma_\Phi(R/p) = R/p \}$.

Suppose that $\Psi$ is another system of ideals and $\Delta_0$ idempotent filter generated by $\Psi$. Hence, as above we can show that

$$\Delta_0 = \{ J \text{ an ideal of } R : V(J) \subseteq U \},$$

where $U = \{ p \in \text{Spec}(R) : \Gamma_\Psi(R/p) = R/p \}$. Set

$$\Phi' = \{ I \text{ an ideal of } R : V(I) \subseteq T \cap U \}$$

and

$$\Phi'' = \{ I \text{ an ideal of } R : V(I) \subseteq T \cup U \}.$$  

It is easy to see that $\Phi'$ and $\Phi''$ are idempotent filters and so systems of ideals.

**Corollary 3.10.** Let $\Phi$ and $\Psi$ be systems of ideals and $\Delta, \Delta_0, \Phi'$ and $\Phi''$ be as Remark 3.9. Then for any $R$-module $M$, there is a long exact sequence

$$0 \rightarrow H^0_{\Phi}(M) \rightarrow H^0_{\Phi}(M) \bigoplus H^0_{\Phi}(M) \rightarrow \cdots \rightarrow H^i_{\Phi}(M) \bigoplus H^i_{\Phi}(M) \rightarrow \cdots$$

Proof. Let $\Delta$ be an idempotent filter. It is easy to see that, if $\sigma$ is a torsion functor corresponding to $\Delta$ then, when $M$ is an $R$-module, $\sigma(M) = \Gamma_\Delta(M)$. Also, by Remark 3.9 $\Gamma_\Delta(M) = \Gamma_\Phi(M)$. Therefore, for all $i \geq 0$ and for all $R$-module $M$, $H^i_{\Phi}(M) = H^i_{\Phi}(M)$. Now, by Theorem 3.6 the result follows.  

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