Bull. Korean Math. Soc. **37** (2000), No. 3, pp. 451–459

ON THE ATTAINABILITY CONDITION AND LOWER BOUND ESTIMATE OF FAIR PRICE FOR HEDGING PORTFOLIO

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ABSTRACT. In this paper, we observe the relation between the attainability of contingent claims and solutions of associated difference-differential equations. And we find the lower bound estimate of *fair price* using the minimax statistical approach.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$ satisfying the usual conditions:

$$\mathcal{F}_0 \text{ contains all the null sets of } P$$

$$\{\mathcal{F}_t\} \text{ is right-continuous.}$$

Suppose $S = \{S_t : 0 \leq t \leq T\}$ is a vector price process whose components $S^0, S^1, \cdots, S^K$ are adapted, right continuous with left limits and strictly positive. Defining $\theta_t = 1/S_t^0$, we call θ the *discount process* for S. Then discounted price process S_t^* for this security is

$$S_t^* = \theta_t S_t.$$

Let P^* be a probability measure on $(\Omega, \mathcal{F})$ which is equivalent to P and such that S^* is a martingale under P^*, denoting by $E^*(\cdot)$ the associated expectation operator with respect to P^*. Let $\{\alpha_t : 0 \leq t \leq T\}$ is

Received July 20, 1999. Revised April 28, 2000.
2000 Mathematics Subject Classification: 60H30, 60H99.
Key words and phrases: contingent claim, attainability, fair price.
This research was supported by Inchon University Research Grant, 2000.

a trading strategy with predictable component $(\alpha^0, \alpha^1, \cdots, \alpha^K)$. A *contingent claim* is defined as a positive random variable X.

We suppose that the investor may invest in two asset : bonds and stocks. The price process of a bond B_t represents values of bond and the price process S_t of a stock is random in nature.
The wealth process V_t^π of portfolio π is equal to

$$V_t^\pi = \beta_t B_t + \gamma_t S_t,$$

where $\pi = (\beta, \gamma)$ is a predictable investor strategy(hedge). We assume that the capital V_t^π is realized without any inflow or outflow. Let $f = (f_t)_{t \geq 0}$ be a nonnegative progressively measurable process on $(\Omega, \mathcal{F})$. We interpret f_t as the payment or the reward of an American option. The strategy π is called a hedging portfolio for f, if

$$V_t^\pi \geq f_t \quad P - a.s.\ t \geq 0.$$

The valuation method by means of difference-differential equations assumes a functional relation of the contingent claim X of the form

$$X_t = E^*[X|\mathcal{F}_t] = h(t, S_t^1, \cdots, S_t^K).$$

In case of European call option, we have $X_t = V_t^\pi$, $X = (S_T - c)^+$ for the exercise price c. The initial wealth of hedging portfolio gives *fair price* or *premium* for corresponding American option. In this case, a claim is said to be *attainable*([2]).

In this note, we observe relation between the attainability of contingent claims and solutions of associated difference- differential equations. And we obtain the lower bound estimate of initial wealth using the minimax statistical approach.

2. The Main Results

Define

$$\mathcal{L}_{P^*} = \Big\{ a + \sum_{k=1}^{K} \int \alpha^k dS^k,\ a \in \mathbf{R},\ \alpha^k \text{ predictable},$$
$$E^*\Big[\int (\alpha^k)^2 d < S^k, S^k > < \infty,\ k \in \{1, \cdots, K\} \Big] \Big\}.$$

Here $< \cdot, \cdot >$ is a quadratic covariational process.

We begin with:

LEMMA 1. *A contingent claim X is P^*-attainable if and only if $X_t = E^*[X|\mathcal{F}_t]$ is an element of $\mathcal{L}_{P^*}$.*

Proof. See Harrison and Pliska [2]. □

LEMMA 2. *Assume that the following conditions hold true:*

$$E^*\Big[\int_0^T \big(\frac{\partial h}{\partial S^k}(s, S_s^1, \cdots, S_s^K)\big)^2 d < S^k, S^k >_s\Big] < \infty, \ k \in \{1, \cdots, K\}$$

$$\int_0^t \frac{\partial h}{\partial s}(s, S_s^1, \cdots, S_s^K)ds + \frac{1}{2}\sum_{k,l=1}^{K}\int_0^t \frac{\partial^2 h}{\partial S^k \partial S^l}(s, S_s^1, \cdots, S_s^K)d < S^k - S_0^k, S^l - S_0^l >= 0.$$

Then X is P^*-attainable.

Proof. From Ito's formula, we have

$$\begin{aligned} X_t = X_0 &+ \int_0^t \frac{\partial h}{\partial s}(s, S_s^1, \cdots, S_s^K)ds \\ &+ \sum_{k=1}^{K}\int_0^t \frac{\partial h}{\partial S^k}(s, S_s^1, \cdots, S_s^K)d(S^k - S_0^k) \\ &+ \frac{1}{2}\sum_{k,l=1}^{K}\int_0^t \frac{\partial^2 h}{\partial S^k \partial S^l}(s, S_s^1, \cdots, S_s^K)d < S^k - S_0^k, S^l - S_0^l > . \end{aligned}$$

$\frac{\partial h}{\partial S^k}(\cdot, S^1, \cdots, S^K)$ is predictable and fulfills the integrability condition for the existence of the stochastic integral. By Lemma 1, the P^*-attainability of X follows. □

We now meet:

THEOREM 3. *Suppose that for all $i \neq j$, $i,j \in \{0,\cdots,K\}$, the stochastic processes $(S^i - S_0^i)$ and $(S^j - S_0^j)$ are strongly orthogonal*[1]. *If X is P^*-attainable, then*

$$\text{(1)} \qquad \int_0^t \frac{\partial h}{\partial s} ds + \frac{1}{2}\sum_{k=1}^K \int_0^t \frac{\partial^2 h}{(\partial S^k)^2} d < S^k - S_0^k, S^k - S_0^k > ds = 0$$

holds true.

Proof. Firstly, let X be P^*-attainable. Then there exist α^k, $k \in \{1,\cdots,K\}$ such that

$$E^*\Big[\int (\alpha^k)^2 d < S^k, S^k > \Big] < \infty \text{ and } X_t = X_0 + \sum_{k=1}^K \int_0^t \alpha^k d(S^k - S_0^k)$$

holds true. Ito's formula yields

$$X_t = X_0 + \sum_{k=1}^K \int_0^t \frac{\partial h}{\partial S^k} d(S^k - S_0^k) + \int_0^t \frac{\partial h}{\partial s} ds + \frac{1}{2}\sum_{k=1}^K \int_0^t \frac{\partial^2 h}{(\partial S^k)^2} d < S^k - S_0^k, S^k - S_0^k > .$$

The two representations of X_t imply

$$\sum_{k=1}^K \int_0^t (\alpha^k - \frac{\partial h}{\partial S^k}) d(S^k - S_0^k) = \int_0^t \frac{\partial h}{\partial s} ds + \frac{1}{2}\sum_{k=1}^K \int_0^t \frac{\partial^2 h}{(\partial S^k)^2} d < S^k - S_0^k, S^k - S_0^k > .$$

The expression on the left-hand side is a continuous square integrable martingale, the expression on the right-hand side is a continuous process of bounded variation. Thus the martingale on the left hand side is

[1] in the means of Hilbert space of square integrable martingale

predictable and of bounded variation and hence constant([1, p.121]). Since

$$\sum_{k=1}^{K}\int_0^t(\alpha^k-\frac{\partial h}{\partial S^k})d(S^k-S_0^k)_0=0,$$

this constant has to be zero and therefore

$$\int_0^t\frac{\partial h}{\partial s}ds+\frac{1}{2}\sum_{k=1}^{K}\int_0^t\frac{\partial^2 h}{(\partial S^k)^2}d<S^k-S_0^k,S^k-S_0^k>=0$$

and the result follows. □

If there exists positive Borel function g^k on $[0,T]\times\mathbf{R}^K$ such that

$$d<S^k-S_0^k,S^k-S_0^k>=\int_0^t g^k(s,S_s^1,\cdots,S_s^K)ds,$$

then (1) becomes

$$\int_0^t\frac{\partial h}{\partial s}ds+\frac{1}{2}\sum_{k=1}^{K}\int_0^t\frac{\partial^2 h}{(\partial S^k)^2}g^k(s,S_s^1,\cdots,S_s^K)ds=0.$$

We consider the Black/Scholes model with volatility coefficient σ_i^k and drift coefficient b^k. The volatility coefficient σ_i^k $(i=1,2,\cdots,d,\ k=1,2,\cdots,K)$ means the i-th source of uncertainty influences the price of the k-th stock.

COROLLARY 4. *In case of the* Black/Scholes *model with volatility coefficient* σ_i^k, *we have*

$$g^k(s,S_s^1,\cdots,S_s^K)=\Big[\sum_i(\sigma_i^k)^2\Big](S_s^k)^2.$$

Proof. Since Black/Scholes model is complete and

$$d < S^k - S_0^k, S^k - S_0^k >_t = (S_t^k)^2 [\sum_i (\sigma_i^k)^2] dt$$

holds true, we have the result. □

In this following example, we observe the existence of the function g in case of $k = 2$ and call option.

EXAMPLE. Consider the classical Black/Scholes model and a European call option on a stock with exercise price $c > 0$ and expiration date T. In this case, we have

$$V_t^\pi = S_t^1 \Phi\Big(\frac{ln\frac{S_t^1}{c} + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\Big) - c\Phi\Big(\frac{ln\frac{S_t^1}{c} - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\Big),$$

where Φ is a Standard Normal distribution function.
In doing this, the following notation will be used:

$$p(t, S^1) = \frac{ln\frac{S^1}{c} + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}, \quad q(t, S^1) = \frac{ln\frac{S^1}{c} - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

and

$$h(t, S^1) = S^1\Phi\big(p(t, S^1)\big) - c\Phi\big(q(t, S^1)\big).$$

Then we have

$$\frac{\partial^2 h}{(\partial S^1)^2} = \frac{n(p)}{S^1\sigma\sqrt{T-t}}$$

and

$$\frac{\partial h}{\partial t} = -\frac{\sigma S^1 n(q)}{2\sqrt{T-t}}.$$

Here n is a density of Φ.
Hence, in this case, we get $g = \sigma^2(S^1)^2$.

Suppose Λ is a collection of admissible stopping times $\lambda \in \Lambda$ with respect to $(\mathcal{F}_t)$. A classical approach to pricing of American options requires choosing a self-financing(SF) hedge π such that

$$V_\lambda^\pi \geq f_\lambda \text{ a.s. for all } \lambda \in \Lambda.$$

Denote $\Pi(x, f)$ a collection of self-financing hedges such that $V_0^\pi = x$ and $V_\lambda^\pi \geq f_\lambda$ a.s. Let

$$C(f) = \inf\{x : \Pi(x, f) \neq \varnothing\}$$

(which is called a *fair price*). In case of Diffusion and Binomial models

$$C(f) = \sup_\lambda E^*(f_\lambda / B_\lambda).$$

We will consider hedges only from the following class:

$$SFS = \Big\{\pi \in SF : \frac{V_\lambda^\pi}{B_\lambda} \geq \frac{f_\lambda}{B_\lambda} - C(f) \text{ for all } \lambda \in \Lambda, \\ \frac{V_\lambda^\pi}{B_\lambda} > C \text{ for all } t \geq 0,\ C \text{ is some constant}\Big\}.$$

Having in mind statistical arguments it seems reasonable instead of condition $V_\lambda^\pi \geq f_\lambda$ a.s. to assume only:

$$\inf_P \inf_\lambda P\{V_\lambda^\pi \geq f_\lambda\} \geq 1 - \alpha,$$

where the constant α is a given significance level.

We consider classical Black/Scholes model.

THEOREM 5. *If* $\pi \in SFS$, *then*

$$\inf_\lambda P^*\{V_\lambda^\pi \geq f_\lambda\} \leq V_0^\pi / C(f).$$

Proof. By completeness of Black/Scholes model, we have for any stopping time λ and localization sequence λ_n

$$E^*(V_{\sigma_n}^\pi / B_{\sigma_n}) = V_0^\pi \text{ with } \sigma_n = min(\lambda, \lambda_n)$$

or

$$E^*(V_{\sigma_n}^\pi / B_{\sigma_n})^+ = E^*(V_{\sigma_n}^\pi / B_{\sigma_n})^- + V_0^\pi,$$

where we use symbols $(\ )^+$ and $(\ )^-$ for positive and negative parts. By assumptions of the class SFS, we have

$$E^*(V_\lambda^\pi/B_\lambda) \le V_0^\pi.$$

Letting

$$\xi = \frac{V_\lambda^\pi}{B_\lambda} - \frac{f_\lambda}{B_\lambda} + E^*(\frac{f_\lambda}{B_\lambda}),$$

by Chebyshev inequality

$$P^*\Big\{V_\lambda^\pi \ge f_\lambda\Big\} = P^*\Big\{\xi \ge E^*(\frac{f_\lambda}{B_\lambda})\Big\} \le \frac{E^*(\xi)^+}{E^*(\frac{f_\lambda}{B_\lambda})}.$$

As $E^*(\xi) \le V_0^\pi$ and $(\xi)^- \le C(f) - E^*(\frac{f_\lambda}{B_\lambda})$. We have

$$P^*\{V_\lambda^\pi \ge f_\lambda\} \le \frac{V_0^\pi + C(f) - E^*(\frac{f_\lambda}{B_\lambda})}{E^*(\frac{f_\lambda}{B_\lambda})}.$$

Taking now infimum over $\lambda \in \Lambda$ we get the result. □

We conclude with:

COROLLARY 6. *If $\pi \in SFS$, then*

$$V_0^\pi \ge (1-\alpha)C(f).$$

Proof. This at once follows from the Theorem 4. □

ACKNOWLEDGMENT. The author expresses deep thanks to referee for his valuable comments.

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