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REMARKS ON APPROXIMATION OF FIXED POINTS OF STRICTLY PSEUDOCONTRACTIVE MAPPINGS

TAE-HWA KIM* AND EUN-SUK KIM

ABSTRACT. In the present paper, we first give some examples of self-mappings which are asymptotically nonexpansive in the intermediate, not strictly hemicontractive, but satisfy the property (H). It is then shown that the modified Mann and Ishikawa iteration processes defined by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n$ and $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n[(1-\beta_n)x_n + \beta_n T^n x_n]$, respectively, converges strongly to the unique fixed point of such a self-mapping in general Banach spaces.

Let X be a Banach space and let K be a nonempty subset of X (not necessarily convex) and $T : K \to K$ a self mapping of K. There appear in the literature two definitions of an asymptotically nonexpansive mapping. The weaker definition (cf. Kirk[14]) requires that

$$\limsup_{n\to\infty} \sup_{y\in K} (\|T^n x - T^n y\| - \|x-y\|) \le 0$$

for every $x \in K$ and that T^N is continuous for some $N \ge 1$. The stronger definition (briefly called *asymptotically nonexpansive* as in [10]) requires each iterate T^n to be Lipschitzian with Lipschitz constants $L \to 1$ as $n \to \infty$. For further generalization of an averaging iteration of Schu [19], Bruck et al. [3] introduced a definition somewhere between these two : T is *asymptotically nonexpansive in the intermediate sense* provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in K} (\|T^n x - T^n y\| - \|x-y\|) \le 0.$$

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A mapping $T : K \to X$ is said to be *pseudocontractive* [20] if for all $x, y \in K$ there exists $j \in J(x-y)$ such that

$$\mathrm{Re}\langle Tx - Ty, j\rangle \le \|x-y\|^2,$$

where J denotes the normalized duality mapping from X to 2^{X^*}, i.e., with each $x \in X$, we associate the set

$$J(x) = \{f \in X^* : \|f\|^2 = \|x\|^2 = \mathrm{Re}\langle x, f\rangle\},$$

where $\mathrm{Re}\langle x, f\rangle$ denotes the real part of $f(x)$, the value of f at x. In [13], Kato discovered the relationship between pseudocontractive mappings and accretive mappings, proving

LEMMA K([13]). *Let $x, y \in X$. Then $\|x\| \le \|x + \alpha y\|$ for every $\alpha > 0$ if and only if there exists $j \in J(x)$ such that* $\mathrm{Re}\langle y, j\rangle \ge 0$.

Applying Lemma K, we know that a mapping T is pseudocontractive if and only if $(I - T)$ is accretive, i.e., the inequality

$$\|x - y\| \le \|x - y + r\{(I-T)x - (I-T)y\}\|$$

holds for all $x, y \in K$ and all $r \ge 0$. A mapping $T : K \to X$ is said to be *strictly pseudocontractive* [7] (or [20]) if there exists $t > 1$ such that for all $x, y \in K$ there exists $j \in J(x-y)$ such that

$$\mathrm{Re}\langle Tx - Ty, j\rangle \le \frac{1}{t}\|x-y\|^2.$$

Let $F(T)$ denotes the set of all fixed points of T, i.e., $F(T) = \{x \in K : Tx = x\}$. If $F(T) \ne \emptyset$, the mapping $T : K \to X$ is said to be *strictly hemicontractive* [7] if there exists $t > 1$ such that for all $x \in K$ and $x^* \in F(T)$ there exists $j \in J(x - x^*)$ such that

$$\mathrm{Re}\langle Tx - x^*, j\rangle \le \frac{1}{t}\|x - x^*\|^2. \tag{1}$$

Using Lemma K, it is easy to check [7] that the strict hemicontractivity of T is equivalent to the following inequality

$$\|x - x^*\| \le \|(1+r)(x - x^*) - rt(Tx - x^*)\|$$

holds for all $x \in K$, $x^* \in F(T)$ and $r > 0$.

We first introduce an example of a Lipschitzian self-mapping which is not strictly pseudocontractive but strictly hemicontractive.

EXAMPLE 1([7]). Take $X = \mathbb{R}$ with the usual norm $|\cdot|$. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \frac{2}{3}x\cos x$$

for all $x \in \mathbb{R}$. Obviously, $F(T) = \{0\}$ and since $\langle Tx, x\rangle = \frac{2}{3}x^2 \cos x \leq \frac{2}{3}|x|^2$ for all $x \in \mathbb{R}$, T is strictly hemicontractive with $t = \frac{3}{2} > 1$. However, if we can take $x = 2\pi$ and $y = \pi$, then

$$\langle Tx - Ty, x - y\rangle = 2\pi^2 > \pi^2 = |x - y|^2.$$

Therefore T is not strictly pseudocontractive. Further, if we can take $K = [-2\pi, 2\pi] \subset \mathbb{R}$, $T : K \to K$ is a Lipschitzian mapping with its Lipschitz constant $\frac{2}{3}(1 + 2\pi)$.

Motivated by the definition of strict hemicontractivity, we can consider a mapping $T : K \to K$ satisfying the following property, i.e., there exists $t > 1$ such that for all $x \in K$ and $x^* \in F(T)(\neq \emptyset)$, there exists $j \in J(x - x^*)$ such that

$$\text{(H)} \qquad \limsup_{n\to\infty} \operatorname{Re}\langle T^n x - x^*, j\rangle \leq \frac{1}{t}\|x - x^*\|^2.$$

Obviously, any mapping $T : K \to K$ which is both strictly hemicontractive and asymptotically nonexpansive (cf. Goebel-Kirk [10]; $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $x, y \in K$ and all $n \in \mathbb{N}$ with $\limsup_{n\to\infty} k_n \leq 1$) satisfies the property (H). Here we shall give two examples of self-mappings which are asymptotically nonexpansive in the intermediate sense, not strictly hemicontractive, but satisfy the above property (H).

EXAMPLE 2. Let $X = \mathbb{R}$ with the usual norm $|\cdot|$ and let $K = [0, 1]$. Let $a_n = \frac{1}{n}$ for each $n \in \mathbb{N}$. Then, construct a continuous mapping T as follows. On the each subinterval $[a_{n+1}, a_n]$, the graph of T consists of the sides of the isosceles triangle with base $[a_{n+1}, a_n]$ and height a_{n+1}. Thus, $Ta_n = 0$ and, if x_n denotes the midpoint of $[a_{n+1}, a_n]$, then $Tx_n = a_{n+1}$. If we further define $T0 = 0$, $T : K \to K$ is uniformly continuous (but, not Lipschitzian) and only $F(T) = \{0\}$.

Since $T^n x \to 0$ uniformly as $n \to \infty$, T is asymptotically nonexpansive in the intermediate sense. Indeed, for each $n \in \mathbb{N}$, there exist $x_n, y_n \in K$ such that

$$\sup_{x,y \in K} (\|T^n x - T^n y\| - \|x - y\|) = \|T^n x_n - T^n y_n\| - \|x_n - y_n\|.$$

Taking $\limsup_{n\to\infty}$ on both sides, we get

$$\limsup_{n\to\infty}(\|T^n x_n - T^n y_n\| - \|x_n - y_n\|) \leq -\liminf_{n\to\infty} \|x_n - y_n\| \leq 0.$$

It is obvious that T satisfies the property (H). Now assume that T is strictly hemicontractive, i.e., there exists $t > 1$ such that

$$\langle Tx, x\rangle \leq \frac{1}{t}|x|^2$$

for all $x \in K$. If we can choose $n \in \mathbb{N}$ so that $\frac{1}{2n+1} < \frac{t-1}{t}$, it is easy to check that $\langle Tx_n, x_n\rangle = a_{n+1}x_n > \frac{1}{t}|x_n|^2$, which gives a contradiction and so T is not strictly hemicontractive.

Also, we shall give an example of a Lipschitzian mapping $T : K \to K$ (in fact, nonexpansive, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$) in the space ℓ_2 which is not strictly hemicontractive but satisfies the property (H).

EXAMPLE 3. Take $X = \ell_2$ with the usual norm $\|\cdot\|$. Let K be the unit ball in ℓ_2 and let $f : [-1, 1] \to [-1, 1]$ be defined by

$$f(x) = \begin{cases} -\frac{1}{2} & \text{if } -1 \leq x \leq -\frac{3}{4}, \\ \frac{(2^{n+1}x - \frac{1}{2^n})}{2^{n+1}+3} & \text{if } -\frac{2^n+1}{2^{2n}} \leq x \leq -\frac{2^{n+1}+1}{2^{2(n+1)}}, \\ 0 & \text{if } x = 0, \\ \frac{(2^{n+1}x + \frac{1}{2^n})}{2^{n+1}+3} & \text{if } \frac{2^{n+1}+1}{2^{2(n+1)}} \leq x \leq \frac{2^n+1}{2^{2n}}, \\ \frac{1}{2} & \text{if } \frac{3}{4} \leq x \leq 1, \end{cases}$$

for all $x \in [-1, 1]$ and all $n \in \mathbb{N}$. Then we readily see that f is nonexpansive, i.e., $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [-1, 1]$ and $F(f) = \{0\}$. Define

$$Tx = (f(x_1), f(x_2), \cdots)$$

for all $x = (x_1, x_2, \cdots) \in K$. Clearly, $T : K \to K$ and $F(T) = \{0\}$, where $0 = (0, 0, \cdots)$. Since

$$T^n x = (f^n(x_1), f^n(x_2), \cdots)$$

for all $x = (x_1, x_2, \cdots) \in K$ and, for each $j \in \mathbb{N}$, $f^n(x_j) \to 0$ as $n \to \infty$, it immediately follows that $T^n x \rightharpoonup 0$ as $n \to \infty$ and so

$$\lim_{n\to\infty} \langle T^n x, x \rangle = 0$$

for all $x = (x_1, x_2, \cdots) \in K$. Thus, $T : K \to K$ satisfies the property (H). However, T is not strictly hemicontractive. Indeed, if there exists $t > 1$ such that

$$\langle Tx, x\rangle = \sum_{j=1}^{\infty} x_j f(x_j) \leq \frac{1}{t}\|x\|^2 \tag{2}$$

for all $x = (x_1, x_2, \cdots) \in K$. Choose $n \geq 2$ so that $\frac{1}{2^n} < (t-1)$. Setting

$$x_j = \frac{2^j + 1}{2^{2j}}$$

for all $j \geq n$ and $u = (x_n, x_{n+1}, \cdots)$, it easily follows that $\|u\| \leq 1$ and

$$\langle Tu, u\rangle = \sum_{j=n}^{\infty} x_j f(x_j) > \frac{1}{t}\|u\|^2$$

because $f(x_j) > \frac{1}{t}x_j$ if and only if $\frac{1}{2^j} < (t-1)$ and $\frac{1}{2^j} \leq \frac{1}{2^n} < (t-1)$ for all $j \geq n$, which contradicts to the assumption (2). Therefore T is not strictly hemicontractive. It is easy to check that T is nonexpansive, i.e.,

$$\|Tx - Ty\| = (\sum_{j=1}^{\infty} |f(x_j) - f(y_j)|^2)^{\frac{1}{2}} \leq \|x - y\|$$

for all $x = (x_1, x_2, \cdots)$, $y = (y_1, y_2, \cdots) \in K$.

Recall that a mapping $T : K \to X$ is said to be *strongly accretive* [2] (or [23]) if there exists a positive number k such that for each $x, y \in K$ there is $j \in J(x-y)$ such that

$$\operatorname{Re}\langle Tx - Ty, j\rangle \geq k\|x-y\|^2.$$

Using Lemma K again, this is equivalent to

$$\|x-y\| \leq \|x-y+r\{(T-kI)x-(T-kI)y\}\|,$$

for all $r > 0$, where I denotes the identity mapping of X. Without loss of generality, we can assume $k \in (0,1)$. Then it was known [1] that the similar connection between strict pseudocontractivity and strong accretivity is that a mapping $T : K \to K$ is strictly pseudocontractive if and only if $I - T$ is strongly accretive, i.e., the inequality

$$\|x-y\| \leq \|x-y+r\{(I-T-kI)x-(I-T-kI)y\}\| \tag{3}$$

holds for any $x, y \in K$ and $r > 0$, where $k = \frac{(t-1)}{t} \in (0,1)$.

Recently, the convergence problems of Ishikawa and Mann iteration sequences (cf. Ishikawa [12] and Mann [17]) have been studied extensively by many authors (see Chidume [4-7], Deng [8], Deng-Ding [9], Haiyun-Yuting [11], Liu [15], Liu [16], Reich [18] and Tan-Xu [22]) for strictly pseudocontractive (or strongly accretive) mappings.

Especially, Liu [15] proved by the inequality (3) that the Mann iteration process converges strongly to the unique fixed point of a Lipschitzian and strictly pseudocontractive mapping, which extends corresponding results of [4-6], [22] and [23] to the general Banach spaces.

THEOREM L([15]). *Let K be a nonempty closed, convex and bounded subset of a* Banach *space X and let $T : K \to K$ be* Lipschitzian *and strictly pseudocontractive mapping. If $F(T) \neq \emptyset$, then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad x_1 \in K,$$

with $\{\alpha_n\} \subset (0,1]$ *satisfying*

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0,$$

converges strongly to $q \in F(T)$ *and* $F(T)$ *is a singleton set.*

The following lemma was recently proved by Haiyun-Yuting [11]. Compare our easy observation with the proof of Lemma 1.1 in [11].

LEMMA H-Y([11]). *For any* $x, y \in X$ *and* $j \in J(x+y)$,

$$\|x+y\|^2 \leq \|x\|^2 + 2Re\langle y, j\rangle.$$

Proof. For any $x, y \in X$ and $j \in J(x+y)$,

$$\begin{aligned} \|x\|^2 + 2\mathrm{Re}\langle y, j\rangle - \|x+y\|^2 &= \|x\|^2 + 2\mathrm{Re}\langle x+y-x, j\rangle - \|x+y\|^2 \\ &= \|x\|^2 - 2\mathrm{Re}\langle x, j\rangle + \|x+y\|^2 \\ &\geq \|x\|^2 - 2\|x\|\|x+y\| + \|x+y\|^2 \\ &= (\|x\| - \|x+y\|)^2 \geq 0. \end{aligned}$$

□

At the same time, Haiyun-Yuting [11] proved, using the above lemma, that the Ishikawa iteration process converges strongly to the unique fixed point of a continuous and strictly pseudocontractive map without Lipschitz assumption in a real uniformly smooth Banach space.

THEOREM H-Y([11]). *Let* K *be a nonempty closed, convex and bounded subset of a real uniformly smooth* Banach *space* X. *Assume that* $T : K \to K$ *is a continuous strictly pseudocontractive mapping. Let* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *be two real sequences satisfying*

(i) $0 < \alpha_n, \beta_n < 1$ *and* $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the Ishikawa *iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *generated by*

$$(4)\qquad \begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \qquad n \geq 1, \end{cases}$$

converges strongly to the unique fixed point of T.

In these respects, it seems natural to ask whether the above two theorems are still valid for any mapping $T : K \to K$ satisfying the property (H). For our affirmative argument, consider the following modified Ishikawa iteration process instead of (4):

$$(4')\qquad \begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \qquad n \geq 1. \end{cases}$$

The above algorithm $(4)'$ was used by Schu [19] and Tan-Xu [21] to show the weak convergence of the Mann and Ishikawa iteration processes to a fixed point of an asymptotically nonexpansive self-mapping.

We first begin with an easy observation of the property (H). The first equivalent is

$$(\mathrm{H}_1)\qquad \liminf_{n\to\infty} \mathrm{Re}\langle x - T^n x, j\rangle \geq \frac{(t-1)}{t}\|x - x^*\|^2.$$

Let $x \neq x^*$. For a fixed ϵ with $0 < \epsilon < \frac{(t-1)}{t}$, it follows from the property (H_1) that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$(\mathrm{H}_2)\qquad \begin{aligned} \mathrm{Re}\langle x - T^n x, j\rangle &\geq \left(\frac{t-1}{t} - \epsilon\right)\|x - x^*\|^2 \\ &= k_\epsilon \|x - x^*\|^2, \end{aligned}$$

where $k_\epsilon := (\frac{t-1}{t} - \epsilon) \in (0,1)$. This inequality is obviously equivalent to

$$(\mathrm{H}_3)\qquad \mathrm{Re}\langle T^n x - x^*, j\rangle \leq (1-k_\epsilon)\|x - x^*\|^2, \qquad \forall n \geq n_0.$$

For employing the method of the proof in [15], we need the following equivalent form of the property (H_2) by virtue of Lemma K:

$$(\mathrm{H}_4)\quad \|x - x^*\| \leq \|x - x^* + r\{(I - T^n - k_\epsilon I)x - (I - T^n - k_\epsilon I)x^*\}\|$$

for all $n \geq n_0$ and all $r > 0$.

Using the property (H_4), we are now ready to present the following

THEOREM 1. *Let K be a nonempty closed, convex and bounded subset of a* Banach *space X. Assume that $T : K \to K$ is asymptotically nonexpansive in the intermediate sense satisfying the property* (H). *Put*

$$c_n = \max\{0, \sup_{x,y \in K} (\|T^n x - T^n y\| - \|x - y\|)\},$$

so that $\lim_{n\to\infty} c_n = 0$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be two real sequences satisfying

(i) $0 \leq \alpha_n, \beta_n \leq 1$ *and* $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$;

(ii) $\sum_{n=1}^\infty \alpha_n = \infty$.

Then the modified Ishikawa *iterative sequence $\{x_n\}_{n=1}^\infty$ generated by* $(4)'$ *converges strongly to the unique fixed point of T in K.*

Proof. We employ the method of the proof of Liu [15]. Since $F(T) \neq \emptyset$, take $q \in F(T)$. From the definition of $\{x_n\}$, we have

$$\begin{aligned}
x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T^n y_n \\
&= (1+\alpha_n)x_{n+1} + \alpha_n(I - T^n - k_\epsilon I)x_{n+1} - (2 - k_\epsilon)\alpha_n x_{n+1} \\
&\qquad + \alpha_n x_n + \alpha_n(T^n x_{n+1} - T^n y_n) \\
&= (1+\alpha_n)x_{n+1} + \alpha_n(I - T^n - k_\epsilon I)x_{n+1} - (2 - k_\epsilon)\alpha_n[(1-\alpha_n)x_n + \alpha_n T^n y_n] \\
&\qquad + \alpha_n x_n + \alpha_n(T^n x_{n+1} - T^n y_n) \\
&= (1+\alpha_n)x_{n+1} + \alpha_n(I - T^n - k_\epsilon I)x_{n+1} - (1 - k_\epsilon)\alpha_n x_n \\
&\qquad + (2 - k_\epsilon)\alpha_n^2(x_n - T^n y_n) + \alpha_n(T^n x_{n+1} - T^n y_n).
\end{aligned}$$

By $Tq = q$, this implies that

$$\begin{aligned}
x_n - q =& (1+\alpha_n)(x_{n+1} - q) + \alpha_n(I - T^n - k_\epsilon I)(x_{n+1} - q) - (1 - k_\epsilon)\alpha_n(x_n - q) \\
&+ (2 - k_\epsilon)\alpha_n^2(x_n - T^n y_n) + \alpha_n(T^n x_{n+1} - T^n y_n).
\end{aligned}$$

By the property (H_4), we obtain

$$\begin{aligned}\|x_n - q\| \geq & (1+\alpha_n)\|x_{n+1} - q\| - (1-k_\epsilon)\alpha_n\|x_n - q\| \\ & - (2-k_\epsilon)\alpha_n^2\|x_n - T^n y_n\| - \alpha_n\|T^n x_{n+1} - T^n y_n\|,\end{aligned}$$

for all $n \geq n_0$. Since

$$\begin{aligned}[1+(1-k_\epsilon)\alpha_n](1+\alpha_n)^{-1} &\leq [1+(1-k_\epsilon)\alpha_n](1-\alpha_n+\alpha_n^2) \\ &= 1 - k_\epsilon\alpha_n + \alpha_n^2 - (1-k_\epsilon)\alpha_n^2(1-\alpha_n) \\ &\leq 1 - k_\epsilon\alpha_n + \alpha_n^2,\end{aligned}$$

it follows from the above inequality that for all $n \geq n_0$,

(5)

$$\begin{aligned}\|x_{n+1} - q\| \leq & [1+(1-k_\epsilon)\alpha_n](1+\alpha_n)^{-1}\|x_n - q\| \\ & + (2-k_\epsilon)\alpha_n^2(1+\alpha_n)^{-1}\|x_n - T^n y_n\| \\ & + \alpha_n(1+\alpha_n)^{-1}\|T^n x_{n+1} - T^n y_n\| \\ \leq & (1-k_\epsilon\alpha_n)\|x_n - q\| + \alpha_n^2[\|x_n - q\| + (2-k_\epsilon)\|x_n - T^n y_n\|] \\ & + \alpha_n d_n \\ \leq & (1-k_\epsilon\alpha_n)\|x_n - q\| + M\alpha_n^2 + \alpha_n d_n,\end{aligned}$$

where $d_n = \|T^n x_{n+1} - T^n y_n\|$ and $M = 3\text{diam}(K)$ (since K is bounded). Since $\{x_n\}$, $\{T^n x_n\}$ and $\{T^n y_n\}$ are all bounded sequences in K,

$$y_n - x_{n+1} = (\alpha_n - \beta_n)x_n + \beta_n T^n x_n - \alpha_n T^n y_n \to 0$$

as $n \to \infty$. Since $c_n \to 0$ as $n \to \infty$ we get

$$\begin{aligned}d_n &= \|T^n y_n - T^n x_{n+1}\| \\ &= [\|T^n y_n - T^n x_{n+1}\| - \|y_n - x_{n+1}\|] + \|y_n - x_{n+1}\| \\ &\leq c_n + \|y_n - x_{n+1}\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Applying Lemma 4 of [23] (or Lemma 1.2 in [11]), we have $x_n \to q$ as $n \to \infty$. Finally, we prove that $F(T) = \{q\}$, a singleton set. If

$p \in F(T)$, by using the property (H), we obtain

$$\begin{aligned}\|p-q\|^2 &= \langle p-q, j\rangle \\ &= \limsup_{n\to\infty} \mathrm{Re}\langle T^n p - q, j\rangle \\ &\le \frac{1}{t}\|p-q\|^2.\end{aligned}$$

Since $t > 1$, we have $q = p$. □

REMARK 1. In view of the example 2 and 3, the above theorem is a new approach of the strong convergence problems of iterative sequences to the unique fixed point of self-mappings which are not strictly hemicontractive (hence, not strictly pseudocontractive). Compare this with Theorem L. Following the steps of the proof of Theorem H-Y, it is easy to see that if X is uniformly smooth and if $T : K \to K$ is a continuous mapping with the property (H) Theorem 1 is still valid.

REMARK 2. Using Lemma H-Y and the property (H_3), we can similarly prove Theorem 1. Indeed,

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n \mathrm{Re}\langle T^n y_n - q, j_n\rangle \\ &\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n \mathrm{Re}\langle T^n y_n - T^n x_{n+1}, j_n\rangle \\ &\quad + 2\alpha_n \mathrm{Re}\langle T^n x_{n+1} - q, j_n\rangle \\ &\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n d_n + 2\alpha_n(1-k_\epsilon)\|x_{n+1} - q\|^2\end{aligned}$$

for $j_n \in J(x_{n+1} - q)$, where $d_n = \mathrm{Re}\langle T^n y_n - T^n x_{n+1}, j_n\rangle \to 0$ as $n \to \infty$. In fact, since K is bounded and $c_n \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned}\|T^n y_n - T^n x_{n+1}\| &= [\|T^n y_n - T^n x_{n+1}\| - \|y_n - x_{n+1}\|] + \|y_n - x_{n+1}\| \\ &\le c_n + \|y_n - x_{n+1}\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

On the other hand, since $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\alpha_n \to 0$ as $n \to \infty$, we can choose n_1 ($\ge n_0$) so that $\alpha_n > 0$, $1 - 2\alpha_n(1-k_\epsilon) > 0$, and $2k_\epsilon - \alpha_n > 0$ for all $n \ge n_1$. Then, the above inequality can be written as follows:

$$(6) \qquad \|x_{n+1} - q\|^2 \le \frac{(1-\alpha_n)^2}{1-2\alpha_n(1-k_\epsilon)}\|x_n - q\|^2 + \frac{2\alpha_n d_n}{1-2\alpha_n(1-k_\epsilon)}.$$

Since $\frac{2k_\epsilon-\alpha_n}{1-2\alpha_n(1-k_\epsilon)} \to 2k_\epsilon$ as $n \to \infty$ and $k_\epsilon \in (0,1)$, there exists a n_2 ($\geq n_1$) such that

$$\left|\frac{2k_\epsilon-\alpha_n}{1-2\alpha_n(1-k_\epsilon)} - 2k_\epsilon\right| \leq k_\epsilon$$

for all $n \geq n_2$. This implies that $k_\epsilon \leq \frac{2k_\epsilon-\alpha_n}{1-2\alpha_n(1-k_\epsilon)}$, that is,

$$\frac{(1-\alpha_n)^2}{1-2\alpha_n(1-k_\epsilon)} \leq (1-k_\epsilon\alpha_n)$$

for all $n \geq n_2$. The inequality (6) can be expressed as follows.

$$\|x_{n+1}-q\|^2 \leq (1-k_\epsilon\alpha_n)\|x_n-q\|^2 + \frac{2\alpha_n d_n}{1-2\alpha_n(1-k_\epsilon)},$$

for all $n \geq n_2$. Then it follows from the lemma 4 of Weng [23] that the sequence $\{x_n\}$ strongly converges to the unique fixed point q of T.

For evaluating the error estimate of the strong convergence of the Ishikawa type iteration $\{x_n\}_{n=1}^{\infty}$ generated by $(4)'$ to the unique fixed point, in addition to all assumptions of Theorem 1, suppose that $T : K \to K$ is uniformly L-Lipschitzian, that is, there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x-y\|,$$

for all $n \in \mathbb{N}$ and $x, y \in K$. Obviously, every nonexpansive mappings are uniformly 1-Lipschitzian.

Now we are ready to present the following error estimate of the convergence.

THEOREM 2. *Let K be a nonempty closed, convex and bounded subset of a* Banach space X *and let $T : K \to K$ a uniformly L-*Lipschitzian *mapping satisfying the property* (H). *If $\alpha_n = \frac{k_\epsilon}{2(3+3L+L^2)}$ and $\beta_n = \frac{k_\epsilon}{4L(L^2+2L-1)}$ where k_ϵ is the positive real number in $(0,1)$ as in the property (H_2) and $F(T) = \{q\}$. Then the modified* Ishikawa *iterative sequence $\{x_n\}_{n=1}^{\infty}$ generated by $(4)'$ converges strongly to the unique fixed point of T in K, and we obtain the estimate*

$$\|x_{n+1}-q\| < \rho^n\|x_1-q\|,$$

where $\rho = 1 - \frac{k_\epsilon^2}{8(3+3L+L^2)}$.

Proof. Let $Tq = q$. Note that

$$\begin{aligned}\|x_n - T^n y_n\| &\le \|x_n - q\| + L\|y_n - q\| \\ &\le \|x_n - q\| + L[1 + \beta_n(L-1)]\|x_n - q\| \\ &= [1 + L + \beta_n L(L-1)]\|x_n - q\|\end{aligned}$$

and

$$\begin{aligned}\|T^n x_{n+1} - T^n y_n\| &\le L(\|x_{n+1} - x_n\| + \|x_n - y_n\|) \\ &= L(\alpha_n\|x_n - T^n y_n\| + \beta_n\|x_n - T^n x_n\|) \\ &\le L[\alpha_n(1 + L + \beta_n L(L-1)) + \beta_n(1+L)]\|x_n - q\| \\ &= [\alpha_n L(L+1) + \alpha_n\beta_n L^2(L-1) + \beta_n L(L+1)]\|x_n - q\|.\end{aligned}$$

Combined with the inequality (5), the required estimate is now obtained.

$$\begin{aligned}\|x_{n+1} - q\| \le &(1 - k_\epsilon\alpha_n)\|x_n - q\| + \alpha_n^2[\|x_n - q\| + (2 - k_\epsilon)\|x_n - T^n y_n\|] \\ &+ \alpha_n\|T^n x_{n+1} - T^n y_n\| \\ \le &(1 - k_\epsilon\alpha_n)\|x_n - q\| + \alpha_n^2[1 + (2 - k_\epsilon)(1 + L + \beta_n L(L-1))]\|x_n - q\| \\ &+ \alpha_n[\alpha_n L(L+1) + \alpha_n\beta_n L^2(L-1) + \beta_n L(L+1)]\|x_n - q\| \\ \le &(1 - k_\epsilon\alpha_n)\|x_n - q\| + \alpha_n^2[1 + (2 - k_\epsilon)(1 + L) + L(L+1)]\|x_n - q\| \\ &+ \alpha_n\beta_n[(2 - k_\epsilon)L(L-1) + L^2(L+1) + L(L+1)]\|x_n - q\| \\ < &(1 - k_\epsilon\alpha_n)\|x_n - q\| + \alpha_n^2(3 + 3L + L^2)\|x_n - q\| \\ &+ \alpha_n\beta_n L(L^2 + 2L - 1)\|x_n - q\| \\ = &\left[1 - \frac{k_\epsilon^2}{8(3 + 3L + L^2)}\right]\|x_n - q\| \\ = &\rho\|x_n - q\|.\end{aligned}$$

Hence $\|x_{n+1} - q\| < \rho^n\|x_1 - q\|$. □

Taking $\beta_n = 0$ for all $n \ge 1$ in Theorem 1, we have the following

COROLLARY 1. *Let K be a nonempty closed, convex and bounded subset of a Banach space X. Assume that $T : K \to K$ is asymptotically nonexpansive in the intermediate sense satisfying the property* (H). *Put*

$$c_n = \max\{0, \sup_{x,y \in K}(\|T^n x - T^n y\| - \|x - y\|)\},$$

so that $\lim_{n\to\infty} c_n = 0$. *Then the modified* Mann *iterative sequence* $\{x_n\}_{n=0}^{\infty}$ *generated by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad x_1 \in K$$

with $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1]$ *satisfying*

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0,$$

strongly converges $q \in F(T)$ *and* $F(T)$ *is a singleton set.*

REMARK 4. Compared with Theorem L and Theorem H-Y, our iterative algorithm can be applicable to the approximating problems of all continuous mappings with the property (H) in the finite dimensional spaces.

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Department of Applied Mathematics, Pukyong National University, Pusan 608-737, Korea
E-mail: taehwa@dolphin.pknu.ac.kr