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LINE GRAPHS OF COVERING GRAPHS ARE COVERING GRAPHS

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ABSTRACT. Let $\tilde{G}$ be a covering graph of G. We show that the line graph of $\tilde{G}$ covers the line graph of G. Moreover, if the first covering is regular, then the line-graph covering is regular.

1. Introduction

Covering graphs are a fascinating and useful area of graph theory. Loosely speaking, a graph $\tilde{G}$ covers a graph G if each vertex of $\tilde{G}$ locally looks like a vertex of G. This local similarity frequently allows one to succinctly describe large graphs in terms of their much smaller local structure. These covering properties also allow concise descriptions of covering maps between surfaces via covering graphs of their 1-skeletons.

In this paper we offer a nice construction of covering graphs. In particular, if a graph $\tilde{G}$ covers a graph G, then we show that the line graph of $\tilde{G}$ covers the line graph of G. Moreover, if the first covering is regular (that is, if it can be described as the quotient map of some subgroup of automorphisms), then the line-graph covering is also regular.

The paper is organized as follows: In Section 2 we give our descriptions of graphs, coverings, voltage assignments, and line graphs. In Section 3 we give our main result.

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2. Basics

Our graphs G can have loops and multiple edges. Each edge in G has two *ends*, one corresponding to each of its two incidences. These two ends *lie in the same edge* and are *opposite*. Let $E^2 = EE(G)$ denote the set of edge ends. Let $\lambda = \lambda(G)$ denote the fixed-point-free involution on E^2 which swaps the two opposite ends on every edge. Two edge ends are *adjacent* if they are incident with a common vertex. The adjacency relation defines a partition $V = V(G)$ on the set of edge ends which corresponds to the vertices of G. Formally, we define a graph G as a triple (V, E^2, λ) where V partitions a set E^2 of edge ends and λ is a fixed-point-free involution of E^2.

Let $G = (V, E^2, \lambda)$ and $\tilde{G} = (\tilde{V}, \tilde{E}^2, \tilde{\lambda})$ be graphs. A *graph homomorphism* ϕ from $\tilde{G}$ to G is a function from the edge ends of $\tilde{G}$ to the edge ends of G such that:

1) opposite edge ends map to opposite edge ends, and
2) adjacent edge ends map to adjacent edge ends.

When Condition 1) holds we say that ϕ *respects* λ, and there is a well-defined map from the edges of $\tilde{G}$ to the edges of G. Likewise, when Condition 2) holds ϕ *respects vertices* and there is a well-defined map from $\tilde{V}$ to V. We also denote both of these induced maps by ϕ.

A graph homomorphism is a *covering map* if additionally

3) edge ends in $\tilde{v} \in \tilde{V}$ map bijectively to edge ends in $v = \phi(\tilde{v})$ for each $\tilde{v}$, and
4) the map is surjective onto the edge-ends E^2 of G.

Condition 3) implies Condition 4) when G is connected, as we will assume in this paper. Condition 3) ensures that a vertex of $\tilde{G}$ has the same degree as the vertex it covers. The vertices in $\phi^{-1}(v)$ form the *fiber* over the vertex v; we similarly define the fiber over an edge of G and over edge ends of G.

Let e and $f = \lambda(e)$ denote opposite edge ends. Then by Condition 1) $\tilde{\lambda}$ gives a bijection between $\phi^{-1}(e)$ and $\phi^{-1}(f)$. Similarly, by Condition 3) there is a bijection between the fibers of adjacent edge ends. Thus, when G is connected, any two vertex or edge-end fibers are of the same cardinality n. This number is called the *fold number* of the covering, and we say that ϕ is an *n-fold covering*.

There is a class of coverings which are of special interest. Suppose that there is a group $\mathcal{A}$ of automorphisms of $\tilde{G}$ such that the orbits of the vertex set $\tilde{V}$ and the edge-end set $\tilde{E}^2$ under the action of $\mathcal{A}$ are exactly the fibers of ϕ over the vertices and edge-ends of G. Moreover, suppose that for each e and f in the same edge-end fiber there is a unique $\alpha \in \mathcal{A}$ with $\alpha(e) = f$. Then we say that ϕ is a *regular* covering, or an $\mathcal{A}$-*covering*. The acting group $\mathcal{A}$ is also called the *covering transformation group* of ϕ. Note that a regular covering has fold number $|\mathcal{A}|$.

There is a convenient way to label the vertex and edge-ends in a regular covering. Specifically, fix one vertex (v, id) in each vertex fiber to correspond to the identity element. Label other vertices $\tilde{v}$ in the fiber over v by (v, α), where α is the unique automorphism in $\mathcal{A}$ carrying (v, id) to $\tilde{v}$. Label an edge end $(e, \alpha) \in \tilde{E}^2$ if it covers an edge end e of G and is incident with (v, α). With this labeling $\alpha \in \mathcal{A}$ acts on $\tilde{E}^2$ by $(e, \beta) \mapsto (e, \beta\alpha)$.

Regular coverings can also be described by assigning a group element to each edge-end of G. Specifically, label the edge ends of $\tilde{G}$ as in the preceding paragraph. Assign the group element α to an edge-end $e \in E^2$ if (e, id) and $(\lambda(e), \alpha)$ are opposite edge-ends in $\tilde{G}$. This α is called the *voltage* on e. The function $\nu : e \mapsto \alpha$ is called a *voltage assignment* into the *voltage group* $\mathcal{A}$. Notice that if $\nu(e) = \alpha$, then $\nu(\lambda(e)) = \alpha^{-1}$. In other words, a voltage assignment maps edge-ends to group elements such that opposite ends receive inverse elements. We say that this function *respects inverses.*

Any function $\nu : E^2 \to \mathcal{A}$ which respects inverses is a voltage assignment of a regular covering. We will show how to construct a *derived covering graph* $\tilde{G} = G \times_\nu \mathcal{A}$ from this ν. The edge-ends $\tilde{E}^2$ of $\tilde{G}$ are $E^2 \times \mathcal{A}$. Two ends (e_1, α_1) and (e_2, α_2) are adjacent if and only if e_1 and e_2 are adjacent and $\alpha_1 = \alpha_2$. Define the opposite end $\tilde{\lambda}((e, \alpha))$ of (e, α) to be $(\lambda(e), \alpha\nu(e))$. Then $(\tilde{V}, \tilde{E}^2, \tilde{\lambda})$ is a covering of G. The covering map projects (e, α) onto the first coordinate. The covering is regular since $\alpha \in \mathcal{A}$ acts on $\tilde{G}$ by mapping (e, β) to $(e, \beta\alpha)$.

We close this section with the concept of line graphs. Let $G = (V, E^2, \lambda)$ be a graph. The *line graph* $L(G) = (V_L, E_L^2, \lambda_L)$ has vertex set corresponding to the unordered edges $\{e, \lambda(e)\}$ of G. The edge ends E_L^2 are ordered pairs (e, f) where e and f are adjacent edge ends in G. This edge-end (e, f) is incident with the vertex corresponding to $\{e, \lambda(e)\}$. It is opposite to the edge end $(f, e) = \lambda_L((e, f))$.

3. Line Graphs and the Main Result

In this section we give our main result.

THEOREM 3.1. *Let $\tilde{G}$ be a connected graph covering of G. Then the line graph of $\tilde{G}$ covers the line graph of G. Moreover, if the first covering is regular, then the covering of the line graphs is also regular.*

Proof. Let $\tilde{v}_1, \ldots, \tilde{v}_n$ denote the vertices of $\tilde{G}$ over the vertex v in G. Label an edge ends $e_i \in \tilde{E}^2$ if it covers $e \in E^2$ and is incident with v_i. Using this notation an edge-end of $L(\tilde{G})$ is an ordered pair (e_i, f_j). Define $\phi_L : L(\tilde{G}) \to L(G)$ by $(e_i, f_j) \mapsto (e, f)$. We will show that ϕ_L is a covering.

First note that $\phi_L(\tilde{\lambda}((e_i, f_j)) = \phi_L((f_j, e_i)) = (f, e)$. On the other hand, $\lambda(\phi_L((e_i, f_j)) = \lambda((e, f)) = (f, e)$. Hence ϕ_L respects λ.

Next, consider the edge ends incident with a vertex $\{e_i, f_j\}$ of $L(\tilde{G})$. These divide into two types: those ordered pairs with first coordinate e_i and those with first coordinate f_j. Because $\tilde{G}$ covers G, the first type are in bijective correspondence with the edge ends of $L(G)$ whose first coordinate is e, and the second are in bijective correspondence with the edge ends with first coordinate f. It follows that their union is in bijection with the edge ends incident with $\{e, f\} \in V(L(G))$. Hence Conditions 2) and 3) hold and $L(\tilde{G})$ covers $L(G)$.

It remains to show that the covering ϕ_L is regular when the covering ϕ is. Let $\mathcal{A}$ be the covering transformation group of ϕ. Label the vertices and edge ends of $\tilde{G}$ as in a derived covering graph of a voltage assignment. For convenience, let e_β denote the edge end (e, β). Define an action of $\mathcal{A}$ on the edge ends in $L(\tilde{G})$ by $\alpha : (e_\beta, f_\gamma) \mapsto (e_{\beta\alpha}, f_{\gamma\alpha})$. These lie in the same fiber $\phi_L^{-1}((e, f))$. Conversely, any two edge ends in this fiber differ by the subscript on their first coordinate. Since $\mathcal{A}$ acts regularly on $\tilde{G}$, there is an $\alpha \in \mathcal{A}$ carrying one to the other. Hence ϕ_L is regular as desired. □

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