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TOEPLITZ OPERATORS ON WEIGHTED ANALYTIC BERGMAN SPACES OF THE HALF-PLANE

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ABSTRACT. On the setting of the half-plane $H = \{x + iy | y > 0\}$ of the complex plane, we study some properties of weighted Bergman spaces and their duality. We also obtain some characterizations of compact Toeplitz operators.

1. Introduction

Let H denote the half-plane in the complex plane $\mathbb{C}$ and let dA denote the usual two-dimensional area measure on H. For $1 \le p < \infty$ and $r \ge 0$, we define $B^{p,r} = \{f | f$ is holomorphic on H and $\|f\|_{p,r}^p = \int_H |f(z)|^p K_H(z,z)^{-r}\, dA(z) < \infty\}$, where $K_H(z,w) = -\frac{1}{\pi(z-\overline{w})^2}$. In fact, Toeplitz operators on holomorphic Bergman spaces of unit disk have been well studied(see [1], [2], [4], [5]) and we study Toeplitz operators of Bergman spaces defined on upper planes(see [3]). Since $B^{2,r}$ is a closed subspace of $L^{2,r}$, there is a unique orthogonal projection $P : L^{2,r} \longrightarrow B^{2,r}$ defined by $P(f)(w) = (2r+1) \int_H f(z)\overline{K_H(z,w)^{1+r}} K_H(z,z)^{-r}\, dA(z)$ for all $f \in L^{2,r}$. Then we can show that the dual space of $B^{p,r}$ is $B^{q,r}$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$. We also study the pseudo-hyperbolic metric on H and Toeplitz operators. For $f \in L^{\infty,r}$, we define $T_f(g) = P(fg)$. Then T_f is bounded. We show that T_f is compact if and only if $f \in C_0(H)$ whenever $f \in H^{\infty,r}$ and $\lim_{z\to\infty} f(z) = 0$.

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2. Weighted Bergman spaces

For $1 \leq p < \infty$ and $r \geq 0$, we define $B^{p,r} = \{f | f$ is holomorphic on H and $\|f\|_{p,r} = \left(\int_H |f(z)|^p K_H(z,z)^{-r}\, dA(z)\right)^{\frac{1}{p}} < \infty\}$, where $K_H(z,w) = -\frac{1}{\pi(z-\overline{w})^2}$. In fact, $K_H(\cdot, w)$ is the reproducing kernel for $B^{2,0}$ and $K_{\mathbb{B}}(z,w) = \frac{1}{\pi(1-z\overline{w})^2}$ is the reproducing kernel for $B^{2,0}(\mathbb{B}) = \{f | f$ is holomorphic on $\mathbb{B}$ $\} \cap L^2(\mathbb{B})$, where $\mathbb{B}$ is the unit disk.

LEMMA 2.1. (1) $(2r+1)K_{\mathbb{B}}(\cdot, w)^{1+r}$ *is the reproducing kernel for* $B^{2,r}(\mathbb{B})$.

(2) *For* $f \in B^{2,r}$ *and* $g(z) = \frac{1+z}{1-z} i (z \in \mathbb{B})$, *let* $h(z) = \frac{f(g(z))}{(1-z)^{2+2r}}$. *Then* $h \in B^{2,r}(\mathbb{B})$.

Proof. (1) Let $f \in B^{2,r}(\mathbb{B})$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{B}$. Then

$$\begin{aligned}
&\int_{\mathbb{B}} f(z)(2r+1)\overline{K_{\mathbb{B}}(z,w)}^{1+r} K_{\mathbb{B}}(z,z)^{-r}\, dA(z) \\
&= \frac{2r+1}{\pi} \int_{\mathbb{B}} \sum_{n=0}^{\infty} a_n z^n \sum_{m=0}^{\infty} \binom{-2-2r}{m} (\overline{z}w)^m (1-|z|^2)^{2r}\, dA(z) \\
&= \frac{2r+1}{\pi} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \binom{-2-2r}{m} a_n w^m \int_{\mathbb{B}} z^n \overline{z}^m (1-|z|^2)^{2r}\, dA(z) \\
&= \frac{2r+1}{\pi} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \binom{m+2r+1}{m} a_n w^m \\
&\qquad \int_0^1 \int_0^{2\pi} s^{n+m+1} e^{i(n-m)\theta} (1-s^2)^{2r}\, d\theta ds \\
&= \frac{2r+1}{\pi} \sum_{n=0}^{\infty} \binom{n+2r+1}{n} a_n w^m 2\pi \int_0^1 s^{2n+1}(1-s^2)^{2r}\, ds \\
&= (2r+1) \sum_{n=0}^{\infty} \frac{(n+2r+1)!}{n!(2r+1)!} \frac{n!\Gamma(2r+1)}{\Gamma(n+2r+2)} a_n w^n \\
&= f(w).
\end{aligned}$$

(2) Clearly h is holomorphic in $\mathbb{B}$ and

$$\begin{aligned}
&\int_{\mathbb{B}} |h(z)|^2 K_{\mathbb{B}}(z,z)^{-r}\, dA(z)\\
&= \int_H |h(g^{-1}(z)|^2 K_{\mathbb{B}}(g^{-1}(z), g^{-1}(z))^{-r} |(g^{-1}(z))'|^2\, dA(z)\\
&= \pi^r \int_H \frac{|f(z)|^2}{|1-\frac{z-i}{z+i}|^{4+4r}} (1-|\frac{z-i}{z+i}|^2)^{2r} |\frac{2i}{(z+i)^2}|^2\, dA(z)\\
&= \pi^r \int_H \frac{|f(z)|^2}{4^{1+r}} (\mathrm{Im} z)^{2r}\, dA(z)\\
&= \frac{1}{4^{1+r}} \int_H |f(z)|^2 K_H(z,z)^{-r}\, dA(z) < \infty.
\end{aligned}$$

Thus $h \in B^{2,r}(\mathbb{B})$. □

PROPOSITION 2.2. (1) $(2r+1)K_H(\cdot, w)^{1+r}$ *is the reproducing kernel for* $B^{2,r}$*. Moreover, it is bounded.*
(2) *For* $1 < p < \infty$ *and* $r \geq 0$, $K_H(\cdot, w)^{1+r} \in B^{p,r}$.

Proof. (1) Let $g(z) = \frac{1+z}{1-z} i$. For $f \in B^{2,r}$,

$$\begin{aligned}
&\int_H f(z)(2r+1)\overline{K_H(z,w)}^{1+r} K_H(z,z)^{-r}\, dA(z)\\
&= (2r+1)\int_{\mathbb{B}} f(g(z))\overline{K_H(g(z),w)}^{1+r} K_H(g(z),g(z))^{-r} |g'(z)|^2\, dA(z)\\
&= \frac{2r+1}{\pi}\int_{\mathbb{B}} f(g(z)) \frac{(1-\overline{z})^{2+2r}(1-|z|^2)^{2r}}{(w+i)^{2+2r}(1-g^{-1}(w)\overline{z})^{2+2r}|1-z|^{4+4r}}\, dA(z)\\
&= \frac{2r+1}{\pi}\int_{\mathbb{B}} \frac{f(g(z))}{(1-z)^{2+2r}} \frac{1}{(w+i)^{2+2r}(1-g^{-1}(w)\overline{z})^{2+2r}} (1-|z|^2)^{2r}\, dA(z)\\
&= \frac{1}{(w+i)^{2+2r}} \frac{f(w)}{(1-g^{-1}(w))^{2+2r}}\\
&= f(w).
\end{aligned}$$

For $w = x + iy \in H$,
$\|(2r+1)K_H(z,w)^{1+r}\|_\infty = \sup_{z\in H} |(2r+1)K_H(z,w)^{1+r}| \leq \frac{2r+1}{\pi^{1+r}} y^{-2-2r}$.
Thus the reproducing kernel is bounded.

(2) For $1 < p < \infty$ and $r \geq 0$,

$$\begin{aligned}
&\int_H |K_H(z,w)^{1+r}|^p K_H(z,z)^{-r}\, dA(z) \\
&\leq \frac{4^r}{\pi^{(1+r)(p-1)}} \int_0^\infty \frac{1}{(y+t)^{2p+2rp-1}} \int_{-\infty}^\infty \frac{y+t}{\pi\{(s-x)^2+(y+t)^2\}}\, dxdy \\
&= \frac{4^r}{\pi^{(1+r)(p-1)}} \int_0^\infty \frac{1}{(y+t)^{2p+2rp-1}}\, dy < \infty.
\end{aligned}$$

□

For $w = x + iy \in H$,

$$\begin{aligned}
&\int_H |K_H(z,w)|^{1+r} K_H(z,z)^{-r}\, dA(z) \\
&= \int_0^\infty \int_{-\infty}^\infty \frac{1}{\pi} \frac{4^r y^{2r}}{\{(s-x)^2+(y+t)^2\}^{1+r}}\, dxdy \\
&\geq \frac{4^r}{\pi} \int_t^\infty \int_{y-t}^\infty \frac{y^{2r}}{\{x^2+(y+t)^2\}^{1+r}}\, dxdy \\
&\geq \frac{4^r}{\pi} \int_t^\infty y^{2r} \int_{2y}^\infty \frac{1}{x^{2+2r}}\, dxdy \\
&= \frac{4^r}{\pi} \int_t^\infty \frac{y^{2r}}{2y^{2r+1}}\, dy \\
&= \infty.
\end{aligned}$$

Thus $K_H(\cdot,w)^{1+r} \notin B^{1,r}$. Since $B^{2,r}$ is a closed subspace of the Hilbert space $L^{2,r}$, there is a unique orthogonal projection $P : L^{2,r} \longrightarrow B^{2,r}$ such that $P(f)(w) = \int_H f(z)(2r+1)\overline{K_H(z,w)^{1+r}} K_H(z,z)^{-r}\, dA(z)$ for all $f \in L^{2,r}$ and we can extend to P to $L^{p,r}$. Since $(2r+1)K_H(\cdot,w)^{1+r}$ is the reproducing kernel for $B^{2,r}$, $P|_{B^{2,r}} = I$. In fact, $P|_{B^{p,r}} = I$. To prove this fact, we need the following:

LEMMA 2.3. *Let $1 \leq p < \infty$ and $r \geq 0$. Then $B^{2,r} \cap B^{p,r}$ is dense in $B^{p,r}$.*

Proof. Take any f in $B^{p,r}$ and $\varepsilon > 0$. For any $\delta > 0$, let $f_\delta(z) = f(z+i\delta)$. Then f_δ is bounded in H and if $g \in C_C(H)$ then $\lim_{\delta\to 0} g_\delta = g$ in $L^{p,r}$ and hence $\lim_{\delta\to 0} f_\delta = f$ in $L^{p,r}$. Since $C_C(H)$ is dense in $L^{p,r}$, there is $g \in C_C(H)$ such that $\|f - g\|_{p,r} < \frac{\varepsilon}{3}$. For each $n \in \mathbb{N}$, let $g_n(z) = \frac{(ni)^{2+r}}{(ni+z)^{2+r}}$. Then

$$\begin{aligned}
&\int_H |g_n(z)|^2 K_H(z,z)^{-r}\, dA(z) \\
&= \int_0^\infty \int_{-\infty}^\infty \frac{\pi^r n^{4+2r} 4^r y^{2r}}{\{x^2+(y+n)^2\}^{2+r}}\, dxdy \\
&\le n^{4+2r}(4\pi)^r \int_{-\infty}^\infty \int_1^\infty \frac{y^{2r}}{(x^2+y^2)^{2+r}}\, dxdy \\
&\le n^{4+2r}(4\pi)^r \int_1^\infty \int_0^\pi \frac{s^{2r}}{(s^2)^{2+r}} s\, d\theta ds \\
&= \frac{n^{4+2r}(4\pi)^r\pi}{2+2r}.
\end{aligned}$$

Since $|g_n(z)| = \frac{n^{2+r}}{|ni+z|^{2+r}} \le 1$, g_n is uniformly bounded on H and hence $f_\delta g_n \in B^{2,r} \cap B^{p,r}$ for all $n \in \mathbb{N}$ and $|f_\delta g_n(z) - f_\delta(z)|^p \le 2^p |f_\delta(z)|^p$. By Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty} \int_H |f_\delta\, g_n(z) - f_\delta(z)|^p\, K_H\, (z,z)^{-r}\, dA(z) = 0$. Since $\|f_\delta g_n - f\|_{p,r} \le \|f_\delta g_n - f_\delta\|_{p,r} + \|f_\delta - f\|_{p,r}$, $B^{2,r} \cap B^{p,r}$ is dense in $B^{p,r}$. □

THEOREM 2.4. *For $1 < p < \infty$ and $r \ge 0$, P is bounded on $L^{p,r}$.*

Proof. For each $z \in H$, we define $h(z) = (\mathrm{Im} z)^{-\frac{r}{pq}}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then h is a positive measurable function and

$$\begin{aligned}
&\int_H h(z)^p |K_H(z,w)|^{1+r} K_H(z,z)^{-r}\, dA(z) \\
&= \int_H (\mathrm{Im} z)^{-\frac{r}{q}+2r} \frac{4^r}{\pi|z-\overline{w}|^{2+2r}}\, dA(z) \\
&= \frac{4^r}{\pi} \int_0^\infty \int_{-\infty}^\infty \frac{y^{2r-\frac{r}{q}}}{\{(x-s)^2+(y+t)^2\}^{1+r}}\, dxdy,
\end{aligned}$$

where $z = x+iy$ and $w = s+it$. Hence $\int_H h(z)^p\, |K_H(z,w)|^{1+r}\, K_H(z,z)^{-r}\, dA(z) \le Ch(w)^p$ for some C and $\int_H h(z)^q\, |K_H(z,w)|^{1+r}\, K_H(z,z)^{-r}\, dA(z)$

$\leq Dh(w)^q$ for some D. Take any f in $L^{p,r}$. Then

$$\begin{aligned}
&|P(f)(w)| \\
&\leq \int_H (2r+1)|f(z)||K_H(z,w)|^{1+r}K_H(z,z)^{-r}\,dA(z) \\
&= (2r+1)\int_H h(z)|f(z)||K_H(z,w)|^{1+r}h(z)^{-1}K_H(z,z)^{-r}\,dA(z) \\
&\leq (2r+1)(\int_H h(z)^q|K_H(z,w)|^{1+r}K_H(z,z)^{-r}\,dA(z))^{\frac{1}{q}} \\
&\qquad (\int_H |f(z)|^p h(z)^{-p}|K_H(z,w)|^{1+r}K_H(z,z)^{-r}\,dA(z))^{\frac{1}{p}}
\end{aligned}$$

and hence $\int_H |P(f)(w)|^p\ K_H(w,w)^{-r}\ dA(w) \leq (2r+1)^p\ C^{\frac{p}{q}} \int_H h(w)^p \int_H |f(z)|^p\ h(z)^{-p}\ |K_H(z,w)|^{1+r}\ K_H(z,z)^{-r}\ dA(z) K_H(w,w)^{-r}\ dA(w) \leq (2r+1)^p\ C^{\frac{p}{q}}\ D \int_H |f(z)|^p\ K_H(z,z)^{-r}\ dA(z) = (2r+1)^p\ C^{\frac{p}{q}} D\ \|f\|_{p,r}^p$ i.e., P is bounded. □

PROPOSITION 2.5. *Suppose* $1 \leq p < \infty$ *and* $r \geq 0$. *Then* $P|_{B^{p,r}}$ *is the identity.*

Proof. Take any f in $B^{p,r}$. By Lemma 2.3, there is a sequence (f_n) in $B^{2,r} \cap B^{p,r}$ such that $\lim_{n\to\infty} \|f_n - f\|_{p,r} = 0$. Put $w = x + iy \in H$. Then

$$\begin{aligned}
&|f_n(w) - f(w)|^p \\
&\leq \frac{1}{|B(w,\frac{y}{2})|}\int_{B(w,\frac{y}{2})} |f_n(z) - f(z)|^p\,dA(z) \\
&\leq \frac{1}{\pi(\frac{y}{2})^2}\int_{B(w,\frac{y}{2})} |f_n(z) - f(z)|^p \frac{(\mathrm{Im} z)^{2r}}{(\frac{y}{2})^{2r}}\,dA(z) \\
&= \frac{1}{4^{r-1}\pi^{1+r}y^{2+2r}}\int_{B(w,\frac{y}{2})} |f_n(z) - f(z)|^p K_H(z,z)^{-r}\,dA(z) \\
&\leq \frac{1}{4^{r-1}\pi^{1+r}y^{2+2r}}\int_H |f_n(z) - f(z)|^p K_H(z,z)^{-r}\,dA(z)
\end{aligned}$$

and hence $\lim_{n\to\infty} f_n(w) = f(w)$. We note that

$$\begin{aligned}
&\left|f_n(w) - \int_H f(z)(2r+1)\overline{K_H(z,w)^{1+r}}K_H(z,z)^{-r}\,dA(z)\right| \\
&\le \int_H |f_n(z) - f(z)|(2r+1)|K_H(z,w)|^{1+r}K_H(z,z)^{-r}\,dA(z) \\
&\le (2r+1)\|f_n - f\|_{p,r}\|K_H(\cdot,w)^{1+r}\|_{q,r}.
\end{aligned}$$

Since $\lim_{n\to\infty}\|f_n - f\|_{p,r} = 0$,

$$\begin{aligned}
f(w) &= \lim_{n\to\infty} f_n(w) \\
&= \int_H f(z)\overline{K_H(z,w)^{1+r}}K_H(z,z)^{-r}\,dA(z) \\
&= P(f)(w).
\end{aligned}$$

□

REMARK 2.6. Since $2i \in H$, $B(2i,1) \subsetneq H$ and

$$\begin{aligned}
&\int_H \chi_{B(2i,1)}K_H(z,z)^{-r}\,dA(z) \\
&= \int_{B(2i,1)} \pi^r(2\mathrm{Im}z)^{2r}\,dA(z) \\
&\le \pi^r \int_{B(2i,1)} 6^{2r}\,dA(z) \\
&= 6^{2r}\pi^{r+1}.
\end{aligned}$$

Hence $\chi_{B(2i,1)} \in L^{1,r}(H)$. We note that

$$\begin{aligned}
&\int_H \left|P(\chi_{B(2i,1)})(w)\right|K_H(w,w)^{-r}\,dA(w) \\
&= \int_H \left|\int_{B(2i,1)} (2r+1)\overline{K_H(z,w)^{1+r}}K_H(z,z)^{-r}\,dA(z)\right|K_H(w,w)^{-r}\,dA(w) \\
&\ge \int_H \left|\int_{B(2i,1)} \pi^r\overline{K_H(z,w)^{1+r}}2^{2r}\,dA(z)\right|K_H(w,w)^{-r}\,dA(w) \\
&= \pi^r 4^r \int_H \pi\left|K_H(2i,w)^{1+r}\right|K_H(w,w)^{-r}\,dA(w) \\
&= \infty.
\end{aligned}$$

Hence $P(\chi_{B(2i,1)}) \notin B^{1,r}$.

3. The dual of $B^{p,r}$ for $1 < p < \infty$

Let $1 < p < \infty$ and let $r \geq 0$. By Theorem 2.4, $P : L^{p,r} \longrightarrow B^{p,r}$ is a bounded linear operator. If $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in B^{q,r}$ then Φ_f is a bounded linear functional, where $\Phi_f(g) = \int_H g(z)\overline{f(z)}K_H(z,z)^{-r}\,dA(z)$ for all $g \in B^{p,r}$. We define $\Phi(f) = \Phi_f$. Then $\Phi : B^{q,r} \longrightarrow (B^{p,r})^*$ is a function. Clearly Φ is linear. For $f \in B^{q,r}$, $\|\Phi_f\| = \sup_{\|g\|_{p,r}=1} |\Phi_f(g)| \leq \sup_{\|g\|_{p,r}=1} \int_H |g(z)||f(z)|K_H(z,z)^{-r}\,dA(z) \leq \|f\|_{q,r}$ and hence Φ is bounded and linear. Take any f in $ker\Phi$. Since $(2r+1)\overline{K_H(\cdot,w)}^{1+r} \in B^{p,r}$, $0 = \Phi_f((2r+1)K_H(\cdot,w)^{1+r}) = \int_H (2r+1)K_H(z,w)^{1+r}\overline{f(z)}K_H(z,z)^{-r}\,dA(z) = \overline{f(z)}$ and hence $f = 0$ i.e., Φ is 1-1. Take any Λ in $(B^{p,r})^*$. By Hahn- Banach extension theorem, there is a bounded linear functional $\tilde{\Lambda} : L^{p,r} \longrightarrow \mathbb{C}$ such that $\tilde{\Lambda}|_{B^{p,r}} = \Lambda$ and $\|\tilde{\Lambda}\| = \|\Lambda\|$. By Riesz Representation Theorem, there is $h \in L^{q,r}$ such that $\tilde{\Lambda}(g) = \int_H g(z)\overline{h(z)}K_H(z,z)^{-r}\,dA(z)$ for all $g \in L^{p,r}$. Then $\Lambda(g) = \int_H g(z)\overline{h(z)}K_H(z,z)^{-r}\,dA(z)$ for all $g \in B^{p,r}$ and $P(h) \in B^{q,r}$ and hence

$$\begin{aligned}
&\Phi_{P(h)}(g)\\
&= \int_H g(w)\overline{P(h)(w)}K_H(w,w)^{-r}\,dA(w)\\
&= \int_H g(w)\overline{\Big(\int_H (2r+1)h(z)\overline{K_H(z,w)^{1+r}}K_H(z,z)^{-r}\,dA(z)\Big)}\\
&\qquad\qquad K_H(w,w)^{-r}\,dA(w)\\
&= \int_H \overline{h(z)}g(z)K_H(z,z)^{-r}\,dA(z)\\
&= \Lambda(g)
\end{aligned}$$

for all $g \in B^{p,r}$. Thus $\Phi_{P(h)}(g) = \Lambda$. By the Open Mapping theorem, this implies the following:

THEOREM 3.1. *For $1 < p < \infty$ and $r \geq 0$, $(B^{p,r})^* \cong B^{q,r}$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

4. The pseudo-hyperbolic metric on H

For $w = x + iy \in H$, let $\varphi_w : H \longrightarrow H$ be defined by $\varphi_w(z) = \varphi_w(s + it) = \frac{s-x}{y} + i\frac{t}{y}$. Then φ_w is a bijective holomorphic function. For $w, z \in H$, $d(w,z) = \frac{|z-w|}{|z-\overline{w}|}$ is the pseudo-hyperbolic distance on H. In fact, we can show that d is a metric on H. Let $B(z,t)$ denote a Euclidean disk and for $w = x + iy \in H$ and $0 < R < 1$, let $D(w,R) = \{z \in \mathbb{C} | \ d(z,w) < R\}$ which is the pseudo-hyperbolic disk with center w and radius R. We note that $z \in D(w,R)$ iff $d(z,w) < R$ iff $z \in B((x, \frac{1+R^2}{1-R^2}y), \frac{2Ry}{1-R^2})$. Thus we have the following:

PROPOSITION 4.1. *Let $w = x + iy \in H$ and let $0 < R < 1$. Then*

$$D(w,R) = B((x, \frac{1+R^2}{1-R^2}y), \frac{2Ry}{1-R^2})$$

and hence

$$|D(w,R)| = \frac{4\pi R^2 y^2}{(1-R^2)^2}.$$

LEMMA 4.2. *For $w = x + iy \in H$, $0 < R < 1$ and $z \in D(w,R)$,*

$$\frac{1}{\pi^{1+r}y^{2+2r}}(\frac{1-R}{2})^{2+2r} \le |K_H(z,w)^{1+r}| \le \frac{1}{\pi^{1+r}y^{2+2r}}(\frac{1+R}{2})^{2+2r}.$$

Proof. This is immediate from the fact that $\varphi_w^{-1}(D(i,R)) = D(w,R)$ and $|K_H(z,w)^{1+r}| = \frac{1}{\pi^{1+r}|z-\overline{w}|^{2+2r}}$. □

LEMMA 4.3. *Let $0 < R < t < 1$ and let $1 \le p < \infty$, for any holomorphic function f on H, there is a constant C such that $|f(z)|^p \le \frac{C}{|D(w,t)|^{1+r}} \int_{D(w,t)} |f(u)|^p K_H(u,u)^{-r}\, dA(u)$ for all $w \in H$ and $z \in D(w,R)$.*

Proof. Suppose $w = x + iy \in H$, $z \in D(w,R) = \varphi_w^{-1}(D(i,R))$ and f is holomorphic on H. Then $z = \varphi_w^{-1}(\lambda)$ for some $\lambda \in D(i,R)$. Put $l = d(\partial D(i,R), \partial D(i,t))$. Then $B(\varphi_w(z), l) \subset D(i,t)$ and hence

$$f(z) = f(\varphi_w^{-1}(\lambda)) = \frac{1}{|B(\varphi_w(z),l)|}\int_{B(\varphi_w(z),l)} f \circ \varphi_w^{-1}\, dA.$$

Thus

$$|f(z)|^p \le \frac{1}{\pi l^2} \int_{D(i,t)} |f \circ \varphi_w^{-1}|^p \, dA$$
$$= \frac{1}{\pi l^2 y^2} \int_{D(w,t)} |f(u)|^p \, dA(u)$$
$$\le \frac{1}{\pi l^2 y^2} \frac{1}{\pi^r (\frac{1-t}{1+t})^{2r} y^{2r} 4^r} \int_{D(w,t)} |f(u)|^p K_H(u,u)^{-r} \, dA(u)$$
$$= \frac{\frac{4t^{2+2r}}{l^2(1-t)^{2+4r}(1+t)^2}}{|D(w,t)|^{1+r}} \int_{D(w,t)} |f(u)|^p K_H(u,u)^{-r} \, dA(u).$$

This completes the proof. □

LEMMA 4.4. *For $0 < R < 1$, there is a sequence $\{w_n\}$ in H such that $\cup_{n=1}^{\infty} D(w_n, R) = H$ and there is a natural number M such that for each $z \in H$, $\left|\{k | z \in D(w_k, \frac{2R+1}{3})\}\right| \le M$.*

Proof. See [3]. □

THEOREM 4.5. *Suppose μ is a positive finite* Borel *measure on H. Then for $0 < R < 1$ and $1 \le p < \infty$, the following are equivalent:*

(1) $\sup_{\substack{f \in B^{p,r} \\ f \ne 0}} \frac{\int_H |f|^p \, d\mu}{\int_H |f(z)|^p K_H(z,z)^{-r} \, dA(z)}$

(2) $\sup_{w \in H} \frac{\mu(D(w,R))}{|D(w,R)|^{1+r}}$.

Proof. Let $w = x + iy \in H$. For $f(z) = \frac{1}{(z-\overline{w})^{\frac{4+4r}{p}}}$,

$$\int_H |f(z)|^p K_H(z,z)^{-r} \, dA(z) = \frac{\pi^{1+r}}{4^{1+r}(2r+1)y^{2+2r}}$$

and hence $f \in B^{p,r}$. Since $\int_H |f(z)|^p \, d\mu(z) \ge \int_{D(w,R)} |f(z)|^p \, d\mu(z) \ge \inf_{z \in D(w,R)} |\pi^{1+r} K_H(z,w)^{1+r}|^2 \mu(D(w,R)) = (\frac{1-R}{2y})^{4+4r} \mu(D(w,R))$,

$$\frac{\int_H |f(z)|^p \, d\mu(z)}{\int_H |f(z)|^p K_H(z,z)^{-r} \, dA(z)} \ge (2r+1) R^{2+2r} (\frac{1-R}{1+R})^{2+2r} \frac{\mu(D(w,R))}{|D(w,R)|^{1+r}}.$$

Take any $f \neq 0$ in $B^{p,r}$. Then

$$\int_H |f(z)|^p \, d\mu(z) \leq \sum_{n=1}^{\infty} \int_{D(w_n,R)} |f(z)|^p \, d\mu(z),$$

where $\{D(w_n, R)\}$ is the sequence in Lemma 4.4

$$\leq \sum_{n=1}^{\infty} \sup_{z \in D(w_n,R)} |f(z)|^p \mu(D(w_n, R))$$

$$\leq C \sum_{n=1}^{\infty} \frac{\mu(D(w_n, R))}{|D(w_n, R)|^{1+r}} \int_{D(w_n, \frac{2R+1}{3})} |f(u)|^p K_H(u,u)^{-r} \, dA(u),$$

where C is the constant in Lemma 4.3

$$\leq CM \sup_{w \in H} \frac{\mu(D(w, R))}{|D(w, R)|^{1+r}} \int_H |f(u)|^p K_H(u,u)^{-r} \, dA(u). \qquad \square$$

5. Toeplitz Operators on $B^{2,r}$

We note that $P : L^{2,r} \longrightarrow B^{2,r}$ is an orthogonal projection. For $f \in L^{\infty,r}(H, dA)$, we define $T_f : B^{2,r} \longrightarrow B^{2,r}$ by $T_f(g) = P(fg)$ for all $g \in B^{2,r}$. In this case, T_f is called the Toeplitz operator with symbol f.

LEMMA 5.1. *For $1 \leq p < \infty$, $B^{p,r} \cap L^{\infty,r}$ is dense in $B^{p,r}$.*

Proof. Take any $\varepsilon > 0$ and any f in $B^{p,r}$. For each $\delta > 0$, let $f_\delta(z) = f(z + i\delta)$ for all $z \in H$. Then f_δ is bounded and $f_\delta \in B^{p,r}$. Since $C_C(H)$ is dense in $L^{p,r}$, there is $g \in C_C(H)$ such that $\|g - f\|_{p,r} < \varepsilon$. Since $\lim_{\delta \to 0} \|g_\delta - g\|_{p,r} = 0$, $\lim_{\delta \to 0} \|f_\delta - f\|_{p,r} = 0$. $\square$

PROPOSITION 5.2. *Let $f \in H^{\infty,r}$. If there is a compact subset K of H such that $f = 0$ on $H \setminus K$ then T_f is compact.*

Proof. Take any a norm bounded sequence $\{g_n\}$ in $B^{2,r}$. For any compact subset G of H and any $w \in G$, $|g_n(w)| = |\int_H g_n(z)(2r+1)\overline{K_H(z,w)^{1+r}}\, K_H(z,z)^{-r} \, dA(z)| \leq (2r+1)\|g_n\|_{2,r} \, \|K_H(\cdot, w)^{1+r}\|_{2,r} \leq \frac{C\|g_n\|_{2,r}}{(d(\partial K, \partial H))^{1+r}}$. Since $\{g_n\}$ is a norm bounded sequence, $\{g_n\}$ is uniformly bounded on each compact subset of H and hence there is a holomorphic function g on H and a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ which converges uniformly on K to g. We note that $\int_H |g_{n_k}(z)f(z) - g(z)f(z)|^2 K_H(z,z)^{-r}\, dA(z) \leq \|f\|_{\infty,r}^2 \int_K |g_{n_k}(z) - g(z)|^2 K_H(z,z)^{-r}\, dA(z)$ and $\int_H |T_f(g_{n_k})(z) - P(gf)(z)|^2 K_H(z,z)^{-r}\, dA(z) \leq \|f\|_{\infty,r}^2 \|g_{n_k} - g\|_{2,r}^2$. This implies that T_f is compact. □

PROPOSITION 5.3. *If $f \in C_0(H)$, then T_f is compact.*

Proof. Since $C_C(H)$ is dense in $C_0(H)$, there is a sequence $\{f_n\}$ in $C_C(H)$ such that $\lim_{n\to\infty} f_n = f$. Then $\|T_{f_n} - T_f\| \leq \|f_n - f\|_{\infty,r}$ and hence T_f is compact. □

LEMMA 5.4. $\frac{K_H(\cdot,w)^{1+r}}{\|K_H(\cdot,w)^{1+r}\|_{2,r}}$ *converges weakly to 0 in $B^{2,r}$ as $\mathrm{Im}w \to 0$.*

Proof. Let $f \in B^{2,r} \cap L^{\infty,r}$ and let $w = x+iy \in H$. Then

$$\left\langle f, \frac{K_H(\cdot,w)^{1+r}}{\|K_H(\cdot,w)^{1+r}\|_{2,r}} \right\rangle_{2,r} = \frac{f(w)}{(2r+1)\|K_H(\cdot,w)^{1+r}\|_{2,r}} = \frac{(4\pi)^{\frac{1+r}{2}}}{2r+1} y^{1+r} f(w).$$

Since $\lim_{\mathrm{Im}w\to 0} \left\langle f, \frac{K_H(\cdot,w)^{1+r}}{\|K_H(\cdot,w)^{1+r}\|_{2,r}} \right\rangle_{2,r} = 0$ and $B^{2,r} \cap L^{\infty,r}$ is dense in $B^{2,r}$, $\frac{K_H(\cdot,w)^{1+r}}{\|K_H(\cdot,w)^{1+r}\|_{2,r}}$ converges weakly to 0 in $B^{2,r}$ as $\mathrm{Im}w \to 0$. □

THEOREM 5.5. *Let f be a nonnegative function in $L^{\infty,r}$.*
Then the following are equivalent:
(1) *T_f is compact*
(2) *There is $R \in (0,1)$ such that $\frac{1}{|D(w,R)|^{1+r}} \int_{D(w,R)} f(z) K_H(z,z)^{-r}\, dA(z) \to 0$ as $\mathrm{Im}w \to 0$*
(3) *For any $R \in (0,1)$, $\frac{1}{|D(w,R)|^{1+r}} \int_{D(w,R)} f(z) K_H(z,z)^{-r}\, dA(z) \to 0$ as $\mathrm{Im}w \to 0$.*

Proof. Take any R in $(0,1)$ and let $w = x + iy \in H$. Then

$$\begin{aligned}
&\frac{1}{|D(w,R)|^{1+r}} \int_{D(w,R)} f(z) K_H(z,z)^{-r}\, dA(z) \\
&= \frac{(1-R^2)^{2+2r}}{(4\pi)^{1+r} R^{2+2r} y^{2+2r}} \int_{D(w,R)} f(z) K_H(z,z)^{-r}\, dA(z) \\
&\le C \int_{D(w,R)} f(z) \frac{|K_H(z,w)^{1+r}|^2}{\|K_H(\cdot,w)^{1+r}\|_{2,r}^2} K_H(z,z)^{-r}\, dA(z) \\
&\le C \Big\langle \frac{T_f(K_H(\cdot,w)^{1+r})}{\|K_H(\cdot,w)^{1+r}\|_{2,r}}, \frac{K_H(\cdot,w)^{1+r}}{\|K_H(\cdot,w)^{1+r}\|_{2,r}} \Big\rangle_{2,r} \\
&\le C \frac{\|T_f(K_H(\cdot,w)^{1+r})\|_{2,r}}{\|K_H(\cdot,w)^{1+r}\|_{2,r}}
\end{aligned}$$

and hence we have (3). It remains to show that (2) implies (1). For each $n \in \mathbb{N}$, let $K_n = \{(x,y) \in \mathbb{C} \mid -n \le x \le n, \frac{1}{n} \le y \le n\}$. For $f_n = f \cdot \chi_{K_n}$, T_{f_n} is compact and

$$\begin{aligned}
&\|T_f - T_{f_n}\|^2 \\
&= \sup_{\|g\|_{2,r}=1} \int_{H \setminus K_n} f(z)^2 |g(z)|^2 K_H(z,z)^{-r}\, dA(z) \\
&\le C \sup_{w \in H} \frac{1}{|D(w,R)|^{1+r}} \int_{(H\setminus K_n)\cap D(w,R)} f(z)^2 K_H(z,z)^{-r}\, dA(z).
\end{aligned}$$

This implies $\lim_{n\to\infty} \|T_f - T_{f_n}\| = 0$. Hence T_f is compact. □

LEMMA 5.6. *For $f \in H^{\infty,r}$, $|f(w) - f(z)| \le 2\|f\|_{\infty,r}\, d(w,z)$ for all $w, z \in H$.*

Proof. Take any $w \in H$. Let $\Phi(z) = \frac{f(z)-f(w)}{\frac{z-w}{z-\bar{w}}}$. Since $\lim_{z\to w} \frac{f(z)-f(w)}{z-w} = f'(w)$, Φ is bounded on $H \setminus \{w\}$. Hence we may assume that Φ is holomorphic and bounded on H. For any sequence $\{z_n\}$ in H such that $z_n \to z_0 \in \partial H$, $\limsup_{n\to\infty} |\Phi(z_n)| \le 2\|f\|_{\infty,r}$. By Generalized Maximum principle, $|f(z) - f(w)| \le 2\|f\|_{\infty,r} d(z,w)$. □

THEOREM 5.7. *Suppose $f \in H^{\infty,r}$ and $\lim_{z\to\infty} f(z) = 0$. Then T_f is compact if and only if $f \in C_0(H)$.*

Proof. Suppose that there is $\delta > 0$ and a sequence $\{w_n\}$ in H such that $\lim_{n\to\infty}(\mathrm{Im} w_n) = 0$ and $|f(w_n)| \geq \delta$ for all n. By Lemma 5.6, there is $R > 0$ such that $|f(z)| \geq \frac{\delta}{2}$ for all $z \in D(w_n, R)$ for all n. Since

$$\int_{D(w_n,R)} |f(w)|^2 \, |K_H(w,w_n)^{1+r}| \int_H |K_H(z,w)^{1+r}| \, |K_H(z,w_n)^{1+r}| \, K_H(z,z)^{-r} \, dA(z) K_H(w,w)^{-r} \, dA(w)$$

$$\leq \|f\|^2_{\infty,r} \|K_H(\cdot,w_n)^{1+r}\|_{2,r} \frac{1}{\pi^{1+r}(\mathrm{Im} w_n)^{2+2r}} \left(\frac{1+R}{2}\right)^{2+2r} \left(\frac{2R}{1-R^2}\mathrm{Im} w_n\right)^2 \pi \frac{(2\pi\frac{(1+R)^2}{1-R^2}\mathrm{Im} w_n)^r}{(2\pi\frac{(1-R)^2}{1-R^2}\mathrm{Im} w_n)} < \infty,$$

$$\left\langle T_{|f|^2} \frac{K_H(\cdot,w_n)^{1+r}}{\|K_H(\cdot,w_n)\|_{2,r}}, \frac{K_H(\cdot,w_n)^{1+r}}{\|K_H(\cdot,w_n)^{1+r}\|_{2,r}} \right\rangle \geq \left(\frac{\delta}{2}\right)^2 (2\pi)^{2+4r} \left(\frac{1-R}{2}\right)^{2+2r} \frac{4R^2(1-R)^{4r}}{\pi^r(1-R^2)^{2+2r}}.$$

For $f \in H^{\infty,r}$, $g \in L^{\infty,r}$ and $h \in B^{2,r}$, $T_g T_f(h) = T_g(P(fh)) = T_g(fh) = T_{gf}(h)$ and hence $T_{|f|^2} = T_{\bar{f}f} = T_{\bar{f}}T_f$ is compact. Since $\lim_{n\to\infty}\langle T_{|f|^2}\frac{K_H(\cdot,w_n)^{1+r}}{\|K_H(\cdot,w_n)^{1+r}\|_{2,r}}, \frac{K_H(\cdot,w_n)^{1+r}}{\|K_H(\cdot,w_n)^{1+r}\|_{2,r}}\rangle = 0$, $f \in C_0(H)$. □

References

[1] S. Axler, *Bergman Spaces and Their Operators, Surveys of Some Recent Results in Operator Theory,* **1**, Pitman Research Notes in Math. **171** (1998).

[2] S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory*, Springer-Verlag, New York, 1992.

[3] S. H. Kang and J. Y. Kim, *Toeplitz Operators on Bergman Spaces defined on upper planes*, Comm. Korean Math. Soc, **14** (1999), no.1, 171–177.

[4] J. Miao, *Toeplitz Operators on Harmonic Bergman Spaces*, Integral Equations and Operator Theory, **27** (1998), 426–438.

[5] K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, Inc., New York and Basel, 1990.

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