REMARKS ON APPROXIMATION OF FIXED POINTS
OF STRICTLY PSEUDOCONTRACTIVE MAPPINGS

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ABSTRACT. In the present paper, we first give some examples of self-mappings which are asymptotically nonexpansive in the intermediate, not strictly hemicontractive, but satisfy the property (H). It is then shown that the modified Mann and Ishikawa iteration processes defined by \( x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n \) and \( x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n [(1 - \beta_n)x_n + \beta_n T^n x_n] \), respectively, converges strongly to the unique fixed point of such a self-mapping in general Banach spaces.

Let \( X \) be a Banach space and let \( K \) be a nonempty subset of \( X \) (not necessarily convex) and \( T : K \to K \) a self mapping of \( K \). There appear in the literature two definitions of an asymptotically nonexpansive mapping. The weaker definition (cf. Kirk[14]) requires that

\[
\limsup_{n \to \infty} \sup_{y \in K} (\|T^n x - T^n y\| - \|x - y\|) \leq 0
\]

for every \( x \in K \) and that \( T^N \) is continuous for some \( N \geq 1 \). The stronger definition (briefly called asymptotically nonexpansive as in [10]) requires each iterate \( T^n \) to be Lipschtizian with Lipschtiz constants \( L \to 1 \) as \( n \to \infty \). For further generalization of an averaging iteration of Schu [19], Bruck et al. [3] introduced a definition somewhere between these two: \( T \) is asymptotically nonexpansive in the intermediate sense provided \( T \) is uniformly continuous and

\[
\limsup_{n \to \infty} \sup_{x, y \in K} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.
\]

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A mapping $T : K \to X$ is said to be pseudocontractive [20] if for all $x, y \in K$ there exists $j \in J(x - y)$ such that
\[ \text{Re}(Tx - Ty, j) \leq \|x - y\|^2, \]
where $J$ denotes the normalized duality mapping from $X$ to $2^{X^*}$, i.e., with each $x \in X$, we associate the set
\[ J(x) = \{ f \in X^* : \|f\|^2 = \|x\|^2 = \text{Re}(x, f) \}, \]
where $\text{Re}(x, f)$ denotes the real part of $f(x)$, the value of $f$ at $x$. In [13], Kato discovered the relationship between pseudocontractive mappings and accretive mappings, proving

**Lemma K([13]).** Let $x, y \in X$. Then $\|x\| \leq \|x + \alpha y\|$ for every $\alpha > 0$ if and only if there exists $j \in J(x)$ such that $\text{Re}(y, j) \geq 0$.

Applying Lemma K, we know that a mapping $T$ is pseudocontractive if and only if $(I - T)$ is accretive, i.e., the inequality
\[ \|x - y\| \leq \|x - y + r((I - T)x - (I - T)y)\| \]
holds for all $x, y \in K$ and all $r \geq 0$. A mapping $T : K \to X$ is said to be strictly pseudocontractive [7] (or [20]) if there exists $t > 1$ such that for all $x, y \in K$ there exists $j \in J(x - y)$ such that
\[ \text{Re}(Tx - Ty, j) \leq \frac{1}{t}\|x - y\|^2. \]

Let $F(T)$ denotes the set of all fixed points of $T$, i.e., $F(T) = \{ x \in K : Tx = x \}$. If $F(T) \neq \emptyset$, the mapping $T : K \to X$ is said to be strictly hemicontractive [7] if there exists $t > 1$ such that for all $x \in K$ and $x^* \in F(T)$ there exists $j \in J(x - x^*)$ such that

\[ \text{Re}(Tx - x^*, j) \leq \frac{1}{t}\|x - x^*\|^2. \]

Using Lemma K, it is easy to check [7] that the strict hemicontractivity of $T$ is equivalent to the following inequality
\[ \|x - x^*\| \leq \|(1 + r)(x - x^*) - rt(Tx - x^*)\| \]
holds for all $x \in K, x^* \in F(T)$ and $r > 0$.

We first introduce an example of a Lipschitzian self-mapping which is not strictly pseudocontractive but strictly hemicontractive.
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**Example 1** ([7]). Take $X = \mathbb{R}$ with the usual norm $| \cdot |$. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \frac{2}{3} x \cos x$$

for all $x \in \mathbb{R}$. Obviously, $F(T) = \{0\}$ and since $\langle Tx, x \rangle = \frac{2}{3} x^2 \cos x \leq \frac{2}{3} |x|^2$ for all $x \in \mathbb{R}$, $T$ is strictly hemicontractive with $t = \frac{3}{2} > 1$. However, if we can take $x = 2\pi$ and $y = \pi$, then

$$\langle Tx - Ty, x - y \rangle = 2\pi^2 > \pi^2 = |x - y|^2.$$  

Therefore $T$ is not strictly pseudocontractive. Further, if we can take $K = [-2\pi, 2\pi] \subset \mathbb{R}$, $T : K \to K$ is a Lipschitzian mapping with its Lipschitz constant $\frac{2}{3}(1 + 2\pi)$.

Motivated by the definition of strict hemicontractivity, we can consider a mapping $T : K \to K$ satisfying the following property, i.e., there exists $t > 1$ such that for all $x \in K$ and $x^* \in F(T)(\neq \emptyset)$, there exists $j \in J(x - x^*)$ such that

$$(H) \quad \lim_{n \to \infty} \sup \text{Re}(T^n x - x^*, j) \leq \frac{1}{t} ||x - x^*||^2.$$ 

Obviously, any mapping $T : K \to K$ which is both strictly hemicontractive and asymptotically nonexpansive (cf. Goebel-Kirk [10]; $||T^n x - T^n y|| \leq k_n ||x - y||$ for all $x, y \in K$ and all $n \in \mathbb{N}$ with $\lim \sup_{n \to \infty} k_n \leq 1$) satisfies the property (H). Here we shall give two examples of self-mappings which are asymptotically nonexpansive in the intermediate sense, not strictly hemicontractive, but satisfy the above property (H).

**Example 2.** Let $X = \mathbb{R}$ with the usual norm $| \cdot |$ and let $K = [0, 1]$. Let $a_n = \frac{1}{n}$ for each $n \in \mathbb{N}$. Then, construct a continuous mapping $T$ as follows. On the each subinterval $[a_{n+1}, a_n]$, the graph of $T$ consists of the sides of the isosceles triangle with base $[a_{n+1}, a_n]$ and height $a_{n+1}$. Thus, $Ta_n = 0$ and, if $x_n$ denotes the midpoint of $[a_{n+1}, a_n]$, then $Tx_n = a_{n+1}$. If we further define $T0 = 0$, $T : K \to K$ is uniformly continuous (but, not Lipschitzian) and only $F(T) = \{0\}$.
Since $T^n x \to 0$ uniformly as $n \to \infty$, $T$ is asymptotically nonexpansive in the intermediate sense. Indeed, for each $n \in \mathbb{N}$, there exist $x_n, y_n \in K$ such that

$$
\sup_{x,y \in K} (\|T^n x - T^n y\| - \|x - y\|) = \|T^n x_n - T^n y_n\| - \|x_n - y_n\|.
$$

Taking $\limsup_{n \to \infty}$ on both sides, we get

$$
\limsup_{n \to \infty}(\|T^n x_n - T^n y_n\| - \|x_n - y_n\|) \leq - \liminf_{n \to \infty} \|x_n - y_n\| \leq 0.
$$

It is obvious that $T$ satisfies the property (H). Now assume that $T$ is strictly hemicontractive, i.e., there exists $t > 1$ such that

$$
\langle Tx, x \rangle \leq \frac{1}{t} \|x\|^2
$$

for all $x \in K$. If we can choose $n \in \mathbb{N}$ so that $\frac{1}{2n+1} < \frac{t-1}{t}$, it is easy to check that $\langle Tx_n, x_n \rangle = a_{n+1} x_n > \frac{1}{t} \|x_n\|^2$, which gives a contradiction and so $T$ is not strictly hemicontractive.

Also, we shall give an example of a Lipschitzian mapping $T : K \to K$ (in fact, nonexpansive, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$) in the space $\ell_2$ which is not strictly hemicontractive but satisfies the property (H).

**Example 3.** Take $X = \ell_2$ with the usual norm $\| \cdot \|$. Let $K$ be the unit ball in $\ell_2$ and let $f : [-1, 1] \to [-1, 1]$ be defined by

$$
f(x) = \begin{cases} 
-\frac{1}{2} & \text{if } -1 \leq x \leq -\frac{3}{4}, \\
\frac{(2^n+1)x - nx^n}{2n+1+3} & \text{if } -\frac{2^n+1}{2n} \leq x \leq -\frac{2^{n+1}+1}{2(n+1)}, \\
0 & \text{if } x = 0, \\
\frac{(2^n+1)x + nx^n}{2n+1+3} & \text{if } \frac{2^{n+1}+1}{2(n+1)} \leq x \leq \frac{2^n+1}{2n}, \\
\frac{1}{2} & \text{if } \frac{3}{4} \leq x \leq 1,
\end{cases}
$$

for all $x \in [-1, 1]$ and all $n \in \mathbb{N}$. Then we readily see that $f$ is nonexpansive, i.e., $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [-1, 1]$ and $F(f) = \{0\}$. Define

$$
Tx = (f(x_1), f(x_2), \cdots)
$$

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for all $x = (x_1, x_2, \cdots) \in K$. Clearly, $T : K \to K$ and $F(T) = \{0\}$, where $0 = (0, 0, \cdots)$. Since

$$T^n x = (f^n(x_1), f^n(x_2), \cdots)$$

for all $x = (x_1, x_2, \cdots) \in K$ and, for each $j \in \mathbb{N}$, $f^n(x_j) \to 0$ as $n \to \infty$, it immediately follows that $T^n x \to 0$ as $n \to \infty$ and so

$$\lim_{n \to \infty} \langle T^n x, x \rangle = 0$$

for all $x = (x_1, x_2, \cdots) \in K$. Thus, $T : K \to K$ satisfies the property (H). However, $T$ is not strictly hemicontractive. Indeed, if there exists $t > 1$ such that

$$\langle Tx, x \rangle = \sum_{j=1}^{\infty} x_j f(x_j) \leq \frac{1}{t} \|x\|^2$$

for all $x = (x_1, x_2, \cdots) \in K$. Choose $n \geq 2$ so that $\frac{1}{2^n} < (t - 1)$. Setting

$$x_j = \frac{2^j + 1}{2^{2j}}$$

for all $j \geq n$ and $u = (x_n, x_{n+1}, \cdots)$, it easily follows that $\|u\| \leq 1$ and

$$\langle Tu, u \rangle = \sum_{j=n}^{\infty} x_j f(x_j) > \frac{1}{t} \|u\|^2$$

because $f(x_j) > \frac{1}{t} x_j$ if and only if $\frac{1}{2^j} < (t - 1)$ and $\frac{1}{2j} \leq \frac{1}{2^n} < (t - 1)$ for all $j \geq n$, which contradicts the assumption (2). Therefore $T$ is not strictly hemicontractive. It is easy to check that $T$ is nonexpansive, i.e.,

$$\|Tx - Ty\| = \left( \sum_{j=1}^{\infty} |f(x_j) - f(y_j)|^2 \right)^{\frac{1}{2}} \leq \|x - y\|$$

for all $x = (x_1, x_2, \cdots)$, $y = (y_1, y_2, \cdots) \in K$. 

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Recall that a mapping $T : K \to X$ is said to be strongly accretive [2] (or [23]) if there exists a positive number $k$ such that for each $x, y \in K$ there is $j \in J(x - y)$ such that

$$\text{Re}(Tx - Ty, j) \geq k\|x - y\|^2.$$

Using Lemma K again, this is equivalent to

$$\|x - y\| \leq \|x - y + r\{(I - kI)x - (I - kI)y\}\|,$$

for all $r > 0$, where $I$ denotes the identity mapping of $X$. Without loss of generality, we can assume $k \in (0, 1)$. Then it was known [1] that the similar connection between strict pseudocontractivity and strong accretivity is that a mapping $T : K \to K$ is strictly pseudocontractive if and only if $I - T$ is strongly accretive, i.e., the inequality

$$(3) \|x - y\| \leq \|x - y + r\{(I - T - kI)x - (I - T - kI)y\}\|$$

holds for any $x, y \in K$ and $r > 0$, where $k = \left(\frac{t-1}{t}\right) \in (0, 1)$.

Recently, the convergence problems of Ishikawa and Mann iteration sequences (cf. Ishikawa [12] and Mann [17]) have been studied extensively by many authors (see Chidume [4-7], Deng [8], Deng-Ding [9], Haiyun-Yuting [11], Liu [15], Liu [16], Reich [18] and Tan-Xu [22]) for strictly pseudocontractive (or strongly accretive) mappings.

Especially, Liu [15] proved by the inequality (3) that the Mann iteration process converges strongly to the unique fixed point of a Lipschitzian and strictly pseudocontractive mapping, which extends corresponding results of [4-6], [22] and [23] to the general Banach spaces.

**Theorem L([15]).** Let $K$ be a nonempty closed, convex and bounded subset of a Banach space $X$ and let $T : K \to K$ be Lipschitzian and strictly pseudocontractive mapping. If $F(T) \neq \emptyset$, then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad x_1 \in K,$$

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with \( \{\alpha_n\} \subseteq (0, 1] \) satisfying

\[
\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0,
\]

converges strongly to \( q \in F(T) \) and \( F(T) \) is a singleton set.

The following lemma was recently proved by Haiyun-Yuting [11]. Compare our easy observation with the proof of Lemma 1.1 in [11].

**Lemma H-Y([11]).** For any \( x, y \in X \) and \( j \in J(x + y) \),

\[
\|x + y\|^2 \leq \|x\|^2 + 2\text{Re}(y, j).
\]

**Proof.** For any \( x, y \in X \) and \( j \in J(x + y) \),

\[
\|x\|^2 + 2\text{Re}(y, j) - \|x + y\|^2 = \|x\|^2 + 2\text{Re}(x + y - x, j) - \|x + y\|^2 \\
= \|x\|^2 - 2\text{Re}(x, j) + \|x + y\|^2 \\
\geq \|x\|^2 - 2\|x\|\|x + y\| + \|x + y\|^2 \\
= (\|x\| - \|x + y\|)^2 \geq 0.
\]

At the same time, Haiyun-Yuting [11] proved, using the above lemma, that the Ishikawa iteration process converges strongly to the unique fixed point of a continuous and strictly pseudocontractive map without Lipschitz assumption in a real uniformly smooth Banach space.

**Theorem H-Y([11]).** Let \( K \) be a nonempty closed, convex and bounded subset of a real uniformly smooth Banach space \( X \). Assume that \( T : K \to K \) is a continuous strictly pseudocontractive mapping. Let \( \{\alpha_n\}_{n=1}^{\infty} \) and \( \{\beta_n\}_{n=1}^{\infty} \) be two real sequences satisfying

(i) \( 0 < \alpha_n, \beta_n < 1 \) and \( \alpha_n \to 0, \beta_n \to 0 \) as \( n \to \infty \);

(ii) \( \sum_{n=1}^{\infty} \alpha_n = \infty \).
Then the Ishikawa iterative sequence \( \{x_n\}_{n=1}^{\infty} \) generated by

\[
\begin{align*}
    x_1 &\in K, \\
    x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n, \\
    y_n &= (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 1,
\end{align*}
\]

converges strongly to the unique fixed point of \( T \).

In these respects, it seems natural to ask whether the above two theorems are still valid for any mapping \( T : K \to K \) satisfying the property (H). For our affirmative argument, consider the following modified Ishikawa iteration process instead of (4):

\[
\begin{align*}
    x_1 &\in K, \\
    x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nT^ny_n, \\
    y_n &= (1 - \beta_n)x_n + \beta_nT^nx_n, \quad n \geq 1.
\end{align*}
\]

The above algorithm (4)' was used by Schu [19] and Tan-Xu [21] to show the weak convergence of the Mann and Ishikawa iteration processes to a fixed point of an asymptotically nonexpansive self-mapping.

We first begin with an easy observation of the property (H). The first equivalent is

\[
(H_1) \quad \lim_{n \to \infty} \inf \Re(x - T^n x, j) \geq \frac{(t - 1)}{t} \|x - x^*\|^2.
\]

Let \( x \neq x^* \). For a fixed \( \epsilon \) with \( 0 < \epsilon < \frac{(t-1)}{t} \), it follows from the property (H1) that there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \),

\[
(H_2) \quad \Re(x - T^n x, j) \geq \left( \frac{t-1}{t} - \epsilon \right) \|x - x^*\|^2
\]

\[
= k_\epsilon \|x - x^*\|^2,
\]

where \( k_\epsilon := \left( \frac{t-1}{t} - \epsilon \right) \in (0, 1) \). This inequality is obviously equivalent to

\[
(H_3) \quad \Re(T^n x - x^*, j) \leq (1 - k_\epsilon) \|x - x^*\|^2, \quad \forall n \geq n_0.
\]

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For employing the method of the proof in [15], we need the following equivalent form of the property \((H_2)\) by virtue of Lemma K:

\[
(H_4) \quad \|x - x^*\| \leq \|x - x^* + r((I - T^n - k_eI)x - (I - T^n - k_eI)x^*)\|
\]

for all \(n \geq n_0\) and all \(r > 0\).

Using the property \((H_4)\), we are now ready to present the following

**Theorem 1.** Let \(K\) be a nonempty closed, convex and bounded subset of a Banach space \(X\). Assume that \(T : K \rightarrow K\) is asymptotically nonexpansive in the intermediate sense satisfying the property \((H)\). Put

\[
c_n = \max\{0, \sup_{x, y \in K} (\|T^n x - T^n y\| - \|x - y\|)\},
\]

so that \(\lim_{n \to \infty} c_n = 0\). Let \(\{\alpha_n\}_{n=1}^{\infty}\) and \(\{\beta_n\}_{n=1}^{\infty}\) be two real sequences satisfying

(i) \(0 \leq \alpha_n, \beta_n \leq 1\) and \(\alpha_n \to 0, \ \beta_n \to 0\) as \(n \to \infty\);

(ii) \(\sum_{n=1}^{\infty} \alpha_n = \infty\).

Then the modified Ishikawa iterative sequence \(\{x_n\}_{n=1}^{\infty}\) generated by (4)' converges strongly to the unique fixed point of \(T\) in \(K\).

**Proof.** We employ the method of the proof of Liu [15]. Since \(F(T) \neq \emptyset\), take \(q \in F(T)\). From the definition of \(\{x_n\}\), we have

\[
x_n = x_{n+1} + \alpha_n x_n - \alpha_n T^n y_n
\]

\[
= (1 + \alpha_n)x_{n+1} + \alpha_n(I - T^n - k_eI)x_{n+1} - (2 - k_e)\alpha_n x_{n+1} + \alpha_n x_n + \alpha_n(T^n x_{n+1} - T^n y_n)
\]

\[
= (1 + \alpha_n)x_{n+1} + \alpha_n(I - T^n - k_eI)x_{n+1} - (2 - k_e)\alpha_n((1 - \alpha_n)x_n + \alpha_n T^n y_n)
\]

\[
+ \alpha_n x_n + \alpha_n(T^n x_{n+1} - T^n y_n)
\]

\[
= (1 + \alpha_n)x_{n+1} + \alpha_n(I - T^n - k_eI)x_{n+1} - (1 - k_e)\alpha_n x_n
\]

\[
+ (2 - k_e)\alpha_n^2(x_n - T^n y_n) + \alpha_n(T^n x_{n+1} - T^n y_n).
\]

By \(Tq = q\), this implies that

\[
x_n - q = (1 + \alpha_n)(x_{n+1} - q) + \alpha_n(I - T^n - k_eI)(x_{n+1} - q) - (1 - k_e)\alpha_n(x_n - q)
\]

\[
+ (2 - k_e)\alpha_n^2(x_n - T^n y_n) + \alpha_n(T^n x_{n+1} - T^n y_n).
\]
By the property (H₄), we obtain

$$\|x_n - q\| \geq (1 + \alpha_n)\|x_{n+1} - q\| - (1 - k_\varepsilon)\alpha_n\|x_n - q\| - (2 - k_\varepsilon)\alpha_n^2\|x_n - T^n y_n\| - \alpha_n\|T^m x_{n+1} - T^m y_n\|,$$

for all $n \geq n_0$. Since

$$[1 + (1 - k_\varepsilon)\alpha_n](1 + \alpha_n)^{-1} \leq [1 + (1 - k_\varepsilon)\alpha_n](1 - \alpha_n + \alpha_n^2) = 1 - k_\varepsilon\alpha_n + \alpha_n^2 - (1 - k_\varepsilon)\alpha_n^2(1 - \alpha_n) \leq 1 - k_\varepsilon\alpha_n + \alpha_n^2,$$

it follows from the above inequality that for all $n \geq n_0$,

$$(5) \quad \|x_{n+1} - q\| \leq [1 + (1 - k_\varepsilon)\alpha_n](1 + \alpha_n)^{-1}\|x_n - q\| + (2 - k_\varepsilon)\alpha_n^2(1 + \alpha_n)^{-1}\|x_n - T^n y_n\| + \alpha_n(1 + \alpha_n)^{-1}\|T^m x_{n+1} - T^m y_n\| \leq (1 - k_\varepsilon\alpha_n)\|x_n - q\| + \alpha_n^2(\|x_n - q\| + (2 - k_\varepsilon)\|x_n - T^n y_n\|) + \alpha_n d_n \leq (1 - k_\varepsilon\alpha_n)\|x_n - q\| + M\alpha_n^2 + \alpha_n d_n,$$

where $d_n = \|T^m x_{n+1} - T^m y_n\|$ and $M = 3\text{diam}(K)$ (since $K$ is bounded). Since $\{x_n\}$, $\{T^m x_n\}$ and $\{T^m y_n\}$ are all bounded sequences in $K$,

$$y_n - x_{n+1} = (\alpha_n - \beta_n)x_n + \beta_n T^m x_n - \alpha_n T^m y_n \to 0$$

as $n \to \infty$. Since $c_n \to 0$ as $n \to \infty$ we get

$$d_n = \|T^m y_n - T^m x_{n+1}\| = \|[T^m y_n - T^m x_{n+1}] - [y_n - x_{n+1}]\| + \|y_n - x_{n+1}\| \leq c_n + \|y_n - x_{n+1}\| \to 0 \quad \text{as} \quad n \to \infty.$$

Applying Lemma 4 of [23] (or Lemma 1.2 in [11]), we have $x_n \to q$ as $n \to \infty$. Finally, we prove that $F(T) = \{q\}$, a singleton set. If
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$p \in F(T)$, by using the property (H), we obtain

$$\|p - q\|^2 = \langle p - q, j \rangle$$

$$= \limsup_{n \to \infty} \text{Re}(T^np - q, j)$$

$$\leq \frac{1}{t} \|p - q\|^2.$$

Since $t > 1$, we have $q = p$. \qed

**Remark 1.** In view of the example 2 and 3, the above theorem is a new approach of the strong convergence problems of iterative sequences to the unique fixed point of self-mappings which are not strictly hemi-contractive (hence, not strictly pseudocontractive). Compare this with Theorem L. Following the steps of the proof of Theorem H-Y, it is easy to see that if $X$ is uniformly smooth and if $T : K \to K$ is a continuous mapping with the property (H) Theorem 1 is still valid.

**Remark 2.** Using Lemma H-Y and the property (H3), we can similarly prove Theorem 1. Indeed,

$$\|x_{n+1} - q\|^2 \leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \text{Re}(T^ny_n - q, j_n)$$

$$\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \text{Re}(T^ny_n - T^nx_{n+1}, j_n)$$

$$+ 2\alpha_n \text{Re}(T^nx_{n+1} - q, j_n)$$

$$\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n d_n + 2\alpha_n (1 - k) \|x_{n+1} - q\|^2$$

for $j_n \in J(x_{n+1} - q)$, where $d_n = \text{Re}(T^ny_n - T^nx_{n+1}, j_n) \to 0$ as $n \to \infty$. In fact, since $K$ is bounded and $c_n \to 0$ as $n \to \infty$, we obtain

$$\|T^ny_n - T^nx_{n+1}\| = \|[T^ny_n - T^nx_{n+1}] - [y_n - x_{n+1}]\| + \|y_n - x_{n+1}\|$$

$$\leq c_n + \|y_n - x_{n+1}\| \to 0 \text{ as } n \to \infty.$$

On the other hand, since $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\alpha_n \to 0$ as $n \to \infty$, we can choose $n_1 (\geq n_0)$ so that $\alpha_n > 0$, $1 - 2\alpha_n (1 - k) > 0$, and $2k - \alpha_n > 0$ for all $n \geq n_1$. Then, the above inequality can be written as follows:

$$\|x_{n+1} - q\|^2 \leq \frac{(1 - \alpha_n)^2}{1 - 2\alpha_n (1 - k)} \|x_n - q\|^2 + \frac{2\alpha_n d_n}{1 - 2\alpha_n (1 - k)}.$$
Since \( \frac{2k_\varepsilon - \alpha_n}{1 - 2\alpha_n (1 - k_\varepsilon)} \to 2k_\varepsilon \) as \( n \to \infty \) and \( k_\varepsilon \in (0, 1) \), there exists a \( n_2 \) (\( \geq n_1 \)) such that
\[
\left| \frac{2k_\varepsilon - \alpha_n}{1 - 2\alpha_n (1 - k_\varepsilon)} - 2k_\varepsilon \right| \leq k_\varepsilon
\]
for all \( n \geq n_2 \). This implies that \( k_\varepsilon \leq \frac{2k_\varepsilon - \alpha_n}{1 - 2\alpha_n (1 - k_\varepsilon)} \), that is,
\[
\frac{(1 - \alpha_n)^2}{1 - 2\alpha_n (1 - k_\varepsilon)} \leq (1 - k_\varepsilon \alpha_n)
\]
for all \( n \geq n_2 \). The inequality (6) can be expressed as follows.
\[
\|x_{n+1} - q\|^2 \leq (1 - k_\varepsilon \alpha_n) \|x_n - q\|^2 + \frac{2\alpha_n d_n}{1 - 2\alpha_n (1 - k_\varepsilon)}
\]
for all \( n \geq n_2 \). Then it follows from the lemma 4 of Weng [23] that the sequence \( \{x_n\} \) strongly converges to the unique fixed point \( q \) of \( T \).

For evaluating the error estimate of the strong convergence of the Ishikawa type iteration \( \{x_n\}_{n=1}^\infty \) generated by (4)' to the unique fixed point, in addition to all assumptions of Theorem 1, suppose that \( T : K \to K \) is uniformly \( L \)-Lipschitzian, that is, there exists a constant \( L > 0 \) such that
\[
\|T^n x - T^n y\| \leq L \|x - y\|,
\]
for all \( n \in \mathbb{N} \) and \( x, y \in K \). Obviously, every nonexpansive mappings are uniformly 1-Lipschitzian.

Now we are ready to present the following error estimate of the convergence.

**Theorem 2.** Let \( K \) be a nonempty closed, convex and bounded subset of a Banach space \( X \) and let \( T : K \to K \) a uniformly \( L \)-Lipschitzian mapping satisfying the property (H). If \( \alpha_n = \frac{k_\varepsilon}{2(3 + 3L + L^2)} \) and \( \beta_n = \frac{k_\varepsilon}{4L(L^2 + 2L - 1)} \) where \( k_\varepsilon \) is the positive real number in \((0, 1)\) as in the property \( (H_2) \) and \( F(T) = \{q\} \). Then the modified Ishikawa iterative sequence \( \{x_n\}_{n=1}^\infty \) generated by (4)' converges strongly to the unique fixed point of \( T \) in \( K \), and we obtain the estimate
\[
\|x_{n+1} - q\| < \rho^n \|x_1 - q\|
\]
where \( \rho = 1 - \frac{k_\varepsilon^2}{8(3 + 3L + L^2)} \).
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Proof. Let \( Tq = q \). Note that

\[
\|x_n - T^m y_n\| \leq \|x_n - q\| + L\|y_n - q\|
\]

\[
\leq \|x_n - q\| + L(1 + \beta_n(L - 1))\|x_n - q\|
\]

\[
= [1 + L + \beta_nL(L - 1)]\|x_n - q\|
\]

and

\[
\|T^m x_{n+1} - T^m y_n\| \leq L(\|x_{n+1} - x_n\| + \|x_n - y_n\|)
\]

\[
= L(\alpha_n\|x_n - T^m y_n\| + \beta_n\|x_n - T^m x_n\|)
\]

\[
\leq L[\alpha_n(1 + L + \beta_nL(L - 1)) + \beta_n(1 + L)]\|x_n - q\|
\]

\[
= [\alpha_nL(L+1) + \alpha_n\beta_nL^2(L-1) + \beta_nL(L+1)]\|x_n - q\|
\]

Combined with the inequality (5), the required estimate is now obtained.

\[
\|x_{n+1} - q\| \leq (1 - k_\varepsilon\alpha_n)\|x_n - q\| + \alpha_n^2\|x_n - q\| + (2 - k_\varepsilon)\|x_n - T^m y_n\|
\]

\[
+ \alpha_n\|T^m x_{n+1} - T^m y_n\|
\]

\[
\leq (1 - k_\varepsilon\alpha_n)\|x_n - q\| + \alpha_n^2[1 + (2 - k_\varepsilon)(1 + L + \beta_nL(L - 1))]\|x_n - q\|
\]

\[
+ \alpha_n[\alpha_nL(L+1) + \alpha_n\beta_nL^2(L-1) + \beta_nL(L+1)]\|x_n - q\|
\]

\[
\leq (1 - k_\varepsilon\alpha_n)\|x_n - q\| + \alpha_n^2[1 + (2 - k_\varepsilon)(1 + L) + L(L+1)]\|x_n - q\|
\]

\[
+ \alpha_n\beta_n[(2 - k_\varepsilon)L(L-1) + L^2(L+1) + L(L+1)]\|x_n - q\|
\]

\[
< (1 - k_\varepsilon\alpha_n)\|x_n - q\| + \alpha_n^2(3 + 3L + L^2)\|x_n - q\|
\]

\[
+ \alpha_n\beta_nL(L^2 + 2L - 1)\|x_n - q\|
\]

\[
= \left[1 - \frac{k_\varepsilon^2}{8(3 + 3L + L^2)}\right]\|x_n - q\|
\]

\[
= \rho\|x_n - q\|
\]

Hence \( \|x_{n+1} - q\| < \rho^n\|x_1 - q\| \). \qed

Taking \( \beta_n = 0 \) for all \( n \geq 1 \) in Theorem 1, we have the following

**Corollary 1.** Let \( K \) be a nonempty closed, convex and bounded subset of a Banach space \( X \). Assume that \( T : K \to K \) is asymptotically nonexpansive in the intermediate sense satisfying the property (H). Put

\[
c_n = \max\{0, \sup_{x, y \in K} (\|T^m x - T^m y\| - \|x - y\|)\},
\]

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so that \( \lim_{n \to \infty} c_n = 0 \). Then the modified Mann iterative sequence \( \{x_n\}_{n=0}^{\infty} \) generated by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nT^m x_n, \quad x_1 \in K
\]

with \( \{\alpha_n\}_{n=1}^{\infty} \subset (0, 1] \) satisfying

\[
\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0,
\]

strongly converges \( q \in F(T) \) and \( F(T) \) is a singleton set.

Remark 4. Compared with Theorem L and Theorem H-Y, our iterative algorithm can be applicable to the approximating problems of all continuous mappings with the property (H) in the finite dimensional spaces.

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