TOEPLITZ OPERATORS ON WEIGHTED ANALYTIC BERGMAN SPACES OF THE HALF-PLANE

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ABSTRACT. On the setting of the half-plane $H = \{x + iy|y > 0\}$ of the complex plane, we study some properties of weighted Bergman spaces and their duality. We also obtain some characterizations of compact Toeplitz operators.

1. Introduction

Let $H$ denote the half-plane in the complex plane $\mathbb{C}$ and let $dA$ denote the usual two-dimensional area measure on $H$. For $1 \leq p < \infty$ and $r \geq 0$, we define $B^{p,r} = \{ f | f$ is holomorphic on $H$ and $\|f\|_{p,r}^{p} = \int_{H} |f(z)|^{p}K_{H}(z,z)^{-r}dA(z) < \infty \}$, where $K_{H}(z,w) = \frac{1}{\pi(x-w)^{2}}$. In fact, Toeplitz operators on holomorphic Bergman spaces of unit disk have been well studied (see [1], [2], [4], [5]) and we study Toeplitz operators of Bergman spaces defined on upper planes (see [3]). Since $B^{2,r}$ is a closed subspace of $L^{2,r}$, there is a unique orthogonal projection $P : L^{2,r} \rightarrow B^{2,r}$ defined by $P(f)(w) = (2r + 1)\int_{H} f(z)\overline{K_{H}(z,w)}^{1+r}K_{H}(z,z)^{-r}dA(z)$ for all $f \in L^{2,r}$. Then we can show that the dual space of $B^{p,r}$ is $B^{q,r}$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$. We also study the pseudo-hyperbolic metric on $H$ and Toeplitz operators. For $f \in L^{\infty,r}$, we define $T_{f}(g) = P(fg)$. Then $T_{f}$ is bounded. We show that $T_{f}$ is compact if and only if $f \in C_{0}(H)$ whenever $f \in H^{\infty,r}$ and $\lim_{z \to \infty} f(z) = 0$.

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2. Weighted Bergman spaces

For $1 \leq p < \infty$ and $r \geq 0$, we define $B^{p,r} = \{ f | f$ is holomorphic on $H$ and $\| f \|_{p,r} = \left( \int_H |f(z)|^p K_H(z, z)^{-r} dA(z) \right)^{1/r} < \infty \}$, where $K_H(z, w) = \frac{1}{\pi(z-w)^2}$. In fact, $K_H(\cdot, w)$ is the reproducing kernel for $B^{2,0}$ and $K_B(z, w) = \frac{1}{\pi(1-|z|^2)^{2r}}$ is the reproducing kernel for $B^{2,0}(\mathbb{B}) = \{ f | f$ is holomorphic on $\mathbb{B}$ \} $\cap L^2(\mathbb{B})$, where $\mathbb{B}$ is the unit disk.

**Lemma 2.1.** (1) $(2r + 1)K_B(\cdot, w)^{1+r}$ is the reproducing kernel for $B^{2,r}(\mathbb{B})$.

(2) For $f \in B^{2,r}$ and $g(z) = \frac{1+z}{1-z}i(z \in \mathbb{B})$, let $h(z) = \frac{f(g(z))}{(1-z)^{2r}}$. Then $h \in B^{2,r}(\mathbb{B})$.

**Proof.** (1) Let $f \in B^{2,r}(\mathbb{B})$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{B}$. Then

$$\int_{\mathbb{B}} f(z)(2r + 1)K_B(z, w)^{1+r} K_B(z, z)^{-r} dA(z)$$

$$= \frac{2r + 1}{\pi} \int_{\mathbb{B}} \sum_{n=0}^{\infty} a_n z^n \sum_{m=0}^{\infty} \binom{-2 - 2r}{m} (\bar{z}w)^m (1 - |z|^2)^{2r} dA(z)$$

$$= \frac{2r + 1}{\pi} \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \binom{-2 - 2r}{m} a_n w^m \int_{\mathbb{B}} z^{m+n} (1 - |z|^2)^{2r} dA(z)$$

$$= \frac{2r + 1}{\pi} \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \binom{m + 2r + 1}{m} a_n w^m$$

$$\int_0^1 \int_0^{2\pi} s^{n+m+1} e^{i(n-m)\theta} (1 - s^2)^{2r} d\theta ds$$

$$= \frac{2r + 1}{\pi} \sum_{n=0}^{\infty} \binom{n + 2r + 1}{n} a_n w^m 2\pi \int_0^1 s^{2n+1} (1 - s^2)^{2r} ds$$

$$= (2r + 1) \sum_{n=0}^{\infty} \frac{(n + 2r + 1)!}{n!(2r + 1)!} \frac{n!\Gamma(2r + 1)}{\Gamma(n + 2r + 2)} a_n w^n$$

$$= f(w).$$

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(2) Clearly \( h \) is holomorphic in \( \mathbb{B} \) and

\[
\int_{\mathbb{B}} |h(z)|^2 K_{\mathbb{B}}^r(z, z)^{-r} dA(z)
\]

\[
= \int_{\mathcal{H}} |h(g^{-1}(z))|^2 K_{\mathbb{B}}^r(g^{-1}(z), g^{-1}(z))^{-r} |(g^{-1}(z))'|^2 dA(z)
\]

\[
= \pi r \int_{\mathcal{H}} \frac{|f(z)|^2}{1 - \frac{z+i}{z+i} \frac{2i}{z+i}^2} \frac{(1 - \frac{z+i}{z+i} \frac{2i}{z+i}^2)^2 r}{(z+i)^2} dA(z)
\]

\[
= \pi r \int_{\mathcal{H}} \frac{|f(z)|^2}{4^{1+r}} (\text{Im} \, z)^{2r} dA(z)
\]

\[
= \frac{1}{4^{1+r}} \int_{\mathcal{H}} |f(z)|^2 K_{\mathbb{B}}^r(z, z)^{-r} dA(z) < \infty.
\]

Thus \( h \in B^{2r}(\mathbb{B}) \).

\[\square\]

**PROPOSITION 2.2.** (1) \((2r+1)K_{\mathbb{H}}(\cdot, w)^{1+r}\) is the reproducing kernel for \(B^{2r}\). Moreover, it is bounded.

(2) For \(1 < p < \infty\) and \(r \geq 0\), \(K_{\mathbb{H}}(\cdot, w)^{1+r} \in B^{p,r}\).

**Proof.** (1) Let \( g(z) = \frac{1 + z+i}{1 - z-i} \). For \( f \in B^{2r}\),

\[
\int_{\mathcal{H}} f(z)(2r+1) K_{\mathbb{H}}(z, w)^{1+r} K_{\mathbb{H}}(z, z)^{-r} dA(z)
\]

\[
= (2r+1) \int_{\mathbb{B}} f(g(z)) K_{\mathbb{H}}(g(z), w)^{1+r} K_{\mathbb{H}}(g(z), g(z))^{-r} |g'(z)|^2 dA(z)
\]

\[
= \frac{2r+1}{\pi} \int_{\mathbb{B}} \frac{f(g(z))}{(1 - \bar{z})^{2+2r}(1 - |z|^2)^{2r}} dA(z)
\]

\[
= \frac{2r+1}{\pi} \int_{\mathbb{B}} \frac{f(g(z))}{(1 - z)^{2+2r}(1 - \frac{1}{g^{-1}(w)\bar{z}})^{2+2r}(1 - |z|^2)^{2r}} dA(z)
\]

\[
= \frac{1}{(w+i)^{2+2r}(1 - g^{-1}(w))^{2+2r}} f(w).
\]

For \(w = x + iy \in H\),

\[
\|(2r+1)K_{\mathbb{H}}(z, w)^{1+r}\|_{\infty} = \sup_{z \in H} |(2r+1)K_{\mathbb{H}}(z, w)^{1+r}| \leq \frac{2r+1}{\pi^{1+r}} y^{-2-2r}.
\]

Thus the reproducing kernel is bounded.
(2) For $1 < p < \infty$ and $r \geq 0$,

$$\int_H |K_H(z, w)|^{1+r} |pK_H(z, z)^{-r} \, dA(z)$$

$$\leq \frac{4^r}{\pi^{(1+r)(p-1)}} \int_0^\infty \frac{1}{(y + t)^{2p+2rp-1}} \int_{-\infty}^\infty \frac{y + t}{\pi \{(s - x)^2 + (y + t)^2\}} \, dx \, dy$$

$$= \frac{4^r}{\pi^{(1+r)(p-1)}} \int_0^\infty \frac{1}{(y + t)^{2p+2rp-1}} \, dy < \infty.$$

For $w = x + iy \in H$,

$$\int_H |K_H(z, w)|^{1+r} K_H(z, z)^{-r} \, dA(z)$$

$$= \int_0^\infty \int_{-\infty}^\infty \frac{1}{\pi \{(s - x)^2 + (y + t)^2\}^{1+r}} \, dx \, dy$$

$$\geq \frac{4^r}{\pi} \int_0^\infty \int_{y-t}^{y+t} \frac{y^{2r}}{\left\{x^2 + (y + t)^2\right\}^{1+r}} \, dx \, dy$$

$$\geq \frac{4^r}{\pi} \int_t^\infty \frac{y^{2r}}{2y^{2r+1}} \, dy$$

$$= \infty.$$

Thus $K_H(\cdot, w)^{1+r} \notin B^{1,r}$. Since $B^{2,r}$ is a closed subspace of the Hilbert space $L^{2,r}$, there is a unique orthogonal projection $P : L^{2,r} \rightarrow B^{2,r}$ such that $P(f)(w) = \int_H f(z)(2r+1)K_H(z, z)^{1+r}K_H(z, z)^{-r} \, dA(z)$ for all $f \in L^{2,r}$ and we can extend to $P$ to $L^{p,r}$. Since $(2r+1)K_H(\cdot, w)^{1+r}$ is the reproducing kernel for $B^{2,r}$, $P|_{B^{2,r}} = I$. In fact, $P|_{B^{2,r}} = I$. To prove this fact, we need the following:

**Lemma 2.3.** Let $1 \leq p < \infty$ and $r \geq 0$. Then $B^{2,r} \cap B^{p,r}$ is dense in $B^{p,r}$.

**Proof.** Take any $f$ in $B^{p,r}$ and $\varepsilon > 0$. For any $\delta > 0$, let $f_\delta(z) = f(z + i\delta)$. Then $f_\delta$ is bounded in $H$ and if $g \in C_c(H)$ then $\lim_{\delta \to 0} g_\delta = g$ in $L^{p,r}$ and hence $\lim_{\delta \to 0} f_\delta = f$ in $L^{p,r}$. Since $C_c(H)$ is dense in $L^{p,r}$,
there is $g \in C_0(H)$ such that $\|f - g\|_{p,r} < \frac{\varepsilon}{3}$. For each $n \in \mathbb{N}$, let $g_n(z) = \frac{(n + z)^{2r}}{(n^2 + z^2)^{2r}}$. Then

\[
\int_H |g_n(z)|^2 K_H(z, z)^{-r} dA(z) = \int_0^\infty \int_{-\infty}^\infty \pi^{r} n^{4+2r} 4^{r} y^{2r} \frac{1}{x^2 + (y + n^2)^{2+2r}} dx dy
\leq n^{4+2r} (4\pi)^r \int_0^\infty \int_1^\infty \frac{y^{2r}}{(x^2 + y^2)^{2+2r}} dx dy
\leq n^{4+2r} (4\pi)^r \int_1^\infty \int_0^\pi \frac{s^{2r}}{(s^2)^{2+2r}} s d\theta ds = \frac{n^{4+2r} (4\pi)^r \pi}{2 + 2r}.
\]

Since $|g_n(z)| = \frac{n^{2r}}{|n^2 + z^2|^{2r}} \leq 1$, $g_n$ is uniformly bounded on $H$ and hence $f \ast g_n \in B^{2r} \cap B^{p,r}$ for all $n \in \mathbb{N}$ and $|f \ast g_n(z) - f(z)|^p \leq 2^p |f_\delta(z)|^p$. By Lebesgue Dominated Convergence Theorem, $\lim_{n \to \infty} \int_H |f \ast g_n(z) - f_\delta(z)|^p K_H(z, z)^{-r} dA(z) = 0$. Since $\|f \ast g_n - f\|_{p,r} \leq \|f \ast g_n - f_\delta\|_{p,r} + \|f_\delta - f\|_{p,r}$, $B^{2r} \cap B^{p,r}$ is dense in $B^{p,r}$.

**Theorem 2.4.** For $1 < p < \infty$ and $r \geq 0$, $P$ is bounded on $L^{p,r}$.

**Proof.** For each $z \in H$, we define $h(z) = (\text{Im}z)^{-\frac{r}{q}},$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then $h$ is a positive measurable function and

\[
\int_H h(z)^p |K_H(z, w)|^{1+r} K_H(z, z)^{-r} dA(z) = \int_H (\text{Im}z)^{-\frac{r}{q} + 2r} \frac{4^r}{\pi |z - \bar{w}|^{2+2r}} dA(z)
= \frac{4^r}{\pi} \int_0^\infty \int_{-\infty}^\infty \frac{y^{2r-\frac{r}{q}}}{\{(x-s)^2 + (y+t)^2\}^{1+r}} dx dy,
\]

where $z = x + iy$ and $w = s + it$. Hence $\int_H h(z)^p |K_H(z, w)|^{1+r} K_H(z, z)^{-r} dA(z) \leq C h(w)^p$ for some $C$ and $\int_H h(z)^q |K_H(z, w)|^{1+r} K_H(z, z)^{-r} dA(z)$
\[ D h(w)^q \] for some \( D \). Take any \( f \) in \( L^{p,r} \). Then

\[
|P(f)(w)| \leq \int_H (2r + 1)|h(z)||f(z)||K_H(z,w)|^{1+r}K_H(z,z)^{-r}dA(z)
\]

\[
= (2r + 1) \int_H h(z)|f(z)||K_H(z,w)|^{1+r}h(z)^{-1}K_H(z,z)^{-r}dA(z)
\]

\[
\leq (2r + 1)(\int_H h(z)^q|K_H(z,w)|^{1+r}K_H(z,z)^{-r}dA(z))\frac{1}{q}
\]

\[
\left( \int_H |f(z)|^p h(z)^{-p}|K_H(z,w)|^{1+r}K_H(z,z)^{-r}dA(z) \right)^{\frac{1}{p}}
\]

and hence \( \int_H |P(f)(w)|^p K_H(w,w)^{-r}dA(w) \leq (2r + 1)^p C^F \int_H h(w)^p
\]

\[
\int_H |f(z)|^p h(z)^{-p}|K_H(z,w)|^{1+r}K_H(z,z)^{-r}dA(z)K_H(w,w)^{-r}dA(w) \leq (2r + 1)^p C^F D \int_H |f(z)|^p K_H(z,z)^{-r}dA(z) = (2r + 1)^p C^F D \|f\|_{p,r}^p
\]

i.e., \( P \) is bounded.

\[
\text{PROPOSITION 2.5. Suppose } 1 \leq p < \infty \text{ and } r \geq 0. \text{ Then } P|_{B^{p,r}} \text{ is the identity.}
\]

**Proof.** Take any \( f \) in \( B^{p,r} \). By Lemma 2.3, there is a sequence \((f_n)\) in \( B^{2,r} \cap B^{p,r} \) such that \( \lim_{n \to \infty} \|f_n - f\|_{p,r} = 0 \). Put \( w = x + iy \in H \). Then

\[
|f_n(w) - f(w)|^p \leq \frac{1}{|B(w, \frac{y}{2})|} \int_{B(w, \frac{y}{2})} |f_n(z) - f(z)|^p dA(z)
\]

\[
\leq \frac{1}{\pi(\frac{y}{2})^2} \int_{B(w, \frac{y}{2})} |f_n(z) - f(z)|^p \left( \frac{\text{Im}z}{{\frac{y}{2}}} \right)^{2r} dA(z)
\]

\[
= \frac{1}{4^{r-1}\pi^{1+r}y^{2+2r}} \int_{B(w, \frac{y}{2})} |f_n(z) - f(z)|^p K_H(z,z)^{-r} dA(z)
\]

\[
\leq \frac{1}{4^{r-1}\pi^{1+r}y^{2+2r}} \int_H |f_n(z) - f(z)|^p K_H(z,z)^{-r} dA(z)
\]
and hence $\lim_{n \to \infty} f_n(w) = f(w)$. We note that
\[
|f_n(w) - f(z)(2r + 1)K_H(z, w)^{1+r}K_H(z, z)^{-r} dA(z)|
\leq \int_H |f_n(z) - f(z)|(2r + 1)|K_H(z, w)|^{1+r}K_H(z, z)^{-r} dA(z)
\leq (2r + 1)\|f_n - f\|_{p,r}\|K_H(\cdot, w)^{1+r}\|_{q,r}.
\]

Since $\lim_{n \to \infty} \|f_n - f\|_{p,r} = 0$,
\[
f(w) = \lim_{n \to \infty} f_n(w)
= \int_H f(z)K_H(z, w)^{1+r}K_H(z, z)^{-r} dA(z)
= P(f)(w).
\]

\[\square\]

**Remark 2.6.** Since $2i \in H$, $B(2i, 1) \subseteq H$ and
\[
\int_H \chi_{B(2i, 1)}K_H(z, z)^{-r} dA(z)
= \int_{B(2i, 1)} \pi^r(2\text{Im}z)^{2r} dA(z)
\leq \pi^r \int_{B(2i, 1)} 6^{2r} dA(z)
= 6^{2r}\pi^{r+1}.
\]

Hence $\chi_{B(2i, 1)} \in L^{1+r}(H)$. We note that
\[
\int_H \left|P(\chi_{B(2i, 1)})(w)K_H(w, w)^{-r} dA(w)
= \int_H \int_{B(2i, 1)} (2r + 1)K_H(z, w)^{1+r}K_H(z, z)^{-r} dA(z) dA(w)
\geq \int_H \int_{B(2i, 1)} rK_H(z, w)^{1+r} 2^{2r} dA(z) dA(w)
\geq \pi^r 4^r \int_H \pi|K_H(2i, w)^{1+r}|K_H(w, w)^{-r} dA(w)
= \infty.
\]
Hence \( P(\chi_{B(2,1)}) \notin B^{1,r}. \)

3. The dual of \( B^{p,r} \) for \( 1 < p < \infty \)

Let \( 1 < p < \infty \) and let \( r \geq 0 \). By Theorem 2.4, \( P : L^{p,r} \longrightarrow B^{p,r} \) is a bounded linear operator. If \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( f \in B^{p,r} \) then \( \Phi_f \) is a bounded linear functional, where \( \Phi_f(g) = \int_H g(z) \overline{f(z)} K_H(z,z)^{-r} dA(z) \) for all \( g \in B^{p,r} \). We define \( \Phi(f) = \Phi_f \). Then \( \Phi : B^{q,r} \longrightarrow (B^{p,r})^* \) is a function. Clearly \( \Phi \) is linear. For \( f \in B^{p,r} \), \( \|\Phi_f\| = \sup_{\|g\|_{p,r}=1} |\Phi_f(g)| \leq \sup_{\|g\|_{p,r}=1} \int_H |g(z)||f(z)| K_H(z,z)^{-r} dA(z) \leq \|f\|_{q,r} \) and hence \( \Phi \) is bounded and linear. Take any \( f \) in \( \ker \Phi \). Since \( (2r+1)K_H(\cdot, w)^{1+r} \in B^{p,r} \), \( 0 = \Phi_f((2r+1)K_H(\cdot, w)^{1+r}) = \int_H (2r+1)K_H(z, w)^{1+r} \overline{f(z)} K_H(z,z)^{-r} dA(z) = \overline{f(z)} \) and hence \( f = 0 \) i.e., \( \Phi \) is 1-1. Take any \( \Lambda \) in \( (B^{p,r})^* \). By Hahn-Banach extension theorem, there is a bounded linear functional \( \tilde{\Lambda} : L^{p,r} \longrightarrow \mathbb{C} \) such that \( \tilde{\Lambda}|_{B^{p,r}} = \Lambda \) and \( \|\tilde{\Lambda}\| = \|\Lambda\| \). By Riesz Representation Theorem, there is \( h \in L^{q,r} \) such that \( \tilde{\Lambda}(g) = \int_H g(z) \overline{h(z)} K_H(z,z)^{-r} dA(z) \) for all \( g \in L^{p,r} \). Then \( \Lambda(g) = \int_H g(z) \overline{h(z)} K_H(z,z)^{-r} dA(z) \) for all \( g \in B^{p,r} \) and \( P(h) \in B^{q,r} \) and hence

\[
\Phi_{P(h)}(g) = \int_H g(w) \overline{P(h)(w)} K_H(w,w)^{-r} dA(w)
= \int_H g(w) \left( \frac{\int_H (2r+1)h(z) K_H(z,w)^{1+r} K_H(z,z)^{-r} dA(z)}{K_H(w,w)^{-r} dA(w)} \right)
= \int_H \overline{h(z)} g(z) K_H(z,z)^{-r} dA(z)
= \Lambda(g)
\]

for all \( g \in B^{p,r} \). Thus \( \Phi_{P(h)}(g) = \Lambda \). By the Open Mapping theorem, this implies the following:

**Theorem 3.1.** For \( 1 < p < \infty \) and \( r \geq 0 \), \( (B^{p,r})^* \cong B^{q,r} \), where \( \frac{1}{p'} + \frac{1}{q} = 1 \).
4. The pseudo-hyperbolic metric on $H$

For $w = x + iy \in H$, let $\varphi_w : H \to H$ be defined by $\varphi_w(z) = \varphi_w(s + it) = \frac{s-x}{y} + i\frac{t}{y}$. Then $\varphi_w$ is a bijective holomorphic function. For $w, z \in H$, $d(w, z) = \frac{|z-w|}{|z-w|}$ is the pseudo-hyperbolic distance on $H$. In fact, we can show that $d$ is a metric on $H$. Let $B(z, t)$ denote a Euclidean disk and for $w = x + iy \in H$ and $0 < R < 1$, let $D(w, R) = \{ z \in \mathbb{C} | d(z, w) < R \}$ which is the pseudo-hyperbolic disk with center $w$ and radius $R$. We note that $z \in D(w, R)$ if and only if $d(z, w) < R$ if $z \in B((x, \frac{1+R^2}{1-R^2}), \frac{2Ry}{1-R^2})$. Thus we have the following:

**Proposition 4.1.** Let $w = x + iy \in H$ and let $0 < R < 1$. Then

$$D(w, R) = B((x, \frac{1+R^2}{1-R^2}y), \frac{2Ry}{1-R^2})$$

and hence

$$|D(w, R)| = \frac{4\pi R^2 y^2}{(1-R^2)^2}.$$

**Lemma 4.2.** For $w = x + iy \in H$, $0 < R < 1$ and $z \in D(w, R)$,

$$\frac{1}{\pi^{1+r} y^{2+2r}} \left( \frac{1-R}{2} \right)^{2+2r} \leq |K_H(z, w)^{1+r}| \leq \frac{1}{\pi^{1+r} y^{2+2r}} \left( \frac{1+R}{2} \right)^{2+2r}.$$

**Proof.** This is immediate from the fact that $\varphi_w^{-1}(D(i, R)) = D(w, R)$ and $|K_H(z, w)^{1+r}| = \frac{1}{\pi^{1+r} |z-w|^{1+2r}}$.

**Lemma 4.3.** Let $0 < R < t < 1$ and let $1 \leq p < \infty$, for any holomorphic function $f$ on $H$, there is a constant $C$ such that $|f(z)|^p \leq \frac{C}{|D(w, t)|^{1+2r}} \int_{D(w, t)} |f(u)|^p K_H(u, u)^{-r} dA(u)$ for all $w \in H$ and $z \in D(w, R)$.

**Proof.** Suppose $w = x + iy \in H$, $z \in D(w, R) = \varphi_w^{-1}(D(i, R))$ and $f$ is holomorphic on $H$. Then $z = \varphi_w^{-1}(\lambda)$ for some $\lambda \in D(i, R)$. Put $I = d(\partial D(i, R), \partial D(i, t))$. Then $B(\varphi_w(z), I) \subset D(i, t)$ and hence

$$f(z) = f(\varphi_w^{-1}(\lambda)) = \frac{1}{|B(\varphi_w(z), I)|} \int_{B(\varphi_w(z), I)} f \circ \varphi_w^{-1} dA.$$
Thus
\[ |f(z)|^p \leq \frac{1}{\pi l^2} \int_{D(t,t)} |f \circ \varphi_w^{-1}|^p \, dA \]
\[ = \frac{1}{\pi l^2 y^2} \int_{D(w,t)} |f(u)|^p \, dA(u) \]
\[ \leq \frac{1}{\pi l^2 y^2} \frac{1}{\pi r \left( \frac{1-t}{1+t} \right)^{2r} y^{2r} 4^r} \int_{D(w,t)} |f(u)|^p K_H(u,u)^{-r} \, dA(u) \]
\[ = \frac{p(1-t)^{2r+\epsilon}(1+t)^{3r+\epsilon}}{|D(w,t)|^{1+r}} \int_{D(w,t)} |f(u)|^p K_H(u,u)^{-r} \, dA(u). \]

This completes the proof. \[\square\]

**Lemma 4.4.** For $0 < R < 1$, there is a sequence $\{w_n\}$ in $H$ such that $\bigcup_{n=1}^\infty D(w_n, R) = H$ and there is a natural number $M$ such that for each $z \in H$, \(|k| z \in D(\bar{w}, \frac{2R+1}{3})\) \(| \leq M.\)

**Proof.** See [3]. \[\square\]

**Theorem 4.5.** Suppose $\mu$ is a positive finite Borel measure on $H$. Then for $0 < R < 1$ and $1 \leq p < \infty$, the following are equivalent:

1. \(\sup_{f \in B_{\mu}^p} \frac{\int_H |f(z)|^p \, d\mu}{\int_H |f(z)|^p K_H(z,z)^{-r} \, dA(z)}\)
2. \(\sup_{w \in H} \frac{\mu(D(w,R))}{|D(w,R)|^{1+r}}\)

**Proof.** Let $w = x + iy \in H$. For $f(z) = \frac{1}{(z-w)^{1+r}}$,
\[ \int_H |f(z)|^p K_H(z,z)^{-r} \, dA(z) = \frac{\pi^{1+r}}{4^{1+r}(2r+1)y^{2+2r}} \]
and hence $f \in B_{\mu}^p$. Since $\int_H |f(z)|^p \, d\mu(z) \geq \int_{D(w,R)} |f(z)|^p \, d\mu(z) \geq \inf_{z \in D(w,R)} |\pi^{1+r} K_H(z,w)^{1+r}|^2 \mu(D(w,R)) = (\frac{1}{2y})^{4+4r} \mu(D(w,R))$,
\[ \frac{\int_H |f(z)|^p \, d\mu(z)}{\int_H |f(z)|^p K_H(z,z)^{-r} \, dA(z)} \leq (2r + 1)R^{2+2r} \left( \frac{1 - R}{1 + R} \right)^{2+2r} \frac{\mu(D(w,R))}{|D(w,R)|^{1+r}}. \]
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Take any \( f \neq 0 \) in \( B^{p,r} \). Then
\[
\int_{H} |f(z)|^{p} \, d\mu(z) \leq \sum_{n=1}^{\infty} \int_{D(w_{n}, R)} |f(z)|^{p} \, d\mu(z),
\]
where \( \{D(w_{n}, R)\} \) is the sequence in Lemma 4.4
\[
\leq \sum_{n=1}^{\infty} \sup_{z \in D(w_{n}, R)} |f(z)|^{p} \mu(D(w_{n}, R))
\leq C \sum_{n=1}^{\infty} \frac{\mu(D(w_{n}, R))}{|D(w_{n}, R)|^{1+r}} \int_{D(w_{n}, \frac{R}{2})} |f(u)|^{p} K_{H}(u, u)^{-r} \, dA(u),
\]
where \( C \) is the constant in Lemma 4.3
\[
\leq CM \sup_{w \in H} \frac{\mu(D(w, R))}{|D(w, R)|^{1+r}} \int_{H} |f(u)|^{p} K_{H}(u, u)^{-r} \, dA(u). \qedhere
\]

5. Toepplitz Operators on \( B^{2,r} \)

We note that \( P : L^{2,r} \longrightarrow B^{2,r} \) is an orthogonal projection. For \( f \in L^{\infty,r}(H, dA) \), we define \( T_{f} : B^{2,r} \longrightarrow B^{2,r} \) by \( T_{f}(g) = P(fg) \) for all \( g \in B^{2,r} \). In this case, \( T_{f} \) is called the Toepplitz operator with symbol \( f \).

**Lemma 5.1.** For \( 1 \leq p < \infty \), \( B^{p,r} \cap L^{\infty,r} \) is dense in \( B^{p,r} \).

**Proof.** Take any \( \varepsilon > 0 \) and any \( f \) in \( B^{p,r} \). For each \( \delta > 0 \), let \( f_{\delta}(z) = f(z + i\delta) \) for all \( z \in H \). Then \( f_{\delta} \) is bounded and \( f_{\delta} \in B^{p,r} \). Since \( C_{C}(H) \) is dense in \( L^{p,r} \), there is \( g \in C_{C}(H) \) such that \( \|g - f\|_{p,r} < \varepsilon \). Since \( \lim_{\delta \to 0} \|g_{\delta} - g\|_{p,r} = 0 \), \( \lim_{\delta \to 0} \|f_{\delta} - f\|_{p,r} = 0 \). \( \square \)

**Proposition 5.2.** Let \( f \in H^{\infty,r} \). If there is a compact subset \( K \) of \( H \) such that \( f = 0 \) on \( H \setminus K \) then \( T_{f} \) is compact.

**Proof.** Take any a norm bounded sequence \( \{g_{n}\} \) in \( B^{2,r} \). For any compact subset \( G \) of \( H \) and any \( w \in G, \|g_{n}(w)\| = \int_{H} g_{n}(z)(2r + 1) \frac{K_{H}(z, w)^{1+r}}{K_{H}(z, z)^{-r}} \, dA(z) \leq (2r + 1)\|g_{n}\|_{2,r} K_{H}(\cdot, w)^{1+r} \|_{2,r} \leq \frac{C\|g_{n}\|_{2,r}}{(d(\partial K, \partial H))^{-r}} \). Since \( \{g_{n}\} \) is a norm bounded sequence, \( \{g_{n}\} \) is uniformly
bounded on each compact subset of $H$ and hence there is a holomorphic function $g$ on $H$ and a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ which converges uniformly on $K$ to $g$. We note that

$$
\int_H |g_{n_k}(z)f(z) - g(z)f(z)|^2 K_H(z, z)^{-r} \, dA(z) \leq \|f\|_{2, r}^2 \int_K |g_{n_k}(z) - g(z)|^2 K_H(z, z)^{-r} \, dA(z) \quad \text{and} \quad \int_H |T_f(g_{n_k})(z) - P(gf)(z)|^2 K_H(z, z)^{-r} \, dA(z) \leq \|f\|_{2, r}^2 \|g_{n_k} - g\|_{2, r}^2.
$$

This implies that $T_f$ is compact. \qed

**Proposition 5.3.** If $f \in C_0(H)$, then $T_f$ is compact.

**Proof.** Since $C_C(H)$ is dense in $C_0(H)$, there is a sequence $\{f_n\}$ in $C_C(H)$ such that $\lim_{n \to \infty} f_n = f$. Then $\|T_{f_n} - T_f\| \leq \|f_n - f\|_{\infty, r}$ and hence $T_f$ is compact. \qed

**Lemma 5.4.** $\frac{K_H(\cdot, w)^{1+r}}{\|K_H(\cdot, w)^{1+r}\|_{2, r}}$ converges weakly to $0$ in $B^{2, r}$ as $\text{Im} w \to 0$.

**Proof.** Let $f \in B^{2, r} \cap L^{\infty, r}$ and let $w = x + iy \in H$. Then

$$
\left\langle f, \frac{K_H(\cdot, w)^{1+r}}{\|K_H(\cdot, w)^{1+r}\|_{2, r}} \right\rangle_{2, r} = \frac{f(w)}{(2r + 1)\|K_H(\cdot, w)^{1+r}\|_{2, r}} = \frac{(4\pi)^{\frac{1}{2r}}}{2r + 1} y^{1+r} f(w).
$$

Since $\lim_{\text{Im} w \to 0} \left\langle f, \frac{K_H(\cdot, w)^{1+r}}{\|K_H(\cdot, w)^{1+r}\|_{2, r}} \right\rangle_{2, r} = 0$ and $B^{2, r} \cap L^{\infty, r}$ is dense in $B^{2, r}$, $\frac{K_H(\cdot, w)^{1+r}}{\|K_H(\cdot, w)^{1+r}\|_{2, r}}$ converges weakly to $0$ in $B^{2, r}$ as $\text{Im} w \to 0$. \qed

**Theorem 5.5.** Let $f$ be a nonnegative function in $L^{\infty, r}$. Then the following are equivalent:

1. $T_f$ is compact
2. There is $R \in (0, 1)$ such that $\left. \frac{1}{|D(w, R)|^{1-r}} \right| \int_{D(w, R)} f(z) K_H(z, z)^{-r} \, dA(z) \to 0$ as $\text{Im} w \to 0$
3. For any $R \in (0, 1)$, $\left. \frac{1}{|D(w, R)|^{1-r}} \right| \int_{D(w, R)} f(z) K_H(z, z)^{-r} \, dA(z) \to 0$ as $\text{Im} w \to 0$.  

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Proof. Take any $R$ in $(0, 1)$ and let $w = x + iy \in H$. Then

$$\frac{1}{|D(w, R)|^{1+r}} \int_{D(w, R)} f(z) K_H(z, z)^{-r} dA(z)$$

$$= \frac{(1 - R^2)^{2+2r}}{(4\pi)^{1+r} R^{2+2r} y^{2+2r}} \int_{D(w, R)} f(z) K_H(z, z)^{-r} dA(z)$$

$$\leq C \int_{D(w, R)} f(z) \frac{|K_H(z, w)^{1+r}|^2}{||K_H(\cdot, w)^{1+r}||_{2,r}^2} K_H(z, z)^{-r} dA(z)$$

$$\leq C \left( \frac{T_f(K_H(\cdot, w)^{1+r})}{||K_H(\cdot, w)^{1+r}||_{2,r}} \frac{K_H(\cdot, w)^{1+r}}{||K_H(\cdot, w)^{1+r}||_{2,r}} \right)_{2,r}$$

$$\leq C \frac{||T_f(K_H(\cdot, w)^{1+r})||_{2,r}}{||K_H(\cdot, w)^{1+r}||_{2,r}}$$

and hence we have (3). It remains to show that (2) implies (1). For each $n \in \mathbb{N}$, let $K_n = \{(x, y) \in \mathbb{C} | -n \leq x \leq n, \frac{1}{n} \leq y \leq n\}$. For $f_n = f \cdot \chi_{K_n}$, $T_{f_n}$ is compact and

$$||T_f - T_{f_n}||^2$$

$$= \sup_{||g||_{2,r} = 1} \int_{H \setminus K_n} f(z) g(z)^2 K_H(z, z)^{-r} dA(z)$$

$$\leq C \sup_{w \in H} \frac{1}{|D(w, R)|^{1+r}} \int_{(H \setminus K_n) \cap D(w, R)} f(z) K_H(z, z)^{-r} dA(z).$$

This implies $\lim_{n \to \infty} ||T_f - T_{f_n}|| = 0$. Hence $T_f$ is compact. \qed

Lemma 5.6. For $f \in H^{\infty, r}$, $|f(w) - f(z)| \leq 2 ||f||_{\infty, r} d(w, z)$ for all $w, z \in H$.

Proof. Take any $w \in H$. Let $\Phi(z) = \frac{f(z) - f(w)}{z - w}$. Since $\lim_{z \to w} \frac{f(z) - f(w)}{z - w} = f'(w)$, $\Phi$ is bounded on $H \setminus \{w\}$. Hence we may assume that $\Phi$ is holomorphic and bounded on $H$. For any sequence $\{z_n\}$ in $H$ such that $z_n \to z_0 \in \partial H$, $\limsup_{n \to \infty} |\Phi(z_n)| \leq 2 ||f||_{\infty, r}$. By Generalized Maximum principle, $|f(z) - f(w)| \leq 2 ||f||_{\infty, r} d(z, w)$. \qed

Theorem 5.7. Suppose $f \in H^{\infty, r}$ and $\lim_{z \to \infty} f(z) = 0$. Then $T_f$ is compact if and only if $f \in C_0(H)$.
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Proof. Suppose that there is \( \delta > 0 \) and a sequence \( \{ w_n \} \) in \( H \) such that \( \lim_{n \to \infty} (\text{Im} w_n) = 0 \) and \( |f(w_n)| \geq \delta \) for all \( n \). By Lemma 5.6, there is \( R > 0 \) such that \( |f(z)| \geq \frac{\delta}{2} \) for all \( z \in D(w_n, R) \) for all \( n \). Since

\[
\int_{D(w_n, R)} |f(w)|^2 |K_H(w, w_n)|^{1+r} \, |K_H(z, w)|^{1+r} \, |K_H(z, w_n)|^{1+r} \, |K_H(z, z)|^{-r} \, dA(z) < \infty,
\]

we have

\[
|f(z)|^2 \left( \frac{1}{1-R^2} \right)^{1+r} \left( \frac{1}{1-R^2} \right)^{1+r} \left( \frac{1}{1-R^2} \right)^{1+r} \left( \frac{2 \pi (1+R)^2 \text{Im} w_n}{(2 \pi (1+R)^2 \text{Im} w_n)} \right)^2 < \infty.
\]

For \( f \in H^{\infty, r} \), \( g \in L^{\infty, r} \) and \( h \in B^{2, r} \), \( T_g T_f(h) = T_g(P(fh)) = T_g(fh) = T_g(f)h \) and hence \( T_f^2 = T_f T_f \) is compact. Since \( \lim_{n \to \infty} \frac{K_H(w_n, w)}{K_H(w_n, w)} \frac{K_H(w_n, w)}{K_H(w_n, w)} = 0 \), \( f \in C_0(H) \).

\[
\begin{align*}
\langle T_f |f|^{1+r}, \frac{K_H(w_n, w)}{K_H(w_n, w)} \frac{K_H(w_n, w)}{K_H(w_n, w)} \rangle & \geq \left( \frac{\delta}{2} \right)^2 (2\pi)^{2-2r} (1-\frac{1}{2r})^{2-2r} \left( \frac{2 \pi (1+R)^2 \text{Im} w_n}{(2 \pi (1+R)^2 \text{Im} w_n)} \right)^2.
\end{align*}
\]

\[
\text{References}
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