SPATIAL NUMERICAL RANGES
OF ELEMENTS OF C*-ALGEBRAS

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Dedicated to Professor Kôzô Yabuta on his sixtieth birthday.

Abstract. When $A$ is a subalgebra of a $C^*$-algebra, the spatial numerical range of element of $A$ can be described in terms of positive linear functionals on the $C^*$-algebra.

1. Introduction and results

Let $A$ be a complex Banach algebra and $A^*$ its dual space. Let $a \in A$. If $A$ is unital, then $V(A, a) \equiv \{f(a) : f \in A^*, \|f\| = f(1) = 1\}$ is called the (algebra) numerical range of $a$ and it is a non-void compact convex subset of the complex plane $C$ (see [1, p.52]).

However if $A$ is non-unital, then the above definition is not meaningful. In this case, we consider the following two sets:

$V_1(A, a) = \{f(xa) : \exists f \in A^* \text{ and } \exists x \in A \text{ such that } \|f\| = \|x\| = f(x) = 1\}$

and

$V_2(A, a) = \{f(ax) : \exists f \in A^* \text{ and } \exists x \in A \text{ such that } \|f\| = \|x\| = f(x) = 1\}$.

It is easy to see that $V(A, a) = V_1(A, a) = V_2(A, a)$ for the unital case.

A. K. Gaur and T. Husain([3]) especially called the spatial numerical range $V_2(A, a)$ for non-unital case and investigated this situation. In particular, they showed that if $A$ is a commutative $C^*$-algebra with maximal ideal space $\Phi_A$, then

$$\text{co}\{\hat{a}(\phi) : \phi \in \Phi_A\} \subseteq V_1(A, a) \subseteq \text{co}\{\hat{a}(\phi) : \phi \in \Phi_A\},$$

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where \( \overline{c} \) and \( \hat{a} \) denote the convex hull, the closed convex hull and the Gelfand transform of \( a \in A \), respectively (see [3, Theorem 4.1]).

The purpose of this paper is to investigate the spatial numerical ranges for \( C^* \)-algebras and obtain an extension of their result.

Our main result is the following.

**Theorem.** Let \( A \) be a \( C^* \)-algebra and \( B \) a subalgebra of \( A \). Let \( b \in B \). Then

\[
V_1(B, b) = \{|f|(b) : \exists f \in A^* \text{ and } \exists x \in B \text{ such that } \|f\| = \|x\| = f(x) = 1\}
\]

and

\[
V_2(B, b) = \{|f|(b) : \exists f \in A^* \text{ and } \exists x \in B \text{ such that } \|f\| = \|x\| = f(x) = 1\}
\]

where \( |f| \) denotes the absolute value of \( f \) (cf. [2, Definition 12.2.8]).

If \( B \) is a \( * \)-subalgebra, then \( V_1(B, b) = V_2(B, b) \).

**Remark 1.** The more detailed for the commutative \( C^* \)-algebra case will be appeared in ([5]).

As a corollary of the main theorem, we have the following result which extends [3, Theorem 4.1].

**Corollary.** Let \( A \) be a \( C^* \)-algebra and \( a \in A \). Then

\[
\co\{f(a) : f \in P(A)\} \subseteq V_1(A, a) = V_2(A, a) \subseteq \overline{\co}\{f(a) : f \in P(A)\},
\]

where \( P(A) \) denotes the set of all pure states of \( A \).

**Remark 2.** We don't know conditions under which \( \co\{f(a) : f \in P(A)\} = V_1(A, a)(= V_2(A, a)) \) holds. Similarly for \( \overline{\co}\{f(a) : f \in P(A)\} = V_1(A, a)(= V_2(A, a)) \).

2. Proof of results

*Proof of Theorem.* Set

\[
W_1 = \{|f|(b) : \exists f \in A^* \text{ and } \exists x \in B \text{ such that } \|f\| = \|x\| = f(x) = 1\}
\]

and let \( \lambda \in V_1(B, b) \). Then there exist \( g \in B^* \) and \( x \in B \) such that \( \lambda = g(xb) \) and \( \|g\| = \|x\| = g(x) = 1 \). Take a functional \( f \in A^* \) such
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that $\|f\| = \|g\|$ and $f(b) = g(b)$ for each $b \in B$, and let $f = u \cdot |f|$ be the enveloping polar decomposition of $f$ (cf. [2, Definition 12.2.8]). Then

$$1 = f(x) = |f|(ux) = (x|u^*)_{|f|} \leq \|x\|_{|f|}\|u^*\|_{|f|} \leq 1 \cdot 1 = 1$$

so that we can find a scalar $\alpha$ satisfying

$$\|u^* - \alpha x\|_{|f|} = 0$$

since the equality of the Cauchy-Schwarz inequality in (1) holds. Note that (1) implies

$$\|u^* - \alpha x\|_{|f|} = (x|u^*)_{|f|} = (u^*|u^*)_{|f|} = (x|x)_{|f|} = 1$$

and hence $1 - \alpha^2 - \alpha + |\alpha|^2 = 0$ by (2). Therefore, $\alpha$ must be equal to 1, and so $\|u^* - x\|_{|f|} = 0$, that is $u^* - x$ belongs to the left kernel (in the enveloping von Neumann algebra of $A$) $N_{|f|} = \{ x \in A^* : |f|(x^*x) = 0 \}$ of $|f|$. Also since $|f|(x^*x) = (x|x)_{|f|} = \|x\|_{|f|}^2 = 1$ by (1), it follows that $1 - x^*x \in N_{|f|}$, where 1 denotes the identity element of $A^*$. Therefore we have

$$\lambda = f(xb) = |f|(uxb) = (xb|u^*)_{|f|} = (xb|x)_{|f|} = |f|(x^*xb) = |f|(b)$$

(the 4th-equality follows from $u^* - x \in N_{|f|}$ and the 6th-equality follows from $1 - x^*x \in N_{|f|}$) and so $\lambda \in W_1$, hence $V_1(B, b) \subseteq W_1$.

Conversely suppose $\lambda \in W_1$. Then there exist $f \in A^*$ and $x \in B$ such that $\lambda = |f|(b)$ and $\|f\| = \|x\| = f(x) = 1$. Let $f = u \cdot |f|$ be the enveloping polar decomposition of $f$. Then we can apply directly the above arguments for $f$, $x$ and $u$. Consequently, we have $f(xb) = |f|(b)$ and hence $\lambda \in V_1(B, b)$, so $W_1 \subseteq V_1(B, b)$. We thus obtain $V_1(B, b) = W_1$.

We next set

$$W_2 = \{ |f|(b) : \exists f \in A^* \text{ and } \exists x \in B \text{ such that } \|f\| = \|x\| = f(x^*) = 1 \},$$

and let $\lambda \in V_2(B, b)$. Then there exist $g \in B^*$ and $x \in B$ such that $\lambda = g(bx)$ and $\|g\| = \|x\| = g(x) = 1$. Take a functional $f \in A^*$ such that $\|f\| = \|g\|$ and $f(b) = g(b)$ for each $b \in B$. Then

$$\|f^*\| = \|f\| = \|x\| = \|x^*\| \text{ and } 1 = f(x) = f^*(x^*),$$

so that $\lambda = |f|(bx) = f^*(x^*b^*)$, $\|f^*\| = \|f\| = \|x\| = \|x^*\|$ and $1 = f(x) = f^*(x^*)$, and hence $\lambda \in V_1(B, b^*)$, where $B = \{ x \in A : x^* \in B \}$. Therefore by the preceding argument, we can find $h \in A^*$ and $y \in B$ such that $\lambda = |h|(b^*)$ and $\|h\| = \|y\| = h(y^*) = 1$. This means that $\lambda \in W_2$, so we have $V_2(B, b) \subseteq W_2$.  

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The inverse inclusion \( W_2 \subseteq V_2(B, b) \) can be easily obtained by tracing the converse of the above argument.

Set

\[ A^*_{1,B} = \{ f \in A^* : \|f\| = 1 \text{ and } \exists x \in B \text{ such that } \|x\| = f(x) = 1 \} \]

and

\[ A^*_{2,B} = \{ f \in A^* : \|f\| = 1 \text{ and } \exists x \in B \text{ such that } \|x\| = f(x^*) = 1 \}. \]

If \( B \) is a *-subalgebra, then \( f \to f^* \) is a bijection of \( A^*_{1,B} \) onto \( A^*_{2,B} \) and hence we have

\[ V_1(B, b) = \{|f|(b) : f \in A^*_{1,B}\} = \{|f|(b) : f \in A^*_{2,B}\} = V_2(B, b) \]

\[ \square \]

Proof of Corollary. Let \( A \) be a C*-algebra and \( a \in A \). Then we have \( V_1(A, a) = V_2(A, a) \) by Theorem. We next show that \( \text{co}\{ f(a) : f \in P(A) \} \subseteq V_1(A, a) \). To do this, let \( \alpha \in \text{co}\{ f(a) : f \in P(A) \} \). Then there exist \( f_{11}, \ldots, f_{1m_1}, \ldots, f_{n1}, \ldots, f_{nm_n} \in P(A) \) and \( \lambda_{11}, \ldots, \lambda_{1m_1}, \ldots, \lambda_{n1}, \ldots, \lambda_{nm_n} \geq 0 \) such that

\[ \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} = 1, \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij}f_{ij}(a) = \alpha, \]

\[ \pi_{f_{11}} \equiv \cdots \equiv \pi_{f_{1m_1}}, \ldots, \pi_{f_{n1}} \equiv \cdots \equiv \pi_{f_{nm_n}} \text{ and } \pi_{f_{ij}} \neq \pi_{f_{ij}}(i \neq j). \]

Let \( \pi_1 \equiv \pi_{f_{11}} \equiv \cdots \equiv \pi_{f_{1m_1}}, \ldots, \pi_n \equiv \pi_{f_{n1}} \equiv \cdots \equiv \pi_{f_{nm_n}} \). For each \( i, j(1 \leq i \leq n, 1 \leq j \leq m_i) \), choose an isomorphism \( U_{ij} \) of the Hilbert space \( H_{\pi_i} \) onto the Hilbert space \( H_{\pi_{f_{ij}}} \) which transforms \( \pi_i(x) \) into \( \pi_{f_{ij}}(x) \) for every \( x \in A \), and set \( \xi_{ij} = U_{ij}^*(\xi_{f_{ij}}) \). Also set \( f = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij}f_{ij} \).

Then we have \( \|f\| = 1, f = |f|, \alpha = f(a) \) and

\[ f(x) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij}(\pi_{f_{ij}}(x)\xi_{f_{ij}}|\xi_{f_{ij}}) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij}(\pi_i(x)\xi_{ij}|\xi_{ij}) \]

for every \( x \in A \). Furthermore since \( \pi_1, \ldots, \pi_n \) are mutually inequivalent, it follows that there exists a hermitian element \( y \in A \) such that \( \pi_i(y)\xi_{ij} = \xi_{ij}(1 \leq i \leq n, 1 \leq j \leq m_i) \) by ([2, Theorem 2.8.3, (i)]). Now consider the continuous function \( h(t) \) on \([0, \infty)\) defined by

\[ h(t) = \begin{cases} 
  t, & \text{if } 0 \leq t \leq 1 \\
  1, & \text{if } t > 1
\end{cases} \]

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and set $z = h(y^2)$. Then $z$ is a positive element of $A$ with $\|z\| \leq 1$. Moreover, we assert that

$$\pi_{ij}(z) = \xi_{ij}(1 \leq i \leq n, 1 \leq j \leq m_i).$$

In fact, let $\varepsilon > 0$ be arbitrary and take a polynomial $p(t)$ such that $p(0) = 0$ and $\sup\{|p(t) - h(t)| : 0 \leq t \leq \|z\|\} < \varepsilon/2$. Let $1 \leq i \leq n$ and $1 \leq j \leq m_i$. Then

$$\|\pi_{ij}(z) - \xi_{ij}\| \leq \|\pi_{ij}(h(y^2)) - \pi_{ij}(p(y^2))\| + \|p(\pi_{ij}(y^2)) - \xi_{ij}\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and hence we obtain (5) since $\varepsilon$ is arbitrary. By (4) and (5), we have

$$f(z) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} (\pi_{ij}(z) - \xi_{ij}) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} = 1.$$

Consequently we have $\alpha \in V_1(A, a)$ and hence $\text{co}\{f(a) : f \in P(A)\} \subseteq V_1(A, a)$.

We next show that $V_1(A, a) \subseteq \overline{\text{co}}\{f(a) : f \in P(A)\}$. To do this, let $\alpha \in V_1(A, a)$ and so there exist $f \in A^*$ and $x \in A$ such that $\alpha = |f|(a)$ and $\|f\| = \|x\| = f(x) = 1$. Note that $|f|(x^*x) = 1$ as observed in the proof of the main theorem and consider the following set:

$$S = \{g \in A^* : g \geq 0 \text{ and } \|g\| = g(x^*x) = 1\}.$$

Then $|f| \in S$ and $S$ is weak*-closed. Moreover, we can easily see that any extreme point of $S$ is also an extreme point of $\{g \in A^* : g \geq 0 \text{ and } \|g\| \leq 1\}$. But since the extreme points of $\{g \in A^* : g \geq 0 \text{ and } \|g\| \leq 1\}$ consist of $0$ and $P(A)$ (cf. [2, Proposition 2.5.5]), it follows by the Krein-Milman theorem that $S \subseteq \overline{\text{co}} P(A)$. Then $\alpha = |f|(a) = \lim \lambda g_\lambda(a)$ for some net $\{g_\lambda\}$ in $\text{co} P(A)$, and hence $\alpha \in \overline{\text{co}}\{f(a) : f \in P(A)\}$.  

\[\square\]

References


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